$f(x)$

$f'(x)$

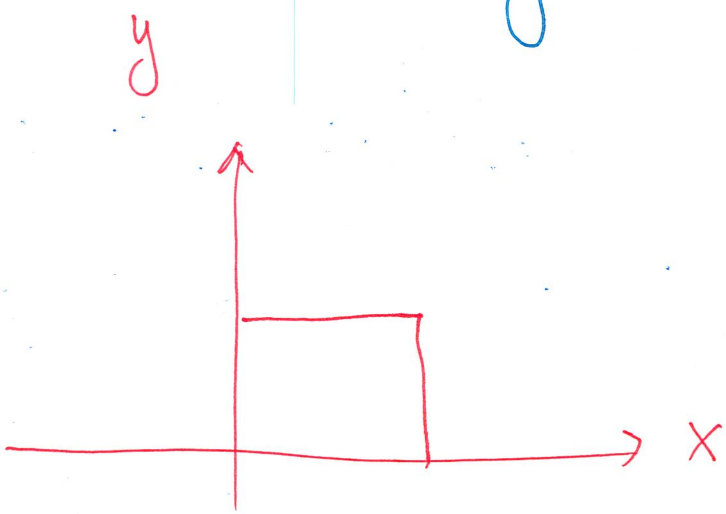
$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b \cancel{+C}$$

$$F'(x) = f(x)$$

$f(x,y)$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\iint_R f(x,y)\,dA$$

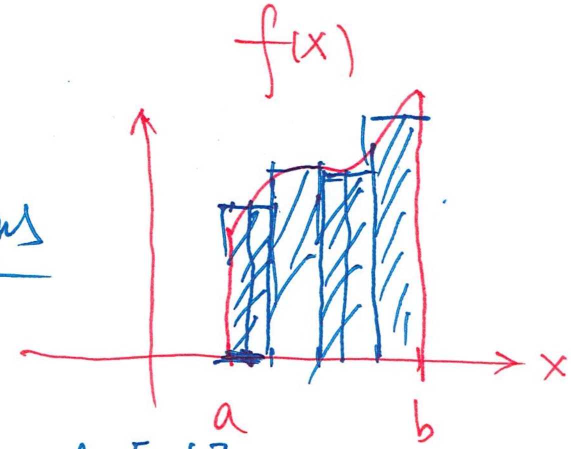


$$\iiint_R f(x,y,z)\,dA$$

Lesson 18

Chapter 16 Multiple Integration

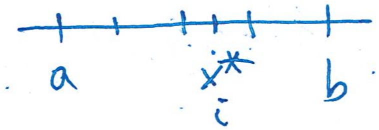
§16.1 Double Integrals over Rectangular Regions



