max/min $f(x)$

$x \in [a,b]$

(1) max/min $f(x)$, $x \in (a,b)$

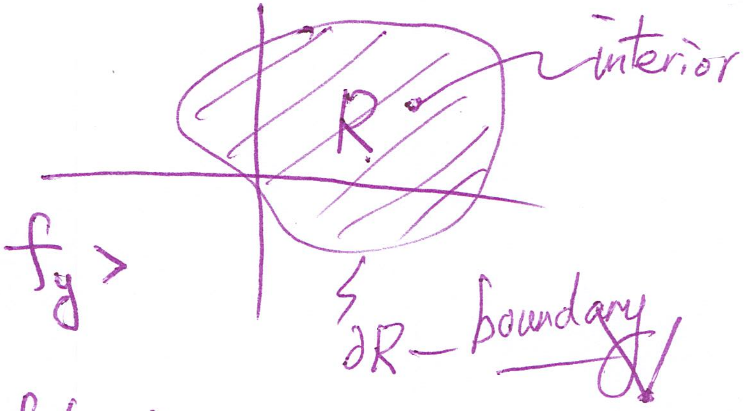
$$\boxed{\text{local max/min}}$$

(2) $f(a), f(b)$

max/min $f(x,y)$

$(x,y) \in R$

$\nabla f = \langle f_x, f_y \rangle$



1st der. test f has a l. max/min at $(a,b) \Rightarrow \nabla f(a,b) = \langle 0, 0 \rangle$

critical pts (a,b): $\nabla f(a,b) = \langle 0,0 \rangle$ or $\nabla f(a,b)$ ~~[illegible]~~ don't exist

saddle pt (a,b): (a,b) is a critical pt but $f(a,b)$ is not a l. max/min

$z = x^2 - y^2$

2nd der. test (a,b) is a critical, $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$

$$\Rightarrow \quad D(a,b) > 0 \quad \begin{cases} f_{xx}(a,b) > 0 & \text{l. min} \\ f_{xx}(a,b) < 0 & \text{l. max} \end{cases}$$

$(z = x^2 + y^2,\ (0,0),\ D(0,0) = 4 > 0)$

$D(a,b) < 0$ saddle pt $(z = x^2 - y^2,\ (0,0),\ D(0,0) = -4 < 0)$

$D(a,b) = 0$

Lesson 16

Definition (Absolute Maximum / Minimum Values)

f is defined on a set $R \subset \mathbb{R}^2$ containing (a,b)

(1) $f(a,b)$ is an absolute maximum value of f $\Longleftrightarrow f(a,b) \geq f(x,y)$, for all $(x,y) \in R$

(2) $f(a,b)$ is an absolute minimum value of f $\Longleftrightarrow f(a,b) \leq f(x,y)$, $\forall (x,y) \in R$

Procedure (Finding abs. max./min. values on closed bounded sets)

R is a closed bounded set in $\mathbb{R}^2$ and $f \in C^0(R)$

(1) max/min $f(x,y)$
inside of R

(2) max/min $f(x,y)$
∂R — boundary

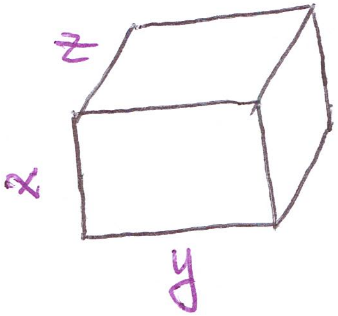
(3) abs. max value of f = ~~max~~ the largest value of (1) & (2) steps

abs. min value of f = the smallest value of ______.

Ex.5 A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

$V = xyz$, constraint $\underbrace{x+y+z=96}$

$\Rightarrow z = 96-x-y$



$V(x,y) = xy(96-x-y)$

$\max V(x,y) = \max\{xy(96-x-y)\}$

$V_x = y\{(96-x-y)+x\cdot(-1)\} = y\cdot(96-2x-y)=0 \Rightarrow \cancel{y=0}$ or $96-2x-y=0$

$V_y = x(96-2y-x)=0 \Rightarrow \cancel{x=0}$ or $96-2y-x=0$

$$\begin{cases} 2x+y=96 \\ x+2y=96 \end{cases}$$

$(x,y) = (32,32)$

$V(32,32) = 32^3$

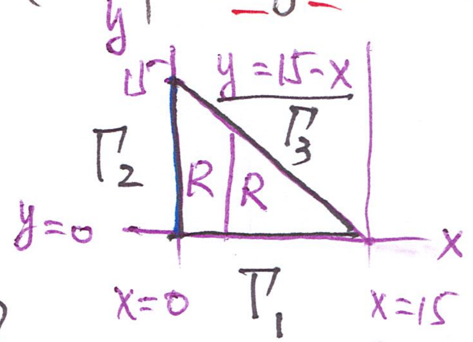
Ex. 6 Find the absolute maximum and minimum values of

