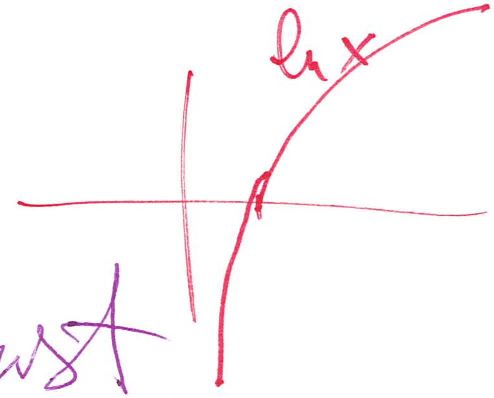
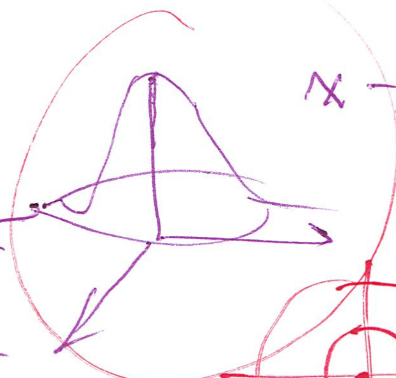
$z = f(x, y)$ — surface



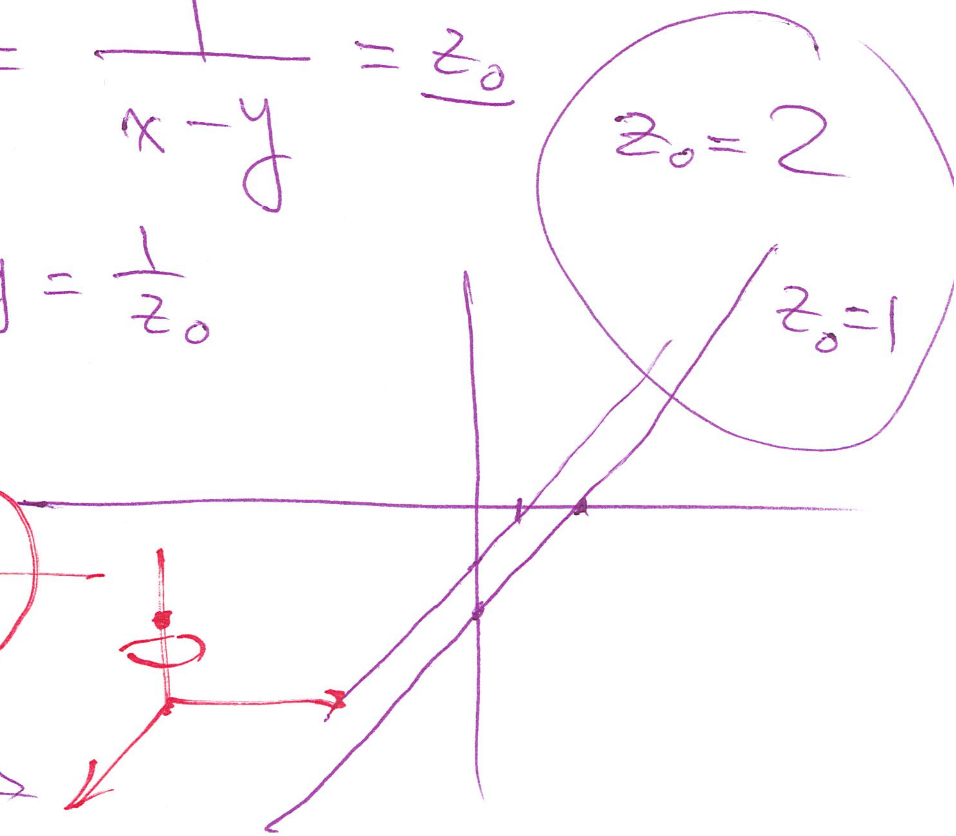
level curves $\quad f(x, y) = z_0$ — const

$z = \cos(xy)$

$z = \dfrac{1}{x - y} = z_0$

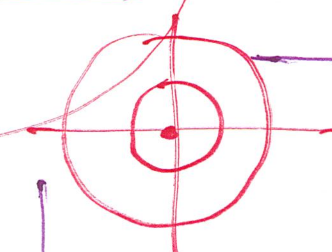


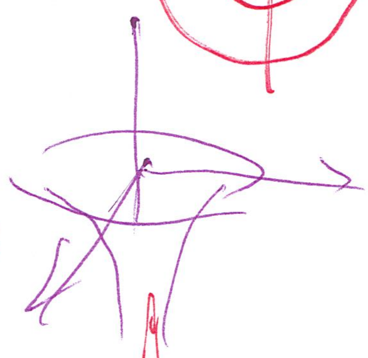
$x - y = \dfrac{1}{z_0}$



$z = \ln(x^2 + y^2)$

$z_0 = \dfrac{1}{1 + x^2 + y^2}$





$x^2 + y^2 = \dfrac{1}{z_0} - 1$

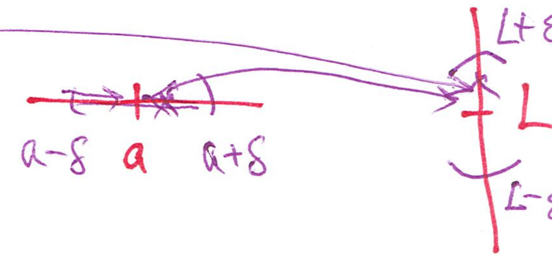
$z_0 = 1 \quad x^2 + y^2 = 0 \qquad (0, 0, 1)$

