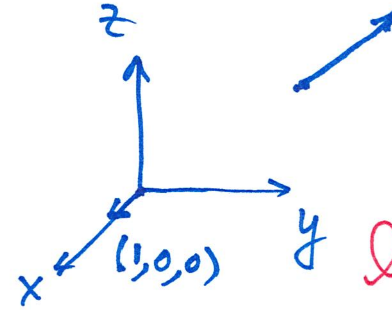
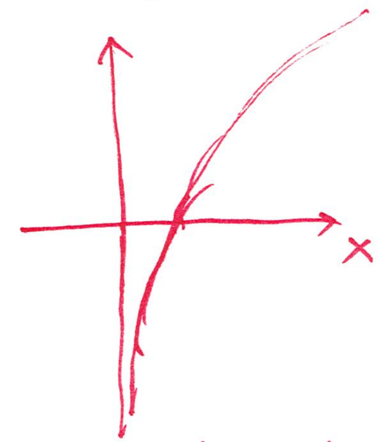
Chapter 13 Vector Functions (4 lectures)

§13.1 Vector Functions and Space Curves

$i = \langle 1, 0, 0 \rangle$





- **vector-valued functions**

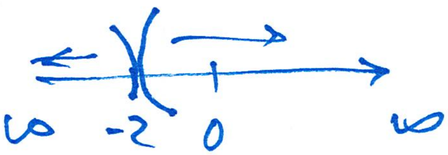
$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

- **Domain** $\quad \text{Dom}(\vec{r}(t)) = \{ t \mid f(t), g(t), h(t) \text{ are well defined} \}$

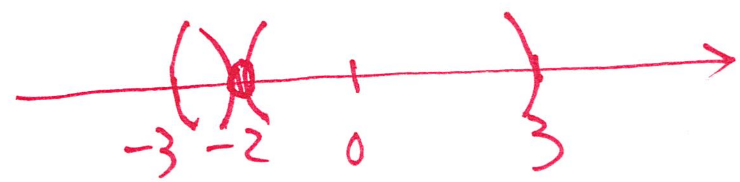
$$= \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$$

$(-3,-2) \cup (-2,3)$

Example (#2) $\quad \vec{r}(t) = \left\langle \frac{t-2}{t+2}, \sin t, \ln(9-t^2) \right\rangle$, $\text{Dom}(\vec{r}) = ?$



$$\text{dom}\left(\frac{t-2}{t+2}\right) = \{ t \mid t \neq -2 \} = (-\infty, -2) \cup (-2, +\infty)$$



$$\text{dom}(\sin t) = (-\infty, +\infty)$$

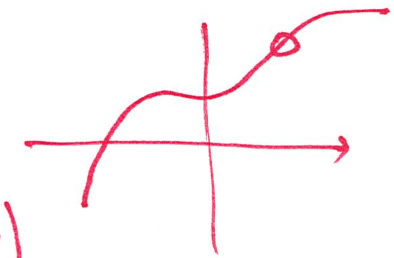
$$\text{dom}(\ln(9-t^2)) = \{ t \mid 9 - t^2 > 0 \} = (-3, 3)$$

$t^2 < 9$, $|t| < 3$, $-3 < t < 3$

• limit

$$\lim_{t\to a}\vec{r}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right\rangle$$

• continuity



$\vec{r}(t)$ is continuous at a $\iff \displaystyle\lim_{t\to a}\vec{r}(t)=\vec{r}(a)$

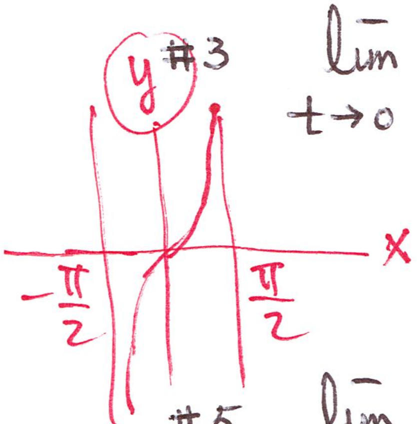
$$\lim_{x\to a} f(x) = f(a) = f\left(\lim_{x\to a} x\right)$$

examples $\displaystyle\lim_{t\to 0}\frac{\sin t}{t} = 1$

$$\frac{0}{0} \qquad \lim \frac{\cancel{2}t}{\cancel{2}\sin t\cdot\cos t} = \lim_{t\to 0}\frac{1}{\cos t}\cdot\lim\frac{t}{\sin t}$$

#3 $\displaystyle\lim_{t\to 0}\left\langle e^{-3t},\ \lim\frac{t^2}{\sin^2 t},\ \cos 2t \right\rangle = \langle e^0, 1, 1\rangle$

