$z \cancel{y} = f(x, y)$

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

$\frac{0}{0}$ <u>exists</u>

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y + x^3}{x+y} = 0$$

$$x^2 y + x^3 = \underline{x^2(y+x)}$$

reduce <u>function of one variable</u>

<u>DNE</u> $\Longleftrightarrow$ $\lim\limits_{\text{along } C_1} f(x, y) \neq \lim\limits_{\text{along } C_2} f$

$$f(x,y) = \begin{cases} \dfrac{1-\cos(x^2+y^2)}{x^2+y^2} & \underline{(x,y)\neq(0,0)} \\ 0 & (x,y)=(0,0) \end{cases}$$

Is $f(x,y)$ continuous?

$\forall$

For all $(x,y)\neq(0,0)$, f is cont.

~~For~~ $(x,y)=(0,0)$ $\quad f(0,0)=0$ $\quad \frac{0}{0}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2 \to 0}$$

$w = x^2+y^2 \to 0$

$$\lim_{w\to 0} \frac{1-\cos w}{w} \overset{\frac{0}{0}}{=} \lim_{w\to 0} \frac{(1-\cos w)'}{(w)'}$$

$$= \lim_{w\to 0} \frac{\sin w}{1} = 0 = f(0,0) \Rightarrow \text{cont.}$$

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos\left[(x^2+y^2)\cancel{(y+2)}(y+2)\right]}{(x^2+y^2)} \frac{(y+2)}{(y+2)}$$

$$= \lim_{(x,y)\to(0,0)} \left\{ \frac{1-\cos\left[(x^2+y^2)(y+2)\right]}{(x^2+y^2)(y+2)} \right\} \cdot (y+2)$$

$$= \lim \left\{ \quad \right\} \lim (y+2)$$

$w=(x^2+y^2)(y+2) \to 0$

$$\lim_{w\to 0} \frac{1-\cos w}{w} \cdot 2 = 0$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

Lesson 11

$\delta \quad \Delta \quad \partial$

§15.3 Partial Derivatives

surface

$z = f(x, y)$

$\frac{df(x)}{dx}$

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{(a+h) - a}$$

$y = b$ plane

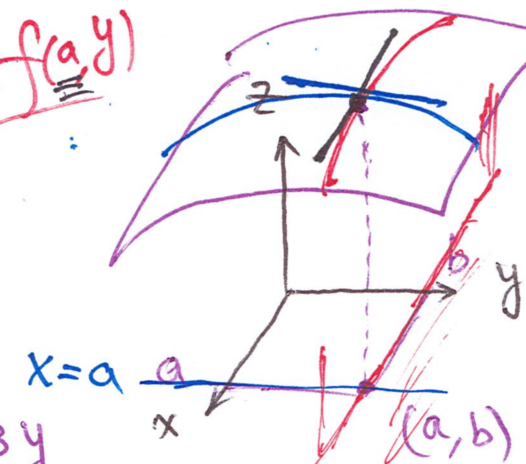
$$\frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{(b+h) - b}$$

$$\lim_{(h_1, h_2) \to (0,0)} \frac{f(a+h_1, b+h_2) - f(a,b)}{|(a+h_1, b+h_2) - (a,b)|}$$

$\begin{cases} x = a \\ z = f(x,y) \end{cases} \Rightarrow z = f(a, y)$

$(a, b, f(a,b))$

$\begin{cases} z = f(x,y) \\ y = b \end{cases}$



$z = f(x, b)$

$y = b$

#16 $f(x,y) = 4x^3y^2 + 3x^2y^3 + 10$

$$\frac{\partial f}{\partial x} = 4 \cdot 3x^2 y^2 + 3 \cdot 2x \cdot y^3 \qquad \frac{\partial f}{\partial y} = 4x^3 \cdot 2y + 3x^2 \cdot 3y^2$$

#22 $f(x,y) = \tan^{-1} \frac{x^2}{y^2}$

$x^2 y^{-2}$

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{x^2}{y^2}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{x^2}{y^2}\right) = \frac{y^4}{y^4 + x^4} \cdot \frac{2x}{y^2}$$

$$\frac{\partial f}{\partial y} = \frac{y^4}{y^4 + x^4} \cdot \frac{\partial}{\partial y}\left(\frac{x^2}{y^2}\right) = \frac{y^4}{y^4 + x^4} \cdot x^2 (-2) y^{-3}$$

#29 $h(x,y) = x - \sqrt{x^2-4y}$

$\cancel{\frac{\partial h}{\partial x} =}$ $\quad L'(u) = \ell(u)$

$f(x) = L(u)\Big|_{h(x)}^{g(x)} = L(g(x)) - L(h(x))$

$f(x) = \int_{h(x)}^{g(x)} \ell(u)\,du$

$\cancel{\frac{\partial h}{\partial y}} = f'(x) = \ell(g(x))\,g'(x) - \ell(h(x))\cdot h'(x)$

$f' = L'(g(x))\,g'(x) - L'(h(x))\,h'(x)$

#31 $f(x,y) = \int_x^{y^3} e^{t^2}\,dt$

$\frac{\partial f}{\partial x} = e^{(y^3)^2}\cdot\frac{\partial y^3}{\partial x} - e^{x^2}\cdot\frac{\partial x}{\partial x}$

