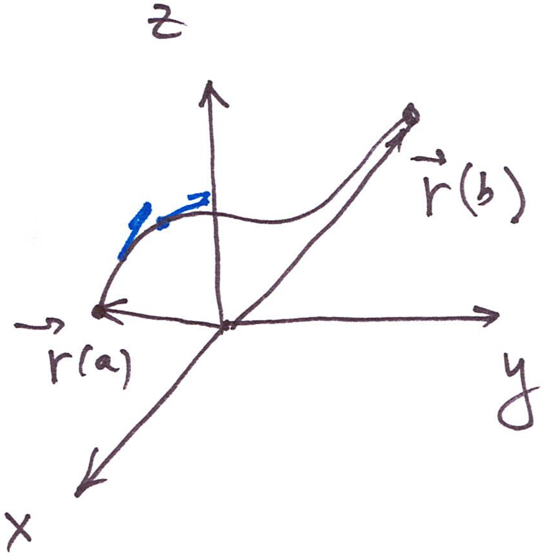
$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \qquad t \in [a, b]$$



$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

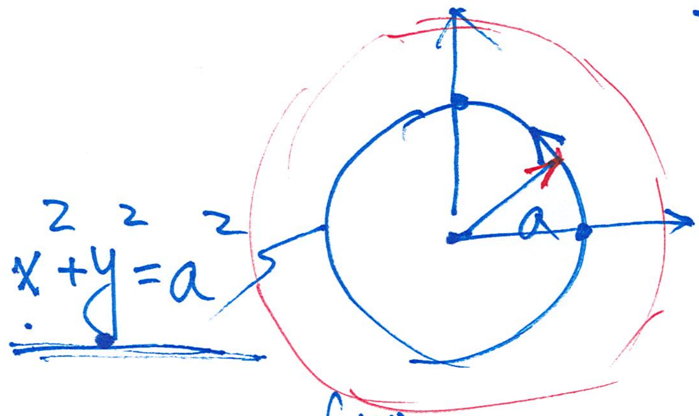
$$T'(t) \neq \vec{r}\,''(t)$$

arc length $\quad L = \displaystyle\int_a^b |\vec{r}\,'(t)|\, dt, \qquad s = s(t) = \int_a^t |\vec{r}\,'(u)|\, du$

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}, \qquad \vec{N} = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \frac{1}{\kappa} \frac{d\vec{T}}{ds}$$

example Find curvature of a circle with radius a.

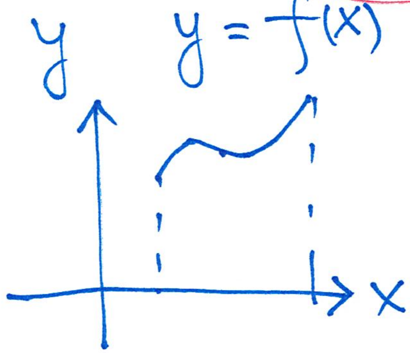
$$\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right|} = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$



$$\vec{r}(t) = \langle a\cos t,\ a\sin t\rangle, \quad 0 \le t < 2\pi$$

$$\vec{r}'(t) = a\langle -\sin t,\ \cos t\rangle$$

$$|\vec{r}'(t)| = a\,|\langle -\sin t,\ \cos t\rangle| = a\sqrt{(-\sin t)^2 + \cos^2 t} = a$$



$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \langle -\sin t,\ \cos t\rangle \qquad \vec{T}'(t) = -\langle \cos t,\ \sin t\rangle$$

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{1}{a}$$

$$s = \int_0^t |\vec{r}'(t)|\,dt = \int_0^t a\,dt = a$$

$$t = \frac{1}{a}s$$

$x = t$

$y = f(t)$

$$\vec{p}(s) = \vec{r}(t(s)) = \vec{r}\left(\frac{1}{a}s\right) = a\left\langle \cos\frac{s}{a},\ \sin\frac{s}{a}\right\rangle$$

$$\vec{T}(t(s)) = \left\langle -\sin\frac{s}{a},\ \cos\frac{s}{a}\right\rangle \qquad \left(\sin\frac{s}{a}\right)' = \frac{1}{a}\cos\frac{s}{a}$$

$$\frac{d\vec{T}(t(s))}{ds} = -\frac{1}{a}\left\langle \cos\frac{s}{a},\ \sin\frac{s}{a}\right\rangle, \quad \kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{a}\sqrt{\cos^2 + \sin^2}$$