$= \langle 1, 1, 1\rangle$



#5 $\displaystyle\lim_{t\to\infty}\left\langle \frac{1+t^2}{1-t^2},\ \tan^{-1}t,\ \frac{1-e^{-2t}}{t} \right\rangle = \left\langle \lim_{t\to\infty}\frac{1+t^2}{1-t^2},\ \lim_{t\to\infty}\tan^{-1}t,\ \lim\frac{1-e^{-2t}}{t} \right\rangle$

$$\frac{\frac{1+t^2}{t^2}}{\frac{1-t^2}{t^2}} = \frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-1}$$

$y=\tan x$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$

$= \left\langle -1,\ \frac{\pi}{2},\ 0 \right\rangle$

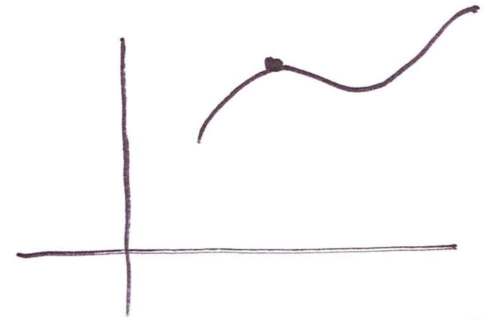
$$\frac{1}{e^{2t}} \to \frac{1}{\infty} = 0$$

$x = \tan^{-1} y$

- representations of curves in 2D

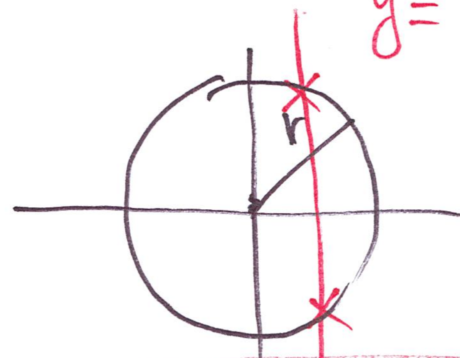
$y = f(x)$ $\quad (x, f(x))$

(1) graphes $\quad C = \{ (x, f(x)) \mid x \in D \}$



$y = \pm\sqrt{r^2 - x^2}$

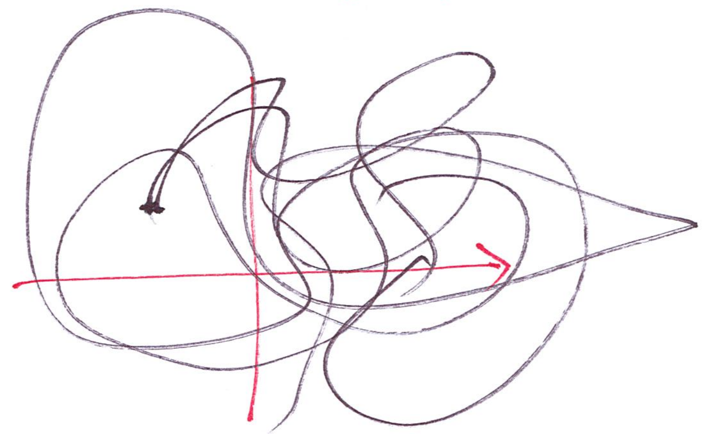
(2) level curves



$f(x,y) \equiv \boxed{x^2 + y^2 = r^2}$

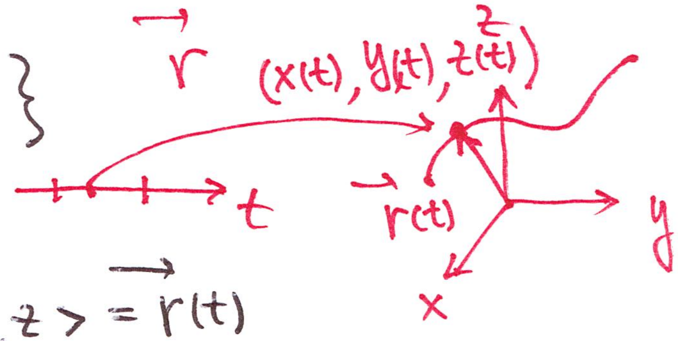
(3) parametric curves

$\vec{r}(t) = \langle x(t), y(t) \rangle$



- space curves

$$C = \{ (f(t), g(t), h(t)) \mid t \in I \}$$

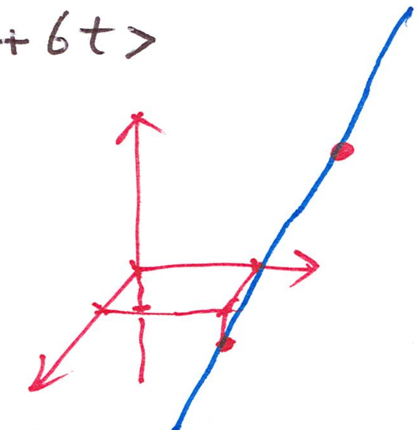


parametric equations of C $\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$ or $\langle x, y, z \rangle = \vec{r}(t)$

