Fall 2016 #5 $\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$, $t \in [0, \pi]$ (Hint: $\cos(2x) = 1 - 2\sin^2 x$)

$$\Rightarrow \boxed{1 - \cos x = 2\sin^2 \frac{x}{2}}$$

$\vec{r}\,'(t) = \langle 1 - \cos t, \sin t \rangle$

$1 - 2\cos t + \cos^2 t + \sin^2 t$

$$|\vec{r}\,'(t)| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2(1-\cos t)} = \sqrt{4\sin^2 \frac{t}{2}} = 2\sin\frac{t}{2}$$

$$\boxed{L = \int_0^\pi |\vec{r}\,'(t)|\,dt} = \int_0^\pi 2\sin\frac{t}{2}\,dt = -4\cos\frac{t}{2}\Big|_0^\pi = -4\left[\cos\frac{\pi}{2} - \cos 0\right] = 4.$$

#5 acceleration $\vec{a}(t) = \langle 0, 2t, \sqrt{2} \rangle$, $\vec{v}(0) = \langle 1, 0, 0 \rangle$. Find the distance traveled for $0 \le t \le 3$.

$$\vec{r}\,'(t) = \vec{v}(t) = \int \vec{a}(t)\,dt = \langle c_1, t^2 + c_2, \sqrt{2}t + c_3 \rangle$$

$$\langle 1, 0, 0 \rangle = \vec{v}(0) = \langle c_1, c_2, c_3 \rangle \Rightarrow \vec{v}(t) = \langle 1, t^2, \sqrt{2}t \rangle$$

$$|\vec{r}\,'(t)| = |\vec{v}(t)| = \sqrt{1^2 + (t^2)^2 + (\sqrt{2}t)^2} = \sqrt{1 + t^4 + 2t^2} = \sqrt{(1+t^2)^2} = 1 + t^2$$

$$\Rightarrow \boxed{L = \int_0^3 |\vec{r}\,'(t)|\,dt} = \int_0^3 (1+t^2)\,dt = \left[t + \frac{1}{3}t^3\right]_0^3 = 3 + \frac{1}{3}\cdot 3^3 = 3 + 9 = 12$$

#10 $g(u,v) = f\left((u+2v)^3+1,\ e^{uv}-1\right)$

(x,y)	f	g	f_x	f_y
$(-1,0)$	8	1	4	2
$(0,0)$	1	3	5	7

$g_u(-1,0) = ?$

$(u,v) = (-1,0)$

$$= f(x,y) \quad \text{with} \quad \boxed{\begin{cases} x^{(u,v)} = (u+2v)^3+1 \\ y^{(u,v)} = e^{uv}-1 \end{cases}}$$

$$(x,y)\Big|_{(u,v)=(-1,0)} \cancel{= \left((1+2)^3+1,\ e-1\right) = (28,}$$

$$= \left((-1+0)^3+1,\ e^0-1\right) = (-1+1,\ 1-1) = (0,0)$$

$$g_u(-1,0) = \left[f_x \cdot \frac{\partial x}{\partial u} + f_y \cdot \frac{\partial y}{\partial u}\right]_{(u,v)=(-1,0)}$$

$$= f_x(0,0)\left[3(u+2v)^2 \cdot 1\right]_{(u,v)=(-1,0)} + f_y(0,0)\left[v e^{uv}\right]_{(u,v)=(-1,0)}$$

$$= 5 \cdot \left[3(-1+0)^2\right] + 7 \cdot \left[0 \cdot e^0\right] = 5 \cdot 3 = 15$$

