- differential and change $\quad z = \underline{f(x,y)} \approx L(x,y) = f(a,b) + \nabla f(a,b)\cdot\begin{pmatrix} x-a \\ y-b\end{pmatrix}$

$$= \underline{f(a,b)} + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(y-b)$$

<u>change</u> $\quad \Delta z = f(x,y) - f(a,b) \approx \frac{\partial f}{\partial x}(a,b)\underbrace{(x-a)}_{dx} + \frac{\partial f}{\partial y}(a,b)\underbrace{(y-b)}_{dy}$

<u>differential</u> $\quad \boxed{dz = \frac{\partial f}{\partial x}(a,b)\,dx + \frac{\partial f}{\partial y}(a,b)\,dy}$

$$\underline{\Delta z \approx dz}$$

$$\Delta x = x-a = dx$$
$$\Delta y = y-b = dy$$

<u>Ex. 4</u> $\quad z = f(x,y) = \frac{5}{x^2+y^2} = 5(x^2+y^2)^{-1}$

~~$\Delta z = f(x,y) - f(-1,2)$~~

Approximate the change in z when the independent variables change from $\underset{(a,b)}{(-1,2)}$ to $\underset{(x,\ y)}{(-0.93, 1.94)}$

$$\Delta z = f(-0.93, 1.94) - f(-1,2)$$
$$\approx dz = f_x(-1,2)\cdot dx + f_y(-1,2)\,dy$$
$$= -\frac{10\cdot(-1)}{25}\cdot(0.07) - \frac{10\cdot 2}{25}\cdot(-0.06)$$

$\underline{\Delta x} = x - a = -0.93 - (-1) = 0.07$

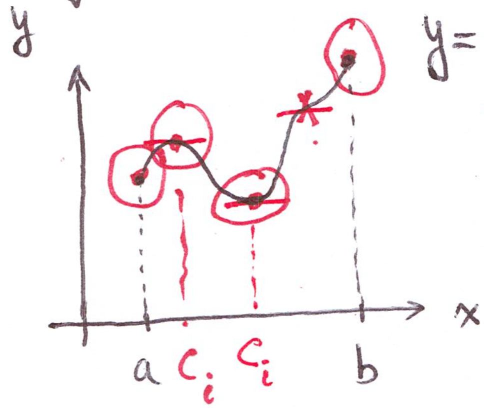
$\underline{\Delta y} = y - b = 1.94 - 2 = -0.06$

$f_x = 5\cdot(-1)\cdot(x^2+y^2)^{-2}\cdot 2x = -\frac{10x}{(x^2+y^2)^2}, \quad f_y = \frac{-10y}{(x^2+y^2)^2}$

Lesson 15

§15.7 Maximum / Minimum Problems

• functions of one variable



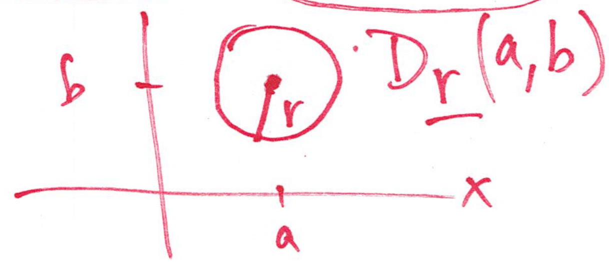
$$\max_{x\in[a,b]}/\min f(x) = \max/\min \{ f(a), f(b), f(c_i) \}$$

$f'(c_i)$ DNE

$[a,b]$

critical pts: $f'(c_i)=0$ or $f'(c_i)$ DNE

• functions of two variables



Definition (Local Max/Min Values)

(1) $f(a,b)$ is a local max value of f $\iff$ $f(a,b) \geq f(x,y)$ for all $(x,y)\in D_r(a,b)$

(2) $f(a,b)$ is a local min value of f $\iff$ $f(a,b) \leq f(x,y)$ $\forall (x,y)\in D_r(a,b)$

Theorem (First Derivative Test)

(1) f has a local max/min at an interior pt (a,b)

(2) $f_x(a,b)$ and $f_y(a,b)$ exist

$$\left.\begin{array}{l}(1)\\(2)\end{array}\right\} \Longrightarrow \begin{array}{l} f_x(a,b) = 0 \\ f_y(a,b) = 0 \end{array}$$

Proof $g(t) = f(a+th_1, b+th_2)$ has a max/min at $t=0$

$x = a+th_1,\ y = b+th_2$

$0 = g'(0) =$

$$0 = \left.\frac{dg(t)}{dt}\right|_{t=0} = \left.\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\right|_{t=0}$$

$$= \left.\frac{\partial f}{\partial x} h_1\right|_{t=0} + \left.\frac{\partial f}{\partial y} h_2\right|_{t=0}$$

$$= \frac{\partial f}{\partial x}(a,b)\,h_1 + \frac{\partial f}{\partial y}(a,b)\,h_2$$

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

Definition (Critical Pt)

(a,b) is a critical pt of f $\Longleftrightarrow$ $\begin{cases} f_x(a,b) = 0 \text{ and} \\ f_y(a,b) = 0 \end{cases}$

or $\nabla f(a,b)$ DNE

Ex. 1 Find critical pts of $f(x,y) = xy(x-2)(y+3)$

$$f(x,y) = xy(x-2)(y+3) = x(x-2)\,y(y+3)$$

$$\nabla f = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\[2ex] \dfrac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} y(x-2)(y+3) + xy(y+3) = 2(x-1)y(y+3) \\[2ex] x(x-2)\left[y+3+y\right] = x(x-2)(2y+3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \cancel{2}(x-1)\,y(y+3) = 0 \implies x=1,\ y=0,\ \text{or } y=-3 \\ x(x-2)(2y+3) = 0 \implies x=0,\ x=2,\ \text{or } y=-\frac{3}{2} \end{cases}$$

$$\left(1,-\tfrac{3}{2}\right),\ (0,0),\ (2,0),\ (0,-3),\ (2,-3)$$

$$\cancel{(0,0)},\ \cancel{(0,-3)}$$

Definition (Saddle Point)

(a,b) is a saddle pt of $f(x,y)$ $\Longleftrightarrow$ (a,b) is a critical pt and $f(a,b)$ is not a local max/min.