$z_0 = \frac{1}{2} \quad x^2 + y^2 = 1$

Lesson 10

$\lim\limits_{x\to a} f(x) = L$

§15.2 Limits and Continuity



- Limits

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \iff \forall \varepsilon>0,\ \exists \delta(\varepsilon)>0,\ \text{s.t.}$$

$$\sqrt{(x-a)^2+(y-b)^2} < \delta(\varepsilon) \quad 0<|(x,y)-(a,b)|<\delta(\varepsilon) \Longrightarrow |f(x,y)-L|<\varepsilon$$

For $a,b,c \in R$ $\quad (x-a)^2+(y-b)^2 < \delta(\varepsilon)$

$$\lim_{(x,y)\to(a,b)} c = c \qquad |f(x,y)-c| = |c-c| = 0$$

$$|f(x,y)-L| = |x-a| \overset{?}{<} \varepsilon.$$

$$\Uparrow$$

$$\sqrt{(x-a)^2+(y-b)^2} < \cancel{\delta(\varepsilon)}\ \varepsilon$$

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

Assume that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$

$$\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = L + M$$

$$\lim_{(x,y)\to(a,b)} [c\, f(x,y)] = cL$$

$$\lim_{(x,y)\to(a,b)} f(x,y)\, g(x,y) = LM$$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \quad \text{provided } M \neq 0$$

$$\lim_{(x,y)\to(a,b)} [f(x,y)]^n = L^n$$

n – even number, $\underline{L \geq 0}$

$$\lim_{(x,y)\to(a,b)} [f(x,y)]^{\frac{1}{n}} = L^{\frac{1}{n}}$$

examples

#19 $\lim_{(x,y)\to(2,0)} \frac{x^2-3xy^2}{x+y} = \frac{2^2-3\cdot 2\cdot 0}{2+0} = 2$

$\frac{0}{0}$

$(-2)^2+2\cdot 1\cdot(-2)=0$

#22 $\lim_{(x,y)\to(1,-2)} \frac{y^2+2xy}{y+2x} = \lim_{(x,y)\to(1,-2)} \frac{\cancel{(y+2x)}\,y}{\cancel{y+2x}} = -2$

0

$x \quad y \qquad -y$

$2x \quad -3y \qquad 3y$

$-\cancel{3y^2}$

$\frac{0}{0}$ #24 $\lim_{(x,y)\to(-1,1)} \frac{2x^2-xy-3y^2}{x+y} = \lim_{(x,y)\to(-1,1)} \frac{\cancel{(x+y)}(2x-3y)}{\cancel{x+y}} = -2-3 = -5$

$a^2-b^2 = (a+b)(a-b)$

$\frac{0}{0}$ #27 $\lim_{(x,y)\to(1,2)} \frac{\sqrt{y}-\sqrt{x+1}}{y-x-1} \cdot \frac{\sqrt{y}+\sqrt{x+1}}{\sqrt{y}+\sqrt{x+1}} = \lim \frac{\cancel{y-(x+1)}}{\cancel{(y-x-1)}(\sqrt{y}+\sqrt{x+1})}$

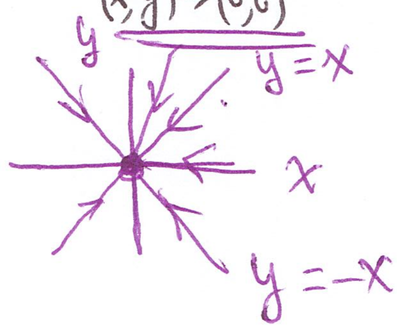
$= \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$

• $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ DNE $\iff$ find two different pathes C_1, C_2

s.t. $\lim\limits_{\text{along } C_1} f(x,y) \neq \lim\limits_{\text{along } C_2} f(x,y)$

examples

#30 $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy}{3x^2+y^2}$ $\quad \lim\limits_{(x,y)\to(0,0)} \dfrac{4xy}{3x^2+y^2}\Bigg|_{y=kx} = \dfrac{4x\cdot kx}{3x^2+k^2x^2} = \dfrac{4k}{3+k^2} = \begin{cases} 0, & k=0 \\ 1, & k=1 \end{cases}$



$y=kx$

#33 $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^3+x^3}{xy^2}$ $\quad y=kx \quad \dfrac{(k^3+1)x^3}{k^2x^3} = \dfrac{k^3+1}{k^2}$ DNE

• continuity

$f(x,y)$ is continuous at (a,b) $\Longleftrightarrow$ $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Is it cont. at $(0,0)$?

examples

#42 $f(x,y) = \begin{cases} \dfrac{y^4-2x^2}{y^4+x^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

$\lim_{(x,y)\to(0,0)} \dfrac{y^4-2x^2}{y^4+x^2}$ $\quad y=kx$

$f(x,y)\Big|_{y=kx} = \dfrac{k^4x^4-2x^2}{k^4x^4+x^2} = \dfrac{k^4x^2-2}{k^4x^2+1} \to -2$

not cont.

$y=k\sqrt{x}$

$f(x,y)\Big|_{y=k\sqrt{x}} = \dfrac{k^4x^2-2x^2}{k^4x^2+x^2} = \dfrac{k^4-2}{k^4+1}$

#54 $f(x,y) = \begin{cases} \dfrac{1-\cos(x^2+y^2)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

$z = x^2+y^2$ $\quad f(x,y) = \begin{cases} \dfrac{1-\cos z}{z} \\ 0 \end{cases}$