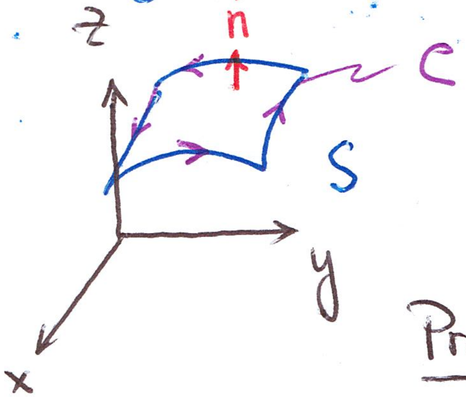
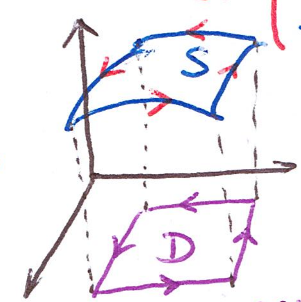
§16.8 Stoke's Theorem

Stoke's Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S. $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$



$$\int_{C=\partial S} \vec{F} \cdot d\vec{r} = \iint_S \overrightarrow{\nabla \times F} \cdot d\vec{S}$$



$C: \begin{cases} x = x(t) \\ y = y(t) \\ z = g(x(t), y(t)) \end{cases}$

$C_1: \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad a \le t \le b$

Proof of S: $z = g(x,y)$, $(x,y) \in D$

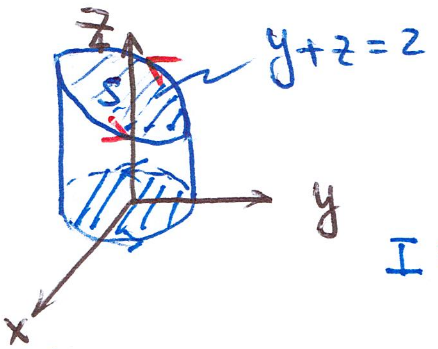
$S: \vec{r}(x,y) = \langle x, y, g(x,y) \rangle$

$$\iint_S \overrightarrow{\nabla \times F} \cdot d\vec{S} = \iint_D \nabla \times \vec{F} \cdot \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle \, dx dy \quad (x,y) \in D$$

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\left(x(t), y(t), g(x(t), y(t))\right) \cdot \langle x'(t), y'(t), \frac{\partial g}{\partial x} \cdot x'(t) + \frac{\partial g}{\partial y} \cdot y'(t) \rangle \, dt$$

Ex. 1 Evaluate $I=\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y,z)=\langle -y^2, x, z^2\rangle$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$.



$C:\ \vec{r}(t)=\langle \cos t, \sin t, 2-\sin t\rangle \qquad t: 0\to 2\pi$

$$I=\int_0^{2\pi}\langle -\sin^2 t, \cos t, (2-\sin t)^2\rangle\cdot\langle -\sin t, \cos t, -\cos t\rangle\, dt$$

$$I=\iint_S (\nabla\times \vec{F})\cdot d\vec{S} = \iint_D \langle 0,0,1+2y\rangle\cdot\langle 0, 1, 1\rangle\, dxdy$$

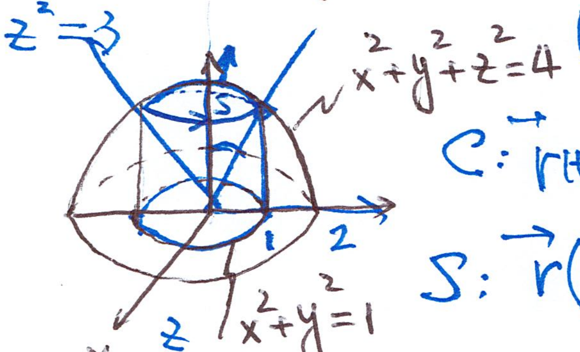
$$=\iint_D (1+2y)\,dxdy = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\, r\,dr\,d\theta$$

$S:\ \vec{r}(x,y)=\langle x, y, 2-y\rangle$

$D: \{(x,y) \mid x^2+y^2\le 1\}$

$$\nabla\times\vec{F}=\begin{vmatrix} i & j & k\\ \partial_x & \partial_y & \partial_z \\ -y^2 & x & z^2\end{vmatrix} = \langle 0, 0, 1+2y\rangle$$

Ex. 2 Use Stoke's ~~Theorem~~ Theorem to compute $\underbrace{\iint_S \nabla\times \vec{F}\cdot d\vec{S}}_{=I}$, where $\vec{F}=\langle xz, yz, xy\rangle$ and S is the part of $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.

