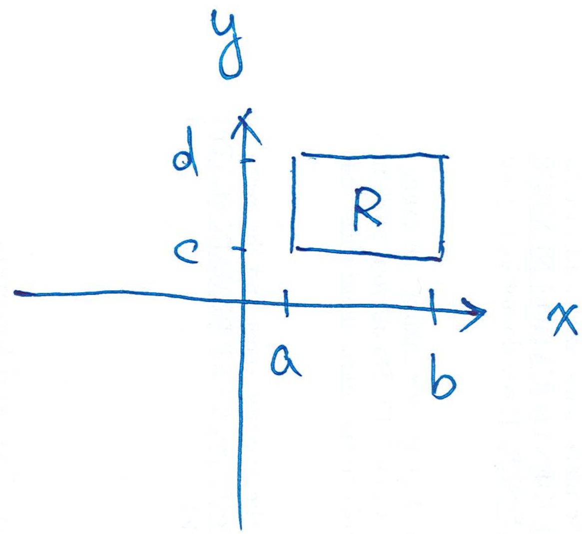
$R = [a,b] \times [c,d]$

$$\begin{cases} a \le x \le b \\ c \le y \le d \end{cases}$$



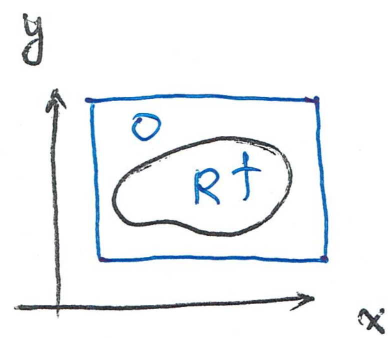
$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Lesson 19

§16.2 Double Integrals over General Regions

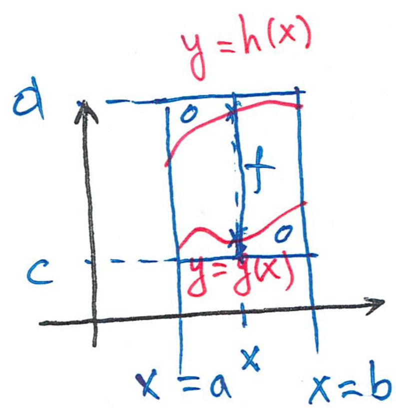
$$\tilde{f}(x,y) = \begin{cases} f(x,y), & \text{in } R \\ 0, & \text{in } D\setminus R \end{cases}$$



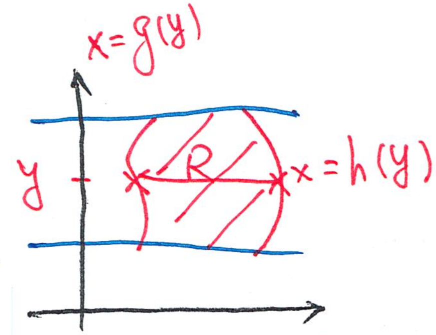
$$\iint_R f(x,y)\,dA = \iint_D \tilde{f}(x,y)\,dA$$

$$= \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx = \int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\,dx \right) dy$$

• elementary regions



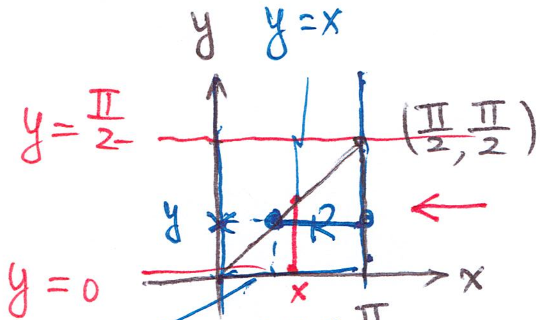
$$\begin{cases} a \le x \le b \\ g(x) \le y \le h(x) \end{cases}$$



$$\begin{cases} c \le y \le d \\ g(y) \le x \le h(y) \end{cases}$$

• iterated integrals

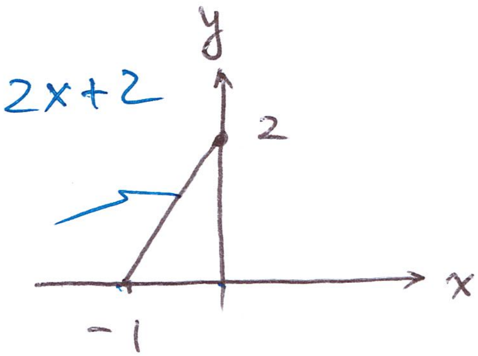
$$\iint_R f(x,y)\,dA = \iint_D \tilde{f}(x,y)\,dA = \int_a^b \int_c^d \tilde{f}(x,y)\,dy\,dx = \int_a^b \left(\int_c^{g(x)} 0 + \int_{g(x)}^{h(x)} f + \int_{h(x)}^{d} 0 \; dy \right) dx$$



