§16.9 The Divergence Theorem

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region that contains E.

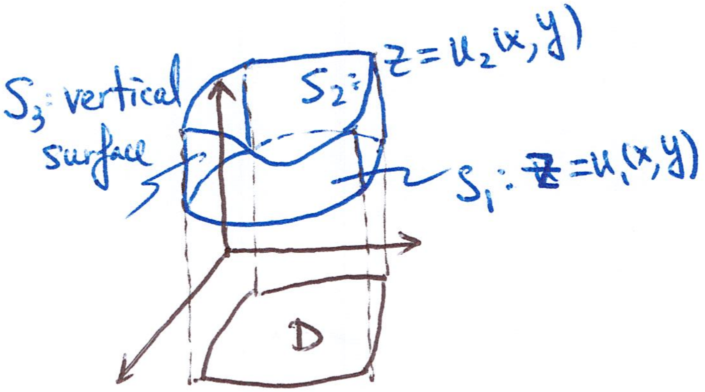
$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{F})\, dV.$$

$$\vec{F} = \langle P, Q, R \rangle$$

Proof of $\displaystyle\iint_S R(\vec{k} \cdot \vec{n})\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV$

$$\int_S \cancel{\boxed{\vec{F} \cdot d\vec{r}}} (\vec{F} \cdot \vec{n})\, ds$$

$$= \iint_E (\nabla \cdot \vec{F})\, dA$$



$$\int_S \vec{F} \cdot d\vec{r} = \int_S (\vec{F} \cdot \vec{T})\, ds$$

E: $u_1(x,y) \le z \le u_2(x,y)$

$(x,y) \in D$

Ex. 1 Find the flux of $\vec{F} = \langle z, y, x\rangle$ over the unit sphere $x^2+y^2+z^2=1$.

$S: \vec{r}(\varphi,\theta) = \langle \overset{x}{\overset{\|}{\sin\varphi\cos\theta}}, \overset{y}{\overset{\|}{\sin\varphi\sin\theta}}, \overset{z}{\overset{\|}{\cos\varphi}} \rangle$

$\vec{r}_\varphi \times \vec{r}_\theta = \sin\varphi \; \vec{r}(\varphi,\theta)$

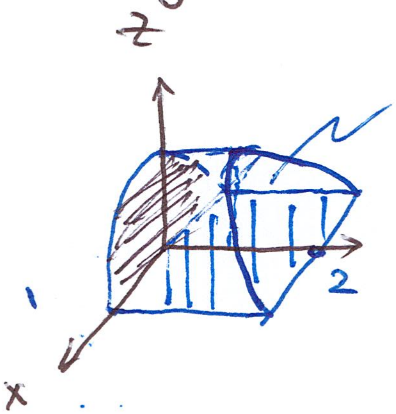
$D = [0,\pi]\times[0,2\pi]$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle z,y,x\rangle \cdot \sin\varphi \, \langle x,y,z\rangle \, d\varphi \, d\theta$$

$$= \iint_D \sin\varphi \, (2xz+y^2) \, d\varphi \, d\theta = \int_0^{\pi}\int_0^{2\pi} \sin\varphi \cdots$$

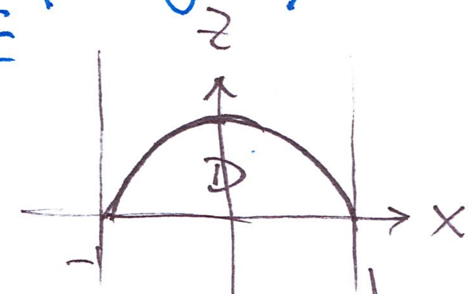
$$= \iiint_E (\nabla\cdot\vec{F}) \, dV = \iiint_E (0+1+0) \, dV = \iiint_E 1 \, dV = |E| = \frac{4}{3}\pi\cdot 1^3 = \frac{4\pi}{3}$$

Ex. 2 Evaluate $\iint_S \vec{F}\cdot d\vec{S}$, $\vec{F} = \langle xy, \, y^2+e^{xz^2}, \, \sin(xy)\rangle$, S is the surface of the region E bounded by $z=1-x^2$ and the planes $z=0$, $y=0$, and $y+z=2$.