Theorem (Second Derivative Test) $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \Delta$

(1) $f_{xx}, f_{xy}, f_{yy} \in C^0(D_r(a,b))$

(2) $f_x(a,b) = f_y(a,b) = 0$

$\Longrightarrow$

(1) $D > 0$ $\begin{cases} f_{xx}(a,b) > 0 \Longrightarrow \text{l. min} \\ f_{xx}(a,b) < 0 \Longrightarrow \text{l. max} \end{cases}$

(2) $D < 0 \Longrightarrow$ saddle pt.

(3) $D = 0 \Longrightarrow$ undetermine

Ex.2 Use the 2^{nd} Der. Test to classify the critical pts of $f(x,y) = x^2 + 2y^2 - 4x + 4y + 6$

$f_x = 2x - 4 = 2(x-2) = 0 \Rightarrow x = 2$

$f_y = 4y + 4 = 4(y+1) = 0 \Rightarrow y = -1$

critical pt : $(2,-1)$ — l. min

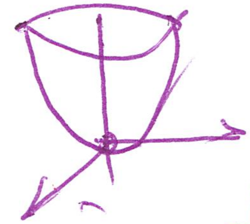
$f_{xx} = 2 > 0$, $f_{yy} = 4$

$f_{xy} = 0$

$\Rightarrow D = 2 \cdot 4 - 0^2 = 8 > 0$

$y = f(x)$

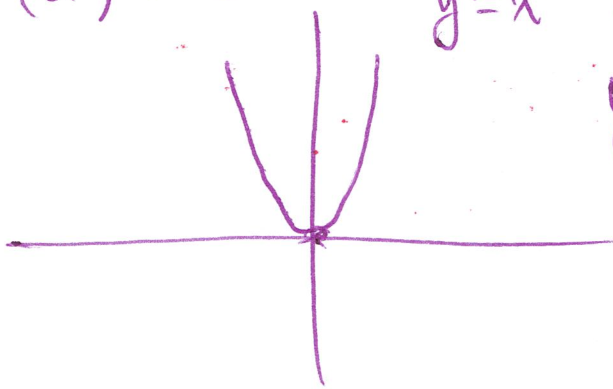
$z = (x^2 + y^2) = f$



a — critical pt $f'(a) = 0$

$f''(a) > 0$

$y = x^2$ $f(a)$ is l. min



$y''(0) = 2 > 0$

$f_x = 2x = 0$

$f_y = 2y = 0$

$\Rightarrow (x, y) = (0, 0)$

$D = f_{xx} f_{yy} - f_{xy}^2$

$= 2 \cdot 2 - 0 = 4 > 0$

$f_{xx} = 2 > 0$

$z = x^2 - y^2$

$f_x = 2x = 0$

$f_y = -2y = 0$ $\Rightarrow (0,0)$ — critical pt.

$f_{xx} = 2 > 0$, $f_{yy} = -2$, $f_{xy} = 0$

$D(x,y) = 2 \cdot (-2) - 0^2 = -4 < 0$

$(0,0)$ — saddle pt.

Ex. 3 Use the 2^{nd} Der. Test to classify the critical pts of $f(x,y) = xy(x-2)(y+3)$.

$= x(x-2)\,y(y+3)$

$(1, -\frac{3}{2}), (0,0), (2,0), (0,-3), (2,-3)$

$f_x = 2(x-1)y(y+3)$, $f_{xx} = 2y(y+3)$

$f_y = x(x-2)(2y+3)$, $f_{yy} = 2x(x-2)$

$f_{xy} = [x-2+x](2y+3) = 2(x-1)(2y+3)$

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4\left[xy(x-2)(y+3) - (x-1)^2(2y+3)^2\right]$

$D(1,-\frac{3}{2}) = 4\cdot(-\frac{3}{2})(-1)(-\frac{3}{2}+3) > 0$

$f_{xx}(1,-\frac{3}{2}) = 2(-\frac{3}{2})(-\frac{3}{2}+3) < 0$

$(1,-\frac{3}{2})$ l. max.

$D(0,0) = 4[-1^2 \cdot 3] < 0$

saddle pt.

Ex. 4 Apply the 2^{nd} Der. Test to the following functions and interpret the results.

(a) $f(x,y) = 2x^4 + y^4$

(b) $f(x,y) = 2 - xy^2$.

~~$D(x,y) = 0$~~

$f_x = 8x^3 = 0 \Rightarrow x = 0$

$f_y = 4y^3 = 0 \Rightarrow y = 0$

$f_{xx} = 24x^2$ $\quad D(x,y) = 24\cdot 12x^2y^2$

$f_{yy} = 12y^2$ $\quad D(0,0) = 0$

$f_{xy} = 0$