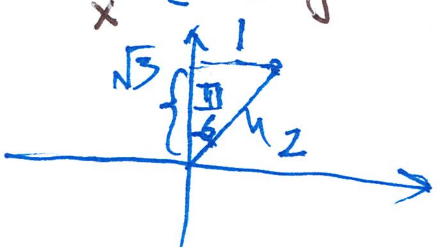


(1) $I=\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\langle \cos t\cdot\sqrt{3}, \sqrt{3}\sin t, \cos t\sin t\rangle\cdot\langle -\sin t, \cos t, 0\rangle\, dt$

$C:\ \vec{r}(t)=\langle \cos t, \sin t, \sqrt{3}\rangle,\ t: 0\to 2\pi \qquad = \sqrt{3}\int_0^{2\pi}(-\cos t\sin t+\sin t\cos t)\,dt = 0$

$S:\ \vec{r}(\varphi,\theta)=\langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle,\quad 0\le\theta\le 2\pi,\quad 0\le\varphi\le\frac{\pi}{6}$

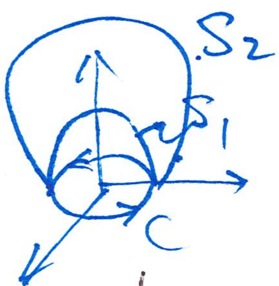
$=\langle x, y, z\rangle$



$$I=\iint_D \langle x-y, x-y, 0\rangle\cdot 2\sin\varphi\, \vec{r}(\varphi,\theta)\, d\varphi\, d\theta \qquad D=[0,\tfrac{\pi}{6}]\times[0,2\pi]$$

$$=\int_0^{\frac{\pi}{6}}\int_0^{2\pi} 2(x^2-y^2)\cdot 2\sin\varphi\, d\varphi\, d\theta = \int_0^{\frac{\pi}{6}}\int_0^{2\pi} 16\sin^3\varphi(\cos 2\theta)\, d\varphi\, d\theta$$

- $\underline{S_1}$ and $\underline{S_2}$ are surfaces having the same orientation, and $\underline{\partial S_1 = \partial S_2 = C}$



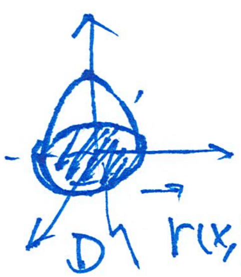
$$\iint_{S_1} \nabla\times\vec{F}\cdot d\vec{S} = \iint_{S_2} \nabla\times\vec{F}\cdot d\vec{S}$$

<u>Examples</u> Use Stoke's Theorem to evaluate $\iint_S \nabla\times\vec{F}\cdot d\vec{S} = I$

<u>#2 (P1139)</u> $\vec{F} = \langle x^2\sin z, y^2, xy\rangle$, S is the part of $z = 1-x^2-y^2$ that lies above the xy-plane, oriented upward.

$$I = \iint_D \langle x^2\sin(1-x^2-y^2), y^2, xy\rangle\cdot\langle +2x, 2y, 1\rangle\,dxdy$$

$$\cancel{= \iint_D \left(2x^3\sin(1-x^2-y^2) + 2y^3 + xy\right)dxdy}$$



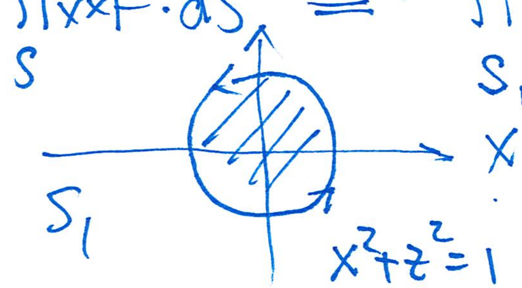
$\vec{r}(x,y) = \langle x, y, 0\rangle$ $\quad I = \iint_D \nabla\times\langle 0, y^2, xy\rangle\cdot\langle 0,0,1\rangle\,dxdy = \iint_D \langle x, y, 0\rangle\cdot\langle 0,0,1\rangle = 0$

<u>#6 (P1139)</u> $\vec{F} = \langle e^{xy}, e^{xz}, x^2z\rangle$, S is the half of $4x^2+y^2+4z^2=4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis.

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ e^{xy} & e^{xz} & x^2z \end{vmatrix} = \langle *, -2xz, * \rangle$$

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_{S_1} \nabla\times\vec{F}\cdot d\vec{S} = \iint_D \langle *, -2xz, *\rangle\cdot\langle 0,1,0\rangle\,dxdz$$

$$= -2\iint_D xz\,dxdy$$



$S_1: \vec{r}(x,z) = \langle x, 0, z\rangle$, $(x,z)\in D$

$$\vec{r}_x\times\vec{r}_z = \left\langle -\frac{\partial y}{\partial x}, 1, -\frac{\partial y}{\partial z}\right\rangle = \langle 0, 1, 0\rangle$$