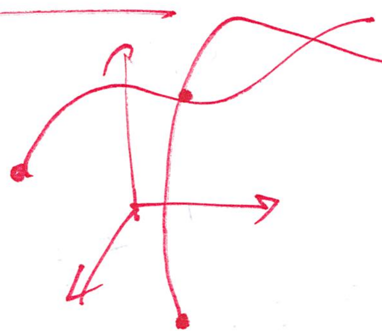
Ex. (#48) Two particles travel along the space curves

$$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad \vec{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle, \quad \text{for } t \geq 0$$

Do the particles collide? Do their paths intersect?

(a) collide $\quad \vec{r}_1(t) = \vec{r}_2(t)$

$$\begin{cases} t = 1+2t \Rightarrow t = -1 < 0 \\ t^2 = 1+6t \\ t^3 = 1+14t \end{cases}$$



(b) intersection $\quad \vec{r}_1(t) = \vec{r}_2(s)$

$$\begin{cases} t = 1+2s \quad \cancel{t^2} \\ t^2 = 1+6s \\ t^3 = 1+14s \end{cases}$$

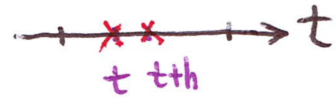
$$\begin{cases} (1+2s)^2 = 1+6s \\ (1+2s)^3 = 1+14s \end{cases}$$

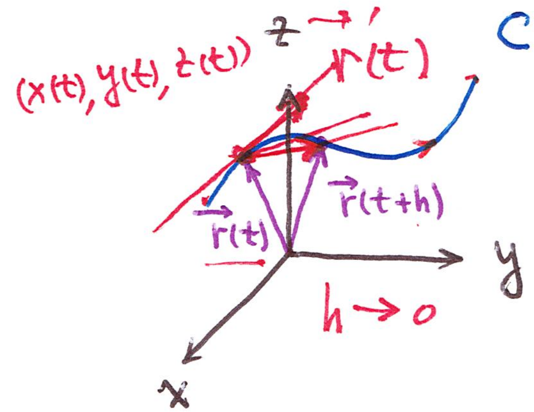
$$\frac{1}{1+2s} = \frac{1+6s}{1+14s}$$

$$1+14s = (1+6s)(1+2s)$$

§13.2 Derivatives and Integrals of Vector Functions

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

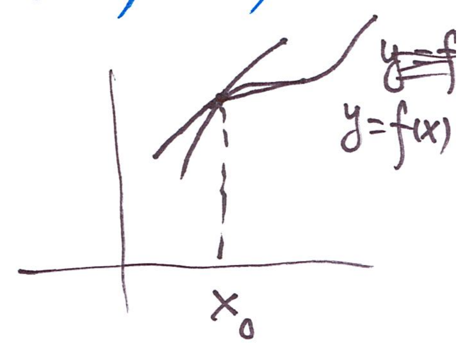




- derivative

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \lim_{h \to 0} \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{h(t+h) - h(t)}{h} \right\rangle = \langle f'(t), g'(t), h'(t) \rangle$$