$= -e^{x^2}$

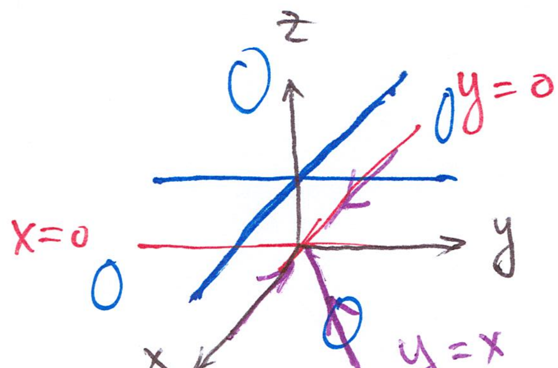
$\frac{\partial f}{\partial y} = e^{(y^3)^2}\cdot\frac{\partial y^3}{\partial y} - e^{x^2}\cdot\frac{\partial x}{\partial y}$

$= 3y^2\cdot e^{y^6}$

- partial derivative and continuity

$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$ $\Leftrightarrow$ $x=0$ or $y=0$

$\lim_{(x,y)\to(0,0)} f(x,y) = \begin{cases} 1 & \text{along } y=0 \\ 0 & \text{along } y=x \end{cases}$



$\frac{\partial f}{\partial x}(0,0) = 0$

$\frac{\partial f}{\partial y}(0,0) = 0$

- higher-order derivative $\quad z = f(x,y)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

mixed 2^{nd}-order der.

#38 $f(x,y) = x^2 \sin y$

$$\frac{\partial f}{\partial x} = 2x \sin y \qquad \frac{\partial^2 f}{\partial x^2} = 2\sin y \qquad \frac{\partial^2 f}{\partial y \partial x} = 2x\cos y$$

$$\parallel$$

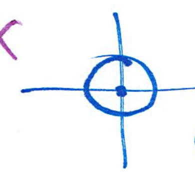
$$\frac{\partial f}{\partial y} = x^2 \cos y \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x\cos y \qquad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin y$$

#39 $h(x,y) = x^3 + xy^2 + 1$

$$\frac{\partial h}{\partial x} = 3x^2 + y^2 \qquad \frac{\partial^2 h}{\partial x^2} = 6x \qquad \frac{\partial^2 h}{\partial y \partial x} = 2y$$

$$\parallel$$

$$\frac{\partial h}{\partial y} = 2xy \qquad \frac{\partial^2 h}{\partial x \partial y} = 2y \qquad \frac{\partial^2 h}{\partial y^2} = 2x$$



(a,b)

Clairaut's Thrm $\quad f$ is defined on a disk $D \ni (a,b)$

f_{xy} and f_{yx} are continuous in $D \Longrightarrow f_{xy}(a,b) = f_{yx}(a,b)$

#59 $G(r,s,t) = \sqrt{rs^3t^5}$

$$\frac{\partial G}{\partial r} = \frac{1}{2}\left(\quad \right)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial r}(rs^3t^5)$$

$$= \frac{1}{2}\left(rs^3t^5\right)^{-\frac{1}{2}} s^3t^5$$

$$\frac{\partial G}{\partial s} = \frac{1}{2}\left(rs^3t^5\right)^{-\frac{1}{2}} \cdot r \cdot 3s^2 \cdot t^5$$

$$\frac{\partial G}{\partial t} = \frac{1}{2}\left(rs^3t^5\right)^{-\frac{1}{2}} \cdot rs^3 \cdot 5t^4$$

#92 Show that $u(x,y) = e^{ax}\cos ay$ satisfies Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for any $a \in \mathbb{R}$

$$\frac{\partial u}{\partial x} = a e^{ax}\cos ay,$$

$$\frac{\partial u}{\partial y} = e^{ax}(-\sin(ay)) \cdot a$$

$$= -a e^{ax}\sin(ay)$$

$$\frac{\partial^2 u}{\partial x^2} = a^2 e^{ax}\cos ay$$

$$+$$

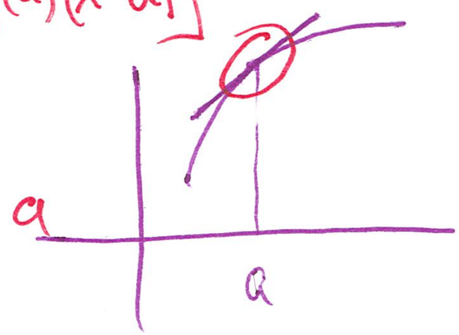
$$\frac{\partial^2 u}{\partial y^2} = -a^2 e^{ax}\cos(ay)$$

$$0$$

• <u>differentiability</u>

$y = f(x)$ is diff. at $x = a$ $\iff$ $y = f(a) + f'(a)(x-a)$

$f(x) - [f(a) + f'(a)(x-a)] \to 0$ as $x \to a$



$z = f(x,y)$ is differentiable at (a,b)

$\iff$

Thrm 15.5

f_x and f_y exist on a disk containing (a,b) and are continuous at (a,b) $\Longrightarrow$ f is differentiable at (a,b)

Thrm 15.6

f is differentiable at (a,b) $\Longrightarrow$ f is continuous at (a,b)