§13.4 Motion in Space: Velocity and Acceleration

$\vec{r}(t)$ — the position of a moving particle

$\underline{\vec{v}(t)} = \underline{\vec{r}\,'(t)}$ — the velocity

$|\vec{v}(t)|$ — the speed $\quad |\vec{v}(t)| = |\vec{r}\,'(t)| = \frac{ds}{dt}$

$\underline{\vec{a}(t)} = \underline{\vec{v}\,'(t)} = \vec{r}\,''(t)$ — the acceleration

<u>Ex. 1</u> The position vector of an object moving in a plane given by $\underline{\vec{r}(t) = \langle t^3, t^2 \rangle}$. Find its velocity, speed, and acceleration when $\underline{t=1}$ and illustrate geometrically.

$\vec{v}(t) = \vec{r}\,'(t) = \langle 3t^2, 2t \rangle$ $\qquad \vec{v}(1) = \langle 3, 2 \rangle$

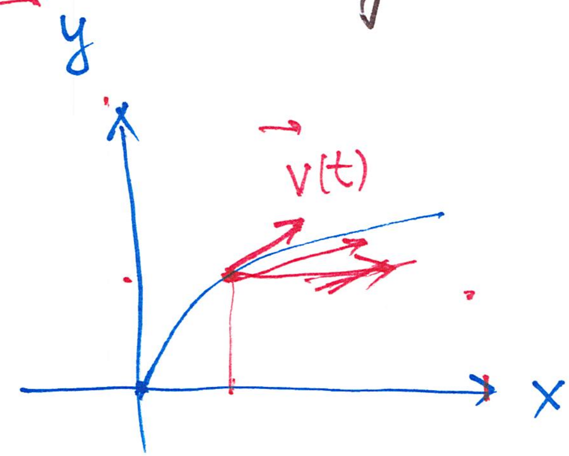
$v = |\vec{v}(t)| = \sqrt{(3t^2)^2 + (2t)^2}$ $\qquad v = \sqrt{13}$

$= \sqrt{9t^4 + 4t^2}$ $\qquad \vec{a}(1) = \langle 6, 2 \rangle$

$\vec{a}(t) = \vec{v}\,'(t) = \langle 6t, 2 \rangle$

$x = t^3 \Rightarrow t = x^{\frac{1}{3}}$

$y = t^2 = x^{\frac{2}{3}}$



Ex. 2 Find the velocity, acceleration, and speed of a particle with position vector $\vec{r}(t) = \langle t^2, e^t, te^t \rangle$.

$\vec{v}(t) = \vec{r}'(t) = \langle 2t, e^t, e^t + te^t \rangle$, $v = |\vec{v}(t)| = \sqrt{(2t)^2 + e^{2t} + e^{2t}(1+t)^2}$

$\vec{a}(t) = \vec{v}'(t) = \langle 2, e^t, 2e^t + te^t \rangle$

Ex. 3 A moving particle starts at an initial position $\vec{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\vec{v}(0) = \langle 1, -1, 1 \rangle$. Its acceleration is $\vec{a}(t) = \langle 4t, 6t, 1 \rangle$.

Find $\vec{v}(t)$ and $\vec{r}(t)$.

$\vec{a}(t) = \vec{v}'(t) \Longrightarrow \vec{v}(t) = \int \vec{a}(t)\,dt = \langle 2t^2 + c_1, 3t^2 + c_2, t + c_3 \rangle$

$= \langle 2t^2, 3t^2, t \rangle + \vec{C}$

$\langle 1, -1, 1 \rangle = \vec{v}(0) = \langle 0, 0, 0 \rangle + \vec{C} \Longrightarrow \vec{v}(t) = \langle 2t^2+1, 3t^2-1, t+1 \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \frac{2}{3}t^3 + t, t^3 - t, \frac{1}{2}t^2 + t \rangle + \vec{C}$

$\langle 1, 0, 0 \rangle = \vec{r}(0) = \vec{C} \Longrightarrow \vec{r}(t) = \langle \qquad \rangle + \langle 1, 0, 0 \rangle$

Ex. 4 An object with mass m that moves in a circular path with constant angula[r] speed ω has position vector $\vec{r}(t) = \langle a\cos(\omega t), a\sin(\omega t)\rangle$.
Find the force acting on the object and show that it is directed toward the origin.