$f(x,y) = xy - 8x - y^2 + 12y + 160$ over the triangular region $R = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le x \le 15 \\ 0 \le y \le 15 - x \end{array} \right\}$

$\Gamma_1 = \{ x \in [0,15] \mid \underline{y=0} \}$

$\Gamma_2 = \{ y \in [0,15] \mid x = 0 \}$

$\Gamma_3 = \{ y = 15 - x \mid 0 \le x \le 15 \}$ $\quad \partial R = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$



(1) interior

$f_x = y - 8 = 0 \Rightarrow y = 8$

$f_y = x - 2y + 12 = 0$

$\Rightarrow x = -12 + 2y = -12 + 16 = 4$

critical pt $(4,8)$, $\boxed{f(4,8) = 192}$ ✓

(2) boundary

on Γ_1, $0 \le x \le 15$: $g_1(x) = f(x,0) = -8x + 160$ ✓, $g_1'(x) = -8 < 0$

$\Rightarrow \boxed{f(0,0) = g_1(0) = 160}$, $\boxed{f(15,0) = g_1(15) = 40}$ ✓ abs. min

On Γ_2 $\{x = 0,\ 0 \le y \le 15\}$ ✓ $g_2(y) = f(0,y) = -y^2 + 12y + 160$, $g_2'(y) = -2y + 12 = 0 \Longrightarrow y = 6$.

abs. max.

$\boxed{f(0,6) = g_2(6) = 196}$, $f(0,0) = g_2(0) = 160$, $\boxed{f(0,15) = g_2(15) = 115}$ ✓

On Γ_3 $\{ y = 15 - x \mid 0 \le x \le 15 \}$ $g_3(x) = f(x, 15-x) = x(15-x) - 8x - (15-x)^2 + 12(15-x) + 160$

$g_3'(x) = 0 \Rightarrow (x,y) = (6.25, 8.75)$, $\boxed{f(6.25, 8.75) = 193.125}$ ✓

$g_3(0) = f(0,15)$, $g_3(15) = f(15,0)$

Ex. 8 Find the abs. max. and min. values of $f(x,y)=4-x^2-y^2$ on $R=\{(x,y) \mid x^2+y^2<1\}$.

$$\left.\begin{array}{l} f_x=-2x=0 \\ f_y=-2y=0 \end{array}\right\} \Rightarrow (x,y)=(0,0) \text{ critical pt.} \quad f(0,0)=4 \ge f(x,y)=4-(x^2+y^2)$$

$\Rightarrow f(0,0)=4$ is the abs. max.

R is an open set and does not contain the boundary: $x^2+y^2=1$

wh~~ere~~en $(x,y) \to \partial R=\{(x,y) \mid x^2+y^2=1\}$ boundary

$\Rightarrow x^2+y^2 \to 1 \Rightarrow f(x,y)=4-(x^2+y^2) \to 4-1=3$, f has no abs. min in R

Ex. 9 Find the point(s) on the plane $x+2y+z=2$ closest to the point $P(2,0,4)$.

$d(x,y,z)=\sqrt{(x-2)^2+(y-0)^2+(z-4)^2}$, for (x,y,z) on the plane, $\underline{z=2-x-2y}$

$$\xRightarrow{z=2-x-2y} d(x,y)=\sqrt{(x-2)^2+y^2+(2+x+2y)^2} \Rightarrow d^2(x,y)=(x-2)^2+y^2+(x+2y+2)^2$$

$$\left.\begin{array}{l} \frac{\partial}{\partial x}d^2=2(x-2)+2(x+2y+2)=2[2x+2y]=0 \Rightarrow x+y=0 \\ \frac{\partial}{\partial y}d^2=2y+2(x+2y+2)\cdot 2=2[2x+5y+4]=0 \Rightarrow 2x+5y=-4 \end{array}\right\} \Rightarrow (x,y)=\left(\frac{4}{3},-\frac{4}{3}\right)$$

$\boxed{\cancel{\Rightarrow d\left(\frac{4}{3},-\frac{4}{3}\right)} = \cancel{D\left(\frac{4}{3},-\frac{4}{3}\right)}} \ne$ 2nd der. test $\Rightarrow d\left(\frac{4}{3},-\frac{4}{3}\right)$ is a local min. Is it abs. min?