$g_v(-1,0) =$

$$g_v \underline{\underline{(-1,0)}} \quad (u,v)=(-1,0)$$

$$(x,y) = (x(-1,0), y(-1,0)) = \underline{\underline{(0,0)}}$$

$$\left[g(u,v) = f(x(u,v), y(u,v)) \right.$$

$$\left. \begin{cases} x(u,v) = (u+2v)^3 + 1 \\ y(u,v) = e^{uv} - 1 \end{cases} \right]$$

$$\left.\frac{\partial g}{\partial v}\right|_{(-1,0)} = \left[\frac{\partial f}{\partial x}\cdot\underline{\frac{\partial x}{\partial v}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}\right]_{(u,v)=(-1,0)}$$

$$= \frac{\partial f}{\partial x}(0,0)\, 3(u+2v)^2 2\Big|_{(-1,0)=(u,v)} + \frac{\partial f}{\partial y}(0,0)\cdot u e^{uv}\Big|_{(u,v)=(-1,0)}$$

$$= 5\cdot 6\cdot(-1)^2 + 7\cdot(-1)\cdot 1$$

$$= 30 + -7 = 23$$

$$f(x, y, z) \qquad \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$f(\vec{r}(t)) = f(x(t), y(t), z(t))$$

$$\frac{d}{dt} f(\vec{r}(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$$= \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) + \frac{\partial f}{\partial z} z'(t)$$

$$= \nabla f(\vec{r}(t)) \circ \underbrace{\langle x', y', z' \rangle}_{\vec{r}\,'(t)}$$

$$= \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

Lesson 17

§15.8 Lagrange Multipliers

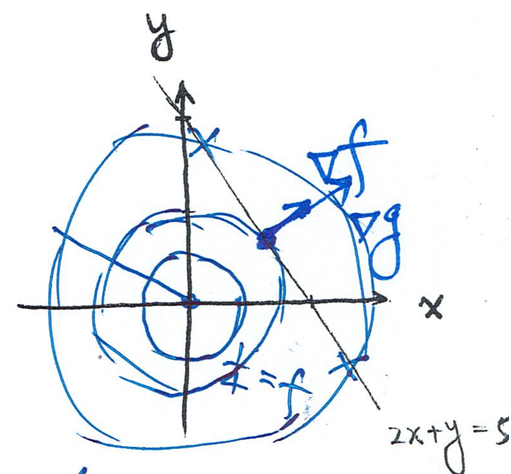
• constrained maximum/minimum problems

$$\max/\min \ f(x,y,z)$$
$$g(x,y,z)=0$$

$f(x,y)=x^2+y^2=k^2$

$k=0 \quad x^2+y^2=0 \Rightarrow (0,0)$

$k=\frac{1}{2} \quad x^2+y^2=\left(\frac{1}{2}\right)^2$



Ex. 1 $\quad \min \ (x^2+y^2) = f(x,y)$

$g(x,y)\equiv 2x+y=5$

Solution 1

$2x+y=5 \Longrightarrow y=5-2x$

$f(x,y)\big|_{g=5} = x^2+y^2\big|_{g=5} = x^2+(5-2x)^2=h(x)$

$h'(x)=2x+2(5-2x)\cdot(-2)$

$= \cancel{\ } \ 10(x-2)=0 \Longrightarrow x=2$

$y\big|_{x=2} = (5-2x)\big|_{x=2} = 5-4 = 1$

$(2,1)$

$f(2,1)=2^2+1^2=5$

Solution 2

$\nabla f \ // \ \nabla g$

$$\boxed{\begin{cases}\nabla f = \lambda \nabla g \\ 2x+y=5\end{cases}}$$

$f=*$

$f_x = 2x = \lambda g_x, \quad g_x=2$

$f_y = 2y = \lambda g_y, \quad g_y = 1$

$$\begin{cases}2x=2\lambda \\ 2y=\lambda \\ 2x+y=5\end{cases}$$

$\frac{2x}{2y}=\frac{2\lambda}{\lambda} \Longrightarrow \frac{x}{y}=2 \Rightarrow x=2y$

$2(2y)+y=5y$

$\Longrightarrow 5 = \cancel{4x+2x=4x} \Rightarrow y=1, \ x=2$

Ex. 2 Find (x, y, z) on $g(x,y,z) = x^2 - z^2 - 1 = 0$ that are closest to $(0,0,0)$.

$$\min_{g=0} \sqrt{x^2+y^2+z^2} \doteq 1 \qquad \min_{g=0} f(x,y,z) = x^2+y^2+z^2 = k^2$$