- Review of a single integral

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

① partition of $[a, b]$



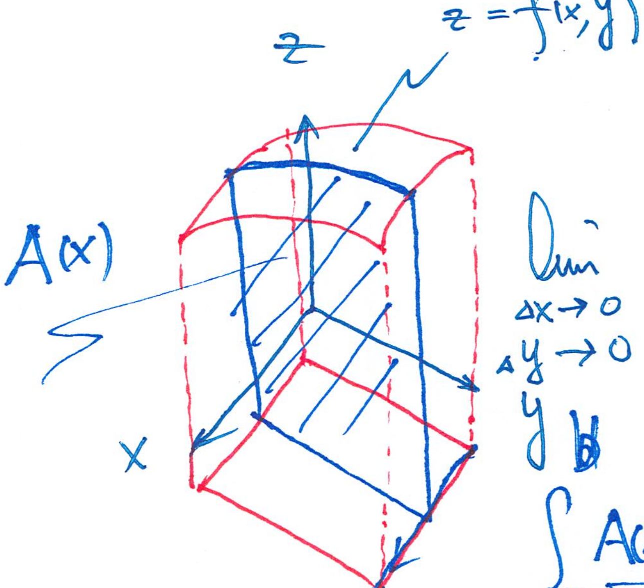
$a = x_0 < x_1 < x_2 \cdots < x_n = b$

$x_1 = x_0 + \Delta x$ $\qquad \Delta x = \frac{b-a}{n} \to 0$

$x_2 = x_0 + 2\Delta x$

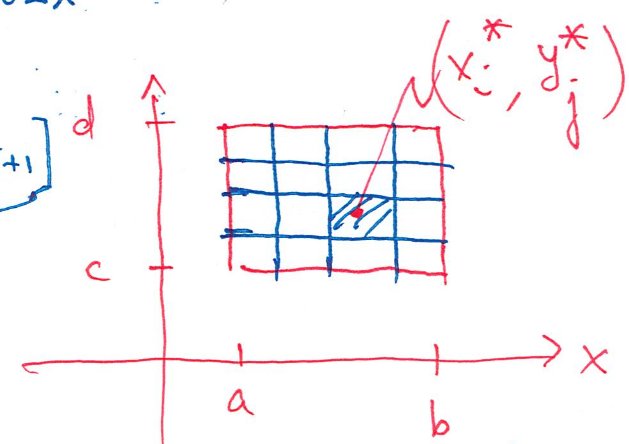
$x_i = x_0 + i\Delta x$ $\qquad$ or $n \to \infty$

- Volume of Solids



$R = [a, b] \times [c, d]$

$[x_i, x_{i+1}] \times [y_j, y_{j+1}]$



$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\,\Delta x\,\Delta y$$

$$\| \qquad \iint_R f(x,y)\,dA$$

$$\int_a A(x)\,dx = V =$$

$c = y_0 < y_1 < \cdots < y_m = d \qquad \Delta y = \frac{d-c}{m}$

- **Iterated Integral**

$$V = \int_a^b A(x)\,dx \qquad A(x) \text{ — area of cross-section}$$

Example 1

$$V = \int_0^1 A(x)\,dx, \qquad A(x) = \int_0^2 \underbrace{(6-2x-y)}_{= f(x,y)}\,dy$$

$$= \int_0^1 \left[\int_0^2 (6-2x-y)\,dy\right] dx = \int_0^1 \left[6y - 2xy - \tfrac{1}{2}y^2\right]_0^2 dx$$

$$= \int_0^1 [\underbrace{12 - 4x - 2}_{= A(x)}]\,dx = \left[10x - 2x^2\right]_0^1 = 10 - 2 = 8$$

Example 2

$$V = \int_0^2 A(y)\,dy, \qquad A(y) = \int_0^1 (6-2x-y)\,dx$$

$$= \left[6x - x^2 - yx\right]_{x=0}^1 = \cancel{6x - x^2 - x}\ (6-1-y) - 0$$

$$= 5 - y$$

$$V = \int_0^2 (5-y)\,dy$$

$$= \left[5y - \tfrac{1}{2}y^2\right]_0^2 = 10 - 2 = 8$$

Fubini Theorem

$R=[a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA=\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx=\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

Ex. 3 Find the volume of the solid bounded by the surface $f(x,y)=4+9x^2y^2$ over the region $R=[-1,1]\times[0,2]$

$$V=\iint_R f\,dA=\int_{-1}^1\left(\int_0^2(4+9x^2y^2)\,dy\right)dx=\int_{-1}^1\left[4y+3x^2y^3\right]_{y=0}^{y=2}dx=\int_{-1}^1 8(1+3x^2)\,dx$$

$$=2\int_0^1 8(1+3x^2)\,dx=16\left[x+x^3\right]_0^1=32$$

$f(-x)=f(x)$ even

$f(-x)=-f(x)$ odd

Ex. 4 $\iint_R ye^{xy}\,dA=\int_0^1\left(\int_0^{\ln 2}\underline{ye^{xy}}\,\underline{dy}\right)dx=\int_0^{\ln 2}\left(\int_0^1 \underline{ye^{xy}\,dx}\right)dy$