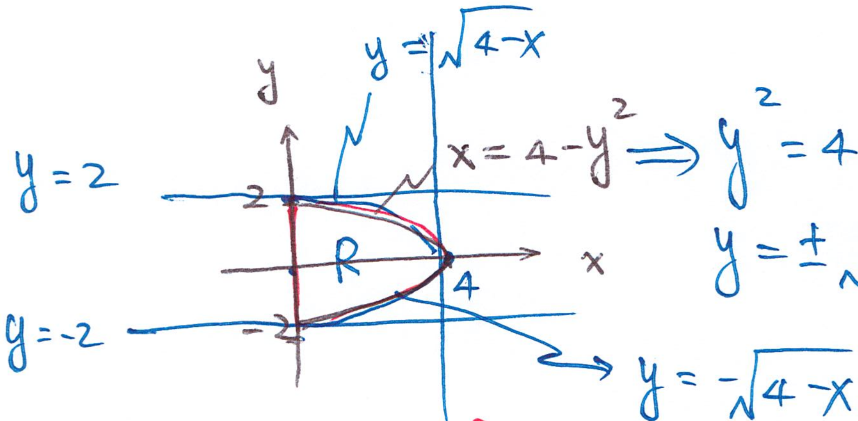
$$\begin{cases} 0 \le x \le \frac{\pi}{2} \\ 0 \le y \le x \end{cases}$$

$$\begin{cases} 0 \le y \le \frac{\pi}{2} \\ \cancel{\ \ } \\ y \le x \le \frac{\pi}{2} \end{cases}$$

$y = 2x+2$



$$\begin{cases} 0 \le x \le 4 \\ -\sqrt{4-x} \le y \le \sqrt{4-x} \end{cases}$$



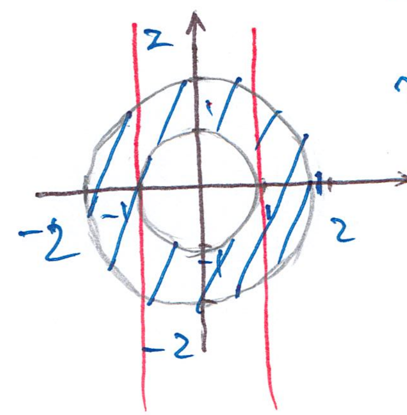
$x = 4-y^2 \Longrightarrow y^2 = 4-x$

$y = \pm\sqrt{4-x}$

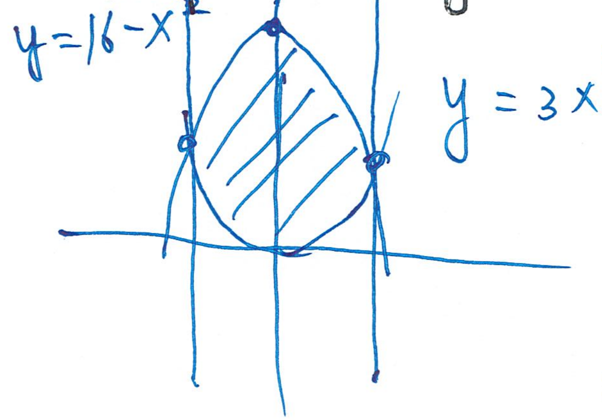
$$\begin{cases} 0 \le x \le 4-y^2 \\ -2 \le y \le 2. \end{cases}$$

$x^2+y^2=2^2$

$x^2+y^2=1^2$



Ex. 1 Express $\iint_R 2x^2 y\, dA$ as an iterated integral, where $\underline{R}$ is the region bounded by the parabolas $y = 3x^2$ and $y = 16 - x^2$. Then evaluate the integral.



$y = 3x^2 = \cancel{y} = 16 - x^2 \Rightarrow 4x^2 = 16 \Rightarrow x = \pm 2$

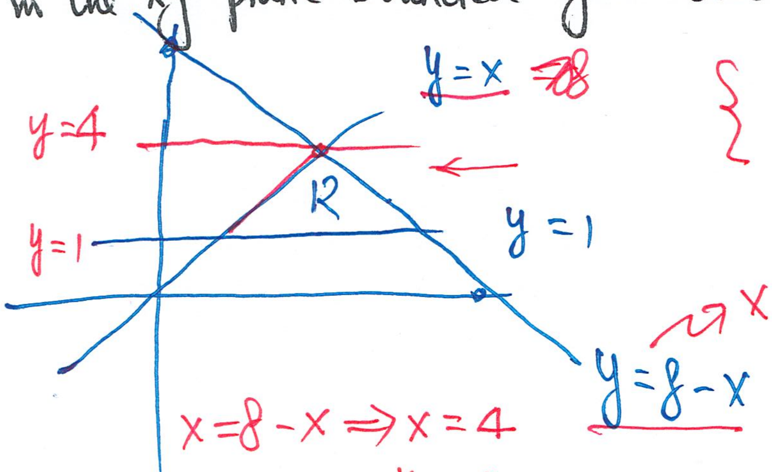
$$\iint_R 2x^2 y\, dA = \int_{-2}^{2} \left(\int_{3x^2}^{16-x^2} \underline{2x^2 y}\, dy \right) dx$$