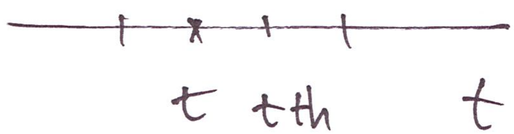
$f'(x_0) =$

- vector equation of tangent line at $\vec{r}(t_0)$

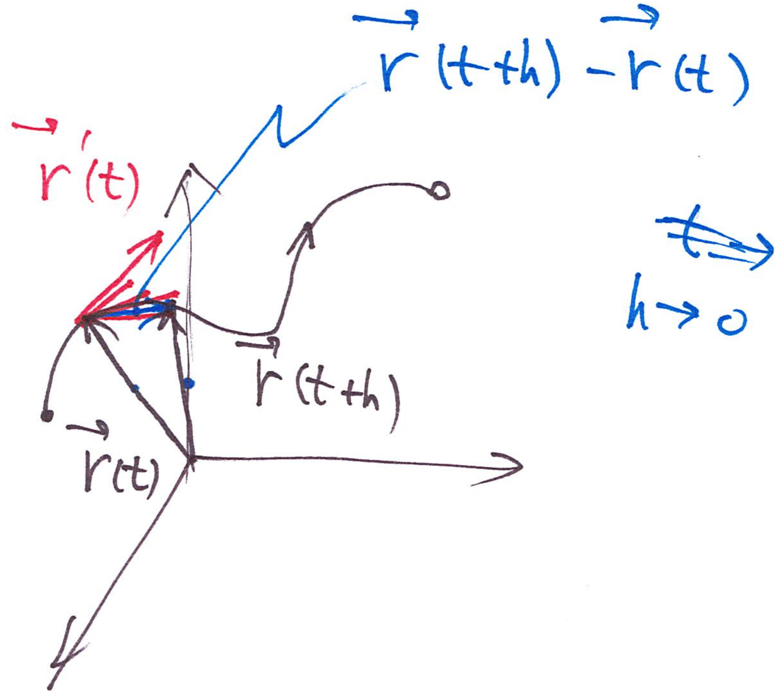
pt. $\vec{r}(t_0)$

vector $\vec{r}\,'(t_0) \parallel \mathcal{L}$

$$\vec{R}(t) = \vec{r}_0 + t\,\vec{r}\,'(t_0)$$



t
t+h
t



$\vec{r}(t+h) - \vec{r}(t)$
$\vec{r}'(t)$
$\vec{r}(t+h)$
$\vec{r}(t)$
$h \to 0$

Ex. 1 Let $\vec{r}(t) = \langle 1+t^3, te^{-t}, \sin 2t \rangle$. (a) $\vec{r}\,'(t) = ?$ (b) Find the unit tangent vector at the point where $t=0$.

$$\vec{r}\,'(t) = \langle (1+t^3)', (te^{-t})', (\sin(2t))' \rangle = \langle 3t^2, e^{-t} - te^{-t}, 2\cos(2t) \rangle$$

$\vec{r}\,'(0) = \langle 0, 1, 2 \rangle$ tangent vector $\rightarrow$ $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, $\vec{T}(0) = \dfrac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$

$$|\vec{r}\,'(0)| = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5}$$

Ex. 2 Let $\vec{r}(t) = \langle \sqrt{t}, 2-t \rangle$, find $\vec{r}\,'(t)$ and sketch the position verctor $\vec{r}(1)$ and the tangent vector $\vec{r}\,'(1)$.

$\sqrt{t} = t^{\frac{1}{2}}$

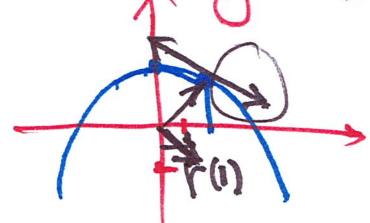
$$\vec{r}\,'(t) = \langle \tfrac{1}{2} t^{-\frac{1}{2}}, -1 \rangle$$

$$\vec{r}(1) = \langle 1, 1 \rangle$$

$$\vec{r}\,'(1) = \langle \tfrac{1}{2}, -1 \rangle$$

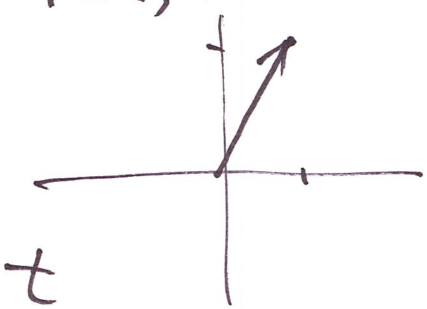
$$\begin{cases} x = \sqrt{t} \\ y = 2 - t \Longrightarrow t = 2 - y \end{cases}$$

$x = \sqrt{2-y}$ $\quad$ $x^2 = \cancel{y} \; 2 - y$



Ex. 3 Find parametric equations for the tangent line to the helix with para. eq. $\begin{cases} x = 2\cos t \\ y = \sin t \\ z = t \end{cases}$ at the point $(0, 1, \frac{\pi}{2}) = (x(t), y(t), z(t)) = \vec{r}(t)$