$R=[0,1]\times[0,\ln 2]$

$\underline{u=y,\ v'=e^{xy}}$

$u'=1,\ v=\frac{1}{x}e^{xy}$

$$\int_0^1\left[\frac{y}{x}e^{xy}\Big|_0^{\ln 2}-\frac{1}{x}\int_0^{\ln 2}e^{xy}\,dy\right]dx$$

$$=\int_0^1\left[\frac{\ln 2}{x}e^{x\ln 2}-\frac{1}{x}\frac{1}{x}e^{xy}\Big|_0^{\ln 2}\right]dx$$

$$=\int_0^{\ln 2}\left[e^{xy}\right]_{x=0}^{x=1}dy=\int_0^{\ln 2}(e^y-1)\,dy$$

$$=\left[e^y-y\right]_0^{\ln 2}=\left(e^{\ln 2}-\ln 2\right)-\left(e^0-0\right)$$

$$=2-\ln 2-1=1-\ln 2$$

- **Average value over the region R**

$$\bar{f} = \frac{1}{|R|} \iint_R f(x,y)\, dA$$

$0 \le x \le 2$

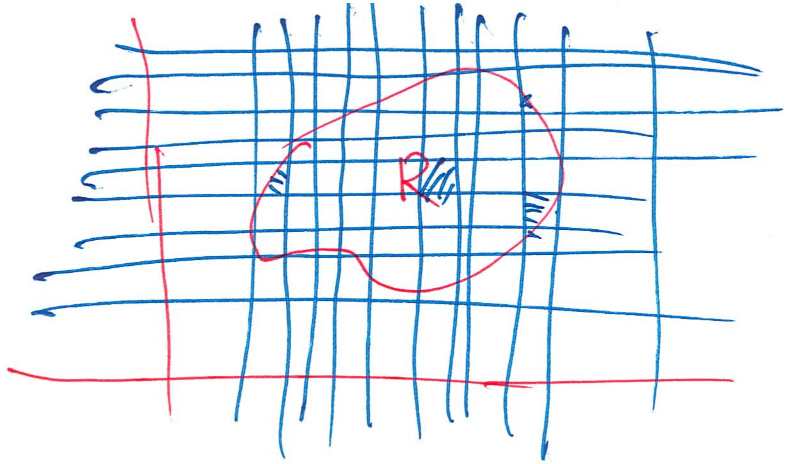
$0 \le y \le 2$

Ex. 5 Find the average value of the quantity $f(x,y) = 2-x-y$ over $R = [0,2]\times[0,2]$

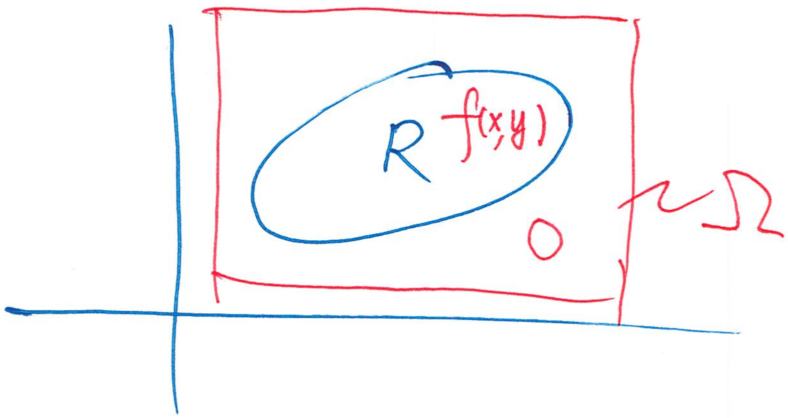
$$\bar{f} = \frac{1}{4} \int_0^2 \left(\int_0^2 (2-x-y)\, dx \right) dy$$

$$= \frac{1}{4} \int_0^2 \left[(2-y)x - \frac{1}{2}x^2 \right]_{x=0}^{x=2} dy = \frac{1}{4} \int_0^2 \left[2(2-y) - 2 \right] dy$$

$$= \frac{1}{2} \int_0^2 (1-y)\, dy = \frac{1}{2} \left[y - \frac{1}{2}y^2 \right]_0^2 = \frac{1}{2}(2-2) = 0$$



$$\underline{\underline{f(x_i^*, y_j^*)\Delta x \Delta y}}$$



$$\tilde{f} = \begin{cases} f(x,y), & (x,y) \in R \\ 0, & (x,y) \in \Omega \setminus R \end{cases}$$

$$\iint_R f(x,y)\, dA = \underline{\underline{\iint_\Omega \tilde{f}(x,y)\, dA}}$$