$x^2 = z^2 + 1$

Solution 1

$g(x,y,z) = 0 \Rightarrow z^2 = x^2 - 1$ or $\boxed{x^2 = z^2+1}$

- $h(x,y) = f(x,y,z)\Big|_{z^2 = x^2-1} = x^2 + y^2 + x^2 - 1 = 2x^2 + y^2 - 1$

$h_x = 4x = 0$
$h_y = 2y = 0$
$\Rightarrow x = 0,\ y = 0 \Rightarrow z^2 = x^2 - 1\Big|_{x=0} = -1$

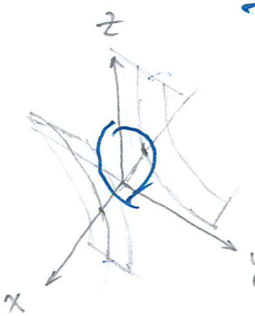
- $k(y,z) = f(x,y,z)\Big|_{x^2 = z^2+1} = (z^2+1) + y^2 + z^2 = 2z^2 + y^2 + 1$

$k_y = 2y = 0$
$k_z = 4z = 0$
$\Rightarrow y = 0,\ z = 0 \Rightarrow x^2 = 0^2 + 1 = 1$

$\Rightarrow x = \pm 1$ $\quad f(-1, 0, 0) = 1$

$f(1, 0, 0) = 1$

Solution 2



$k = 0 \Rightarrow (0,0,0)$

$k = 1 \Rightarrow x^2 + y^2 + z^2 = 1$

$\langle 2x, 2y, 2z \rangle = \lambda \langle 2x, 0, -2z$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

$x = \pm 1,\ y = 0,\ z = 0$

$(1, 0, 0)$, $(-1, 0, 0)$

$$\begin{cases} x = \lambda x \\ y = 0 \\ z = -\lambda z \\ x^2 - z^2 - 1 = 0 \end{cases}$$

$\dfrac{x}{z} = -\dfrac{x}{z} \Rightarrow xz = -xz$

$\Rightarrow 2xz = 0 \Rightarrow xz = 0$

$x = 0$ or $z = 0$

$x = 0 \Downarrow -z^2 - 1 = 0$

$z = 0 \Downarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$

- the method of Lagrange multipliers

Assume that
$$\begin{cases} (1)\ f(x,y,z) \text{ and } g(x,y,z) \text{ are differentiable} \\ (2)\ \nabla g \neq 0 \text{ where } g=0 \end{cases}$$

$$\Longrightarrow \text{critical pts of } \max_{g=c}/\min f \text{ satisfy } \begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases}$$

Ex.3 $\max_{\frac{x^2}{8}+\frac{y^2}{2}=1}/\min\ xy$, $f = xy$, $g = \underline{\frac{x^2}{8}+\frac{y^2}{2}}$

$$\boxed{\begin{aligned} f_x &= y = \lambda \tfrac{1}{4}x = g_x \\ f_y &= x = \lambda y = g_y \end{aligned}} \Big\rangle\ \frac{y}{x} = \frac{\frac{1}{4}\cancel{\lambda} x}{\cancel{\lambda} y} \Rightarrow \frac{y}{x} = \frac{1}{4}\frac{x}{y} \Rightarrow y^2 = \frac{1}{4}x^2$$

$$\frac{x^2}{8} + \frac{y^2}{2} = 1 \longrightarrow 1 = \frac{x^2}{8} + \frac{\frac{1}{4}x^2}{2} = \frac{1}{4}x^2$$

$$y^2 = \frac{1}{4}x^2 \Rightarrow x^2 = 4 \Rightarrow \underline{x = \pm 2},\quad y^2 = \frac{1}{4}\cdot 4 = 1 \quad \text{max}$$

$$\Rightarrow \underline{y = \pm 1} \quad \boxed{f(-2,-1) = 2},\ \underbrace{f(-2,1) = -2}_{\text{min}},\ \underbrace{f(2,-1) = -2}_{\text{min}},\ \boxed{f(2,1) = 2}$$