$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F} \, dV = \iiint_E (y + 2y + 0) \, dV$$

$0 \le y \le 2-z$, $(x,z)\in D$

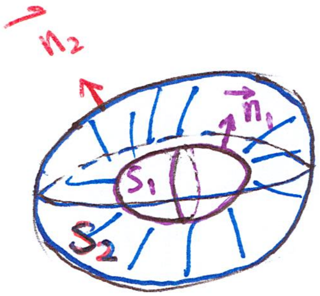


$-1 \le x \le 1$

$0 \le z \le 1-x^2$

$$= \int_{-1}^{1} dx \int_0^{1-x^2} dz \int_0^{2-z} 3y \, dy$$

$$= \int_{-1}^{1} dx \int_0^{1-x^2} \frac{3}{2}(2-z)^2 \, dz$$



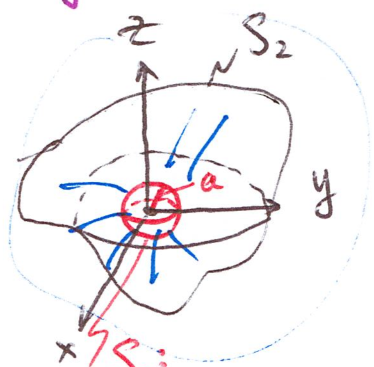
The region E lies between the closed surfaces S_1 and S_2.

$\partial E = S = S_1 \cup S_2$.

$$\iiint_E \operatorname{div} \vec{F}\, dV = \underline{-\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}}$$

Ex. 3 The electric field $\underline{\vec{E}(\vec{x}) = \frac{\varepsilon Q}{|\vec{x}|^3}\vec{x}}$, the electric charge Q is located at the origion, $\vec{x} = \langle x, y, z\rangle$ is a position vector. Prove that $\underline{|\vec{x}| = \sqrt{x^2+y^2+z^2}}$, $\vec{x} = \langle x, y, z\rangle$

the electric flux of $\vec{E}$ $\quad \underline{\iint_{S_2} \vec{E}\cdot d\vec{S}} = \underline{4\pi\varepsilon Q}$, S_2 is any closed surface containing the origion.

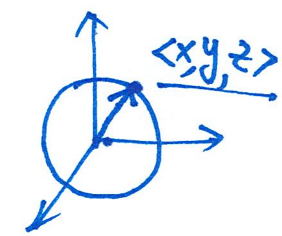


Proof

$$\iint_{S_2} \vec{E}\cdot d\vec{S} = \iiint_E \underset{0}{\underset{\parallel}{\operatorname{div} \vec{E}}}\, dV + \iint_{S_1} \vec{E}\cdot d\vec{S} = \iint_{S_1} \vec{E}\cdot d\vec{S}$$

$$= \iint_{S_1} (\vec{E}\cdot\vec{n})\, dS = \frac{\varepsilon Q}{a^2}\cdot|S_1| = \frac{\varepsilon Q}{a^2}\cdot 4\pi a^2$$

$$= \varepsilon Q 4\pi$$



$$\frac{\varepsilon Q}{|\vec{x}|^3}\vec{x}\cdot\frac{\vec{x}}{|\vec{x}|} = \frac{\varepsilon Q\,|\vec{x}|^2}{|\vec{x}|^4} = \frac{\varepsilon Q}{|\vec{x}|^2} = \frac{\varepsilon Q}{a^2}$$

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2$$

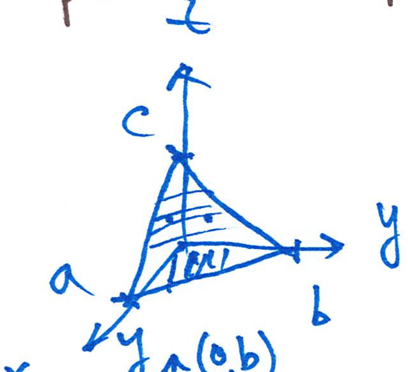
Evaluate $\iint_S \vec{F}\cdot d\vec{S}$

#6 (P1145) $\vec{F} = \langle x^2yz,\ xy^2z,\ xyz^2 \rangle$, S is the surface of the box enclosed by the planes $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, and $z=c$. $(a>0,\ b>0,\ c>0)$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = \int_0^a \int_0^b \int_0^c (2xyz + 2xyz + 2xyz)\, dz\, dy\, dx$$

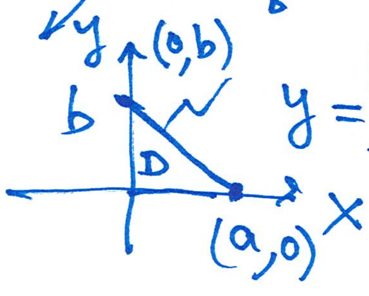
$$= 6 \int_0^a x\,dx \int_0^b y\,dy \int_0^c z\,dz = \frac{3}{4} a^2 b^2 c^2$$

#10 (P1145) $\vec{F} = \langle z,\ y,\ zx \rangle$, S is the surface of the tetrahedron enclosed by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. $(a>0,\ b>0,\ c>0)$.



$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E (0 + 1 + x)\, dV$$

$$0 \le z \le c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \qquad = \int_0^a dx \int_0^{b - \frac{b}{a}x} dy \int_0^{c\left(1 - \frac{x}{a} - \frac{y}{b}\right)} (1+x)\, dz$$



$$y = \frac{b}{-a}(x-a) = b - \frac{b}{a}x$$