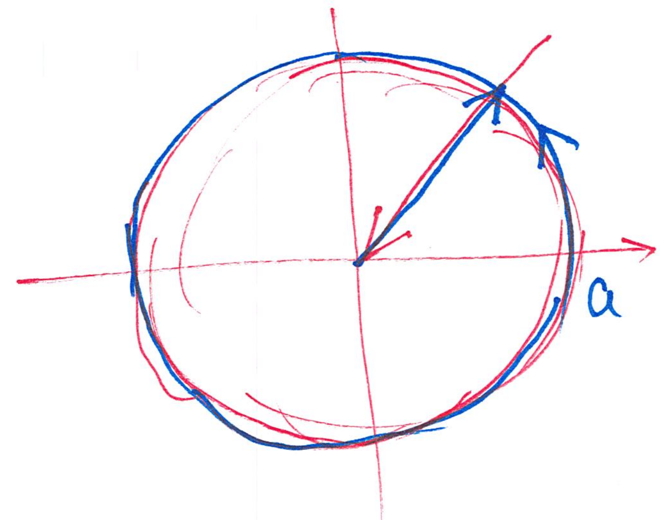
Newton's Second Law of Motion $\vec{F}(t) = m\,\vec{a}(t)$

$a\langle \cos(\omega t), \sin(\omega t)$

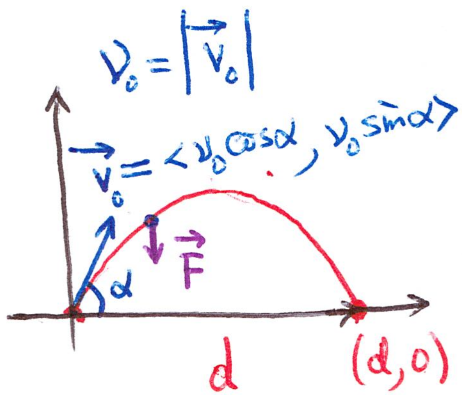
$\vec{v}(t) = a\langle -\omega\sin(\omega t), \omega\cos\omega t\rangle$

$\vec{a}(t) = \vec{v}'(t) = -a\omega^2\langle \cos(\omega t), \sin(\omega t)\rangle$

$\vec{F} = -am\omega^2\langle \cos(\omega t), \sin(\omega t)\rangle$



Ex. 5 A projectile is fired with angle of elevation α and initial velocity $\vec{v}_0$.
Assume that air resistance is negligible and the only external force is due to gravity.
Find the position function $\vec{r}(t)$ of the projectile.
What value of α maximizes the range (the horizontal distance traveled)?



$$\vec{F} = m\vec{a} = \langle 0, -mg\rangle, \quad g = 9.8 \ m/s^2$$

$$\vec{a} = \langle 0, -g\rangle, \quad \vec{v} = \int \vec{a}\,dt = \langle c_1, -gt + c_2\rangle$$
$$= \langle 0, -gt\rangle + \vec{v}_0$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 0, -\tfrac{1}{2}gt^2\rangle + \vec{v}_0 t + \vec{r}_0 = \langle v_0\cos\alpha\, t, -\tfrac{1}{2}gt^2 + v_0\sin\alpha\, t$$

$$\begin{cases} x = d \\ y = 0 = -\tfrac{1}{2}gt^2 + v_0(\sin\alpha)t = t\left(-\tfrac{1}{2}gt + v_0\sin\alpha\right) \Rightarrow t = \dfrac{2v_0\sin\alpha}{g} \end{cases}$$

$$d = x(t) = v_0\cos\alpha \cdot \frac{2v_0\sin\alpha}{g} = \frac{2v_0^2\cos\alpha\sin\alpha}{g} = \frac{v_0^2}{g}\sin(2\alpha)$$

$$\sin(2\alpha) = 1, \quad \alpha = \frac{\pi}{4}$$