Ex. 4 max/min $\left(\underset{f}{\underset{\parallel}{3x+4y}}\right)$

$g = x^2+y^2 = 1$

$$\left.\boxed{\begin{aligned} f_x &= 3 = \lambda\, 2x = g_x \\ f_y &= 4 = \lambda\, 2y = g_y \end{aligned}}\right\} \frac{3}{4} = \frac{x}{y} \Longrightarrow x = \frac{3}{4}y$$

$$x^2+y^2 = 1 = \left(\frac{3}{4}\right)^2 y^2 + y^2 = \left(\frac{9}{16}+1\right)y^2 = \boxed{\frac{25}{16}y^2 = 1}$$

$$y^2 = \frac{16}{25} \Longrightarrow y = \pm\frac{4}{5} \Longrightarrow x = \frac{3}{4}y = \pm\frac{3}{5}$$

$$f\left(-\frac{3}{5}, \frac{4}{5}\right),\ f\left(-\frac{3}{5}, -\frac{4}{5}\right),\ f\left(\frac{3}{5}, \frac{4}{5}\right),\ f\left(\frac{3}{5}, -\frac{4}{5}\right)$$

$f\left(-\frac{3}{5}, \frac{4}{5}\right) = -\frac{9}{5}+\frac{16}{5} = \frac{7}{5}$

$f\left(-\frac{3}{5}, -\frac{4}{5}\right) = -\frac{9}{5}-\frac{16}{5} = -5$ min

$f\left(\frac{3}{5}, \frac{4}{5}\right) = 5$ max

$f\left(\frac{3}{5}, -\frac{4}{5}\right) = \frac{9}{5}-\frac{16}{5} = -\frac{7}{5}$

#24 max/min $f(x,y,z) = xy - z$

$g \equiv x^2+y^2+z^2-xy = 1$

$\nabla f = \langle y, x, -1 \rangle$

$\nabla g = \langle 2x-y, 2y-x, 2z \rangle$

$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} y = \lambda(2x-y) \\ x = \lambda(2y-x) \\ -1 = \lambda 2z \end{cases}$ (first two) $\Rightarrow \dfrac{y}{x} = \dfrac{2x-y}{2y-x}$

$\Rightarrow 2y^2 - xy = 2x^2 - xy$

$\Rightarrow y^2 = x^2 \Rightarrow y = \pm x$

Case 1 $y = x$

$1 = g(x,y,z)\Big|_{y=x} = x^2 + z^2$, $\quad y = \lambda(2x-y)\Big|_{y=x} \Rightarrow x = \lambda x \Rightarrow (\lambda - 1)x = 0 \Rightarrow x = 0$ or $\lambda = 1$

$\cancel{\Rightarrow x \neq 0}$ $\quad \underline{x=0}$ $\quad \left[x^2+z^2=1\right]_{x=0} \Rightarrow z^2 = 1 \Rightarrow z = \pm 1 \Rightarrow f(0,0,1) = -1$ min

$f(0,0,-1) = 1$

$\underline{\lambda = 1}$ $\quad \left[-1 = \lambda 2z\right]_{\lambda=1} \Rightarrow z = -\frac{1}{2} \xrightarrow{1=x^2+z^2} x^2 = 1 - \left(-\frac{1}{2}\right)^2 = \frac{3}{4}$

$\Rightarrow x = \pm\frac{\sqrt{3}}{2} \Rightarrow f\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ max, $\quad f\left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{5}{4}$ max

Case 2 $y = -x$

$1 = g(x,y,z)\Big|_{y=-x} = 3x^2 + z^2$, $\quad y = \lambda(2x-y)\Big|_{y=-x} \Rightarrow (3\lambda+1)x = 0 \Rightarrow x = 0$ or $\lambda = -\frac{1}{3}$

$\underline{\lambda = -\frac{1}{3}}$ $\quad \left[-1 = \lambda 2z\right]_{\lambda = -\frac{1}{3}} \Rightarrow z = \frac{3}{2} \xrightarrow{1=3x^2+z^2} 1 = 3x^2 + \frac{9}{4}$ no solution