Ex. 3 Describe the curve defined by $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$

$$\begin{cases} x = 1+t \\ y = 2+5t \\ z = -1+6t \end{cases} \qquad \vec{r} = \vec{r}_0 + t\vec{v} \qquad \begin{aligned} \vec{r}_0 &= \langle 1, 2, -1 \rangle \\ \vec{v} &= \langle 1, 5, 6 \rangle \end{aligned}$$

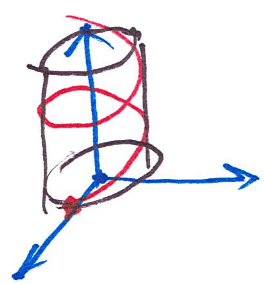
$\vec{r}(1) = \langle 2, 7, 5 \rangle$



Ex. 4 Sketch the curve whose vector equation is $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

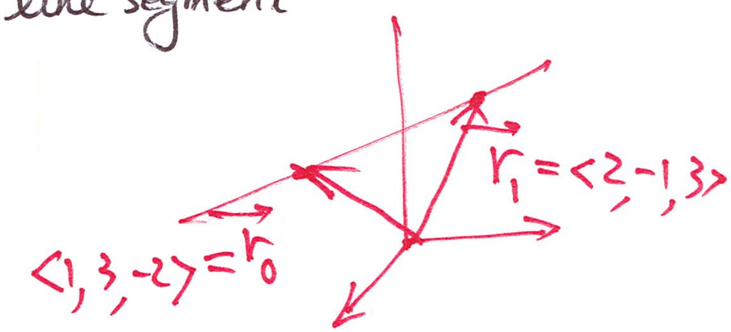
$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \qquad x^2 + y^2 = 1$$

$\langle 1, 0, 0 \rangle$



helix

Ex. 5 Find a vector equation and parametric equations for the line segment that joins the points $(1,3,-2)$ and $(2,-1,3)$.



$$\vec{r}(t) = t\,\vec{r}_0 + (1-t)\,\vec{r}_1$$

$$0 \le t \le 1$$

Ex. (#8) Sketch the curve of $\vec{r}(t) = \langle t^3, t^2\rangle$. Indicate the direction in which t increases

$$\begin{cases} x = t^3 \Rightarrow t = x^{\frac{1}{3}} \\ y = t^2 = x^{\frac{2}{3}} \end{cases}$$

$$y = x^{\frac{2}{3}}$$

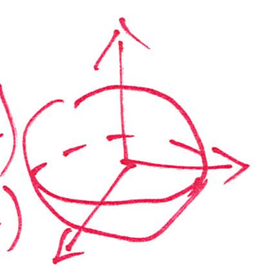
Ex. (#30) At what points does the helix $\vec{r}(t) = \langle \sin t, \cos t, t\rangle$ intersect the sphere $x^2+y^2+z^2=5$?

$$\begin{cases} x = \sin t \\ y = \cos t \\ z = t \end{cases}$$

$$\underline{\sin^2 t + \cos^2 t} + t^2 = 5$$

$$t^2 = 5-1 = 4 \qquad t = \pm 2$$

$$\begin{cases} x = \sin(-2) \\ y = \cos(-2) \\ z = -2 \end{cases}$$



Ex. (#48) Two particles travel along the space curves

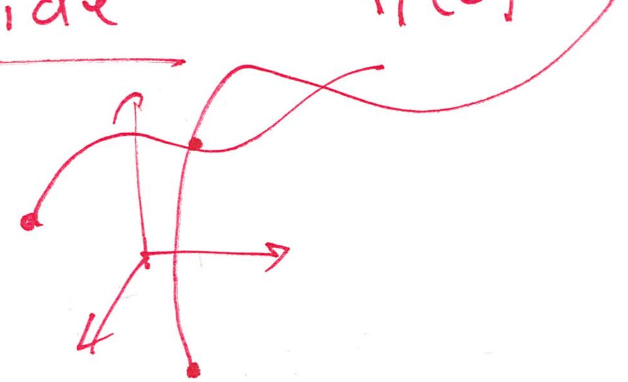
$$\vec{r}_1(t) = \langle t,\ t^2,\ t^3 \rangle, \quad \vec{r}_2(t) = \langle 1+2t,\ 1+6t,\ 1+14t \rangle, \quad \text{for } t \geq 0$$

Do the particles collide? Do their paths intersect?

(a) collide

$$\vec{r}_1(t) = \vec{r}_2(t)$$

$$\begin{cases} t = 1+2t \implies t = -1 < 0 \\ t^2 = 1+6t \\ t^3 = 1+14t \end{cases}$$



(b) intersection

$$\vec{r}_1(t) = \vec{r}_2(s)$$

$$\begin{cases} t = 1+2s \quad \cancel{t \geq 0} \\ t^2 = 1+6s \\ t^3 = 1+14s \end{cases}$$

$$\begin{cases} (1+2s)^2 = 1+6s \\ \left((1+2s)\right)^3 = 1+14s \end{cases}$$