$\vec{r} = \langle 1, 3 \rangle$



$$\vec{r}\,'(t) = \langle -2\sin t, \cos t, 1 \rangle$$

$$\begin{cases} 2\cos t = 0 \\ \sin t = 1 \\ \frac{\pi}{2} = t \end{cases}$$

$t = \frac{\pi}{2}$ $\quad$ $\vec{r}(\frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2} \rangle$

$$\vec{r}\,'(\tfrac{\pi}{2}) = \langle -2, 0, 1 \rangle$$

$$\begin{cases} x = 0 + (-2)t \\ y = 1 + 0 \\ z = \frac{\pi}{2} + t \end{cases}$$

Differentiation Rules

$\vec{u}(t), \vec{v}(t)$ — diff. vector functions, $f(t)$ — diff. scalar function
c — constant

(1) $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \vec{u}'(t) \pm \vec{v}'(t)$; (2) $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\vec{u}'(t)$

(3) $\frac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$

(4) $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

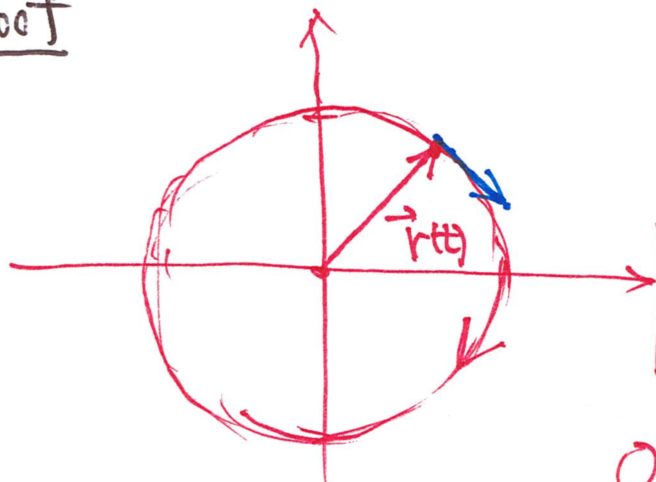
(5) $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

(6) $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}'(f(t))$

Proof of (4)

Ex. 4 Show that if $|\vec{r}(t)| = c$ (a constant), then $\vec{r}\,'(t) \perp \vec{r}(t)$ for all t.

Proof



$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

$$c = |\vec{r}(t)| = \sqrt{x^2(t) + y^2(t)}$$

$$\boxed{c^2 = x^2(t) + y^2(t)}$$

$$0 = 2x(t)x'(t) + 2y(t)y'(t)$$

$$\vec{r}\,'(t) \perp \vec{r}(t)$$

$$\Updownarrow$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\langle x(t), y(t) \rangle \cdot \langle x'(t), y'(t) \rangle$$

$$\parallel$$

$$x(t)x'(t) + y(t)y'(t)$$

Ex. (#49) Find $f'(2)$, where $f(t) = \vec{u}(t) \cdot \vec{v}(t)$, $\vec{u}(2) = \langle 1, 2, -1 \rangle$, $\vec{u}\,'(2) = \langle 3, 0, 4 \rangle$, and $\vec{v}(t) = \langle t, t^2, t^3 \rangle$.

$$f'(2) = f'(t)\Big|_{t=2} = \Big[\vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)\Big]_{t=2}$$

$$= \vec{u}\,'(2) \cdot \vec{v}(2) + \vec{u}(2) \cdot \vec{v}\,'(2) = \langle 3, 0, 4 \rangle \cdot \langle 2, 4, 8 \rangle$$

$$+ \langle 1, 2, -1 \rangle \cdot \langle 1, 4, 12 \rangle$$

- **Integrals**

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

example (#35)

$$\int_0^2 \langle t, -t^3, 3t^5 \rangle\,dt$$

$$= \left\langle \int_0^2 t\,dt, -\int_0^2 t^3\,dt, 3\int_0^2 t^5\,dt \right\rangle$$

$$= \left\langle \frac{1}{2}t^2 \Big|_0^2, -\frac{1}{4}t^4 \Big|_0^2, \frac{3}{6} t^6 \Big|_0^2 \right\rangle$$

$$= \langle 2, -4, 2^5 \rangle$$