$$= \int_{-2}^{2} x^2 y^2 \Big|_{y=3x^2}^{16-x^2} dx = \int_{-2}^{2} x^2 \left[(16-x^2)^2 - 9x^4 \right] dx$$

$-2 \le x \le 2$

$3x^2 \le y \le 16 - x^2$

Ex. 2 Find the volume of the solid below the surface $f(x,y) = 2 + \frac{1}{y}$ and above the region R in the xy-plane bounded by the lines $y = x$, $y = 8 - x$, and $y = 1$



$x = 8 - x \Rightarrow x = 4$

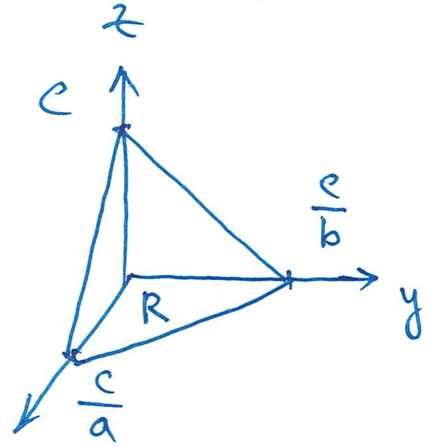
$y = 4$

$$\begin{cases} 1 \le y \le 4 \\ y \le x \le 8-y \end{cases}$$

$$V = \iint_R \left(2 + \frac{1}{y}\right) dA$$

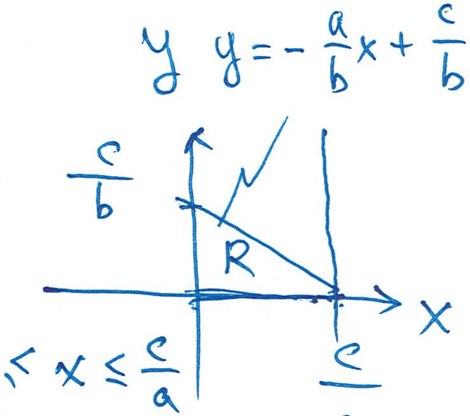
$$= \int_1^4 \left(\int_y^{8-y} \left(2 + \frac{1}{y}\right) dx \right) dy$$

Ex.3 Find the volume of the tetrahedron in the first octant bounded by the plane $z = c - ax - by$ and the coordinate planes ($x=0$, $y=0$, and $z=0$). Assume $a, b, c > 0$.



$$V = \iint_R (c - ax - by)\,dA$$

$$= \int_0^{\frac{c}{a}} dx \int_0^{-\frac{a}{b}x + \frac{c}{b}} (c - ax - by)\,dy$$



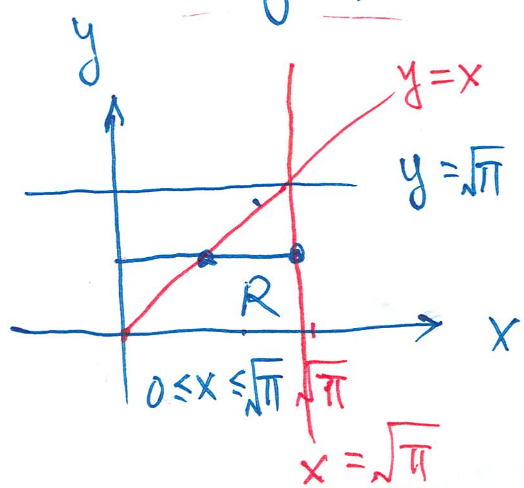
$$\begin{cases} 0 \le x \le \frac{c}{a} \\ 0 \le y \le -\frac{a}{b}x + \frac{c}{b} \end{cases}$$

$z = y = 0 \Longrightarrow c - ax = 0,\ x = \frac{c}{a}$

Ex.4 $$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin x^2\,dx\,dy = \int_0^{\sqrt{\pi}} \left(\int_0^x \sin x^2\,dy \right) dx = \int_0^{\sqrt{\pi}} (\sin x^2)\, y \Big|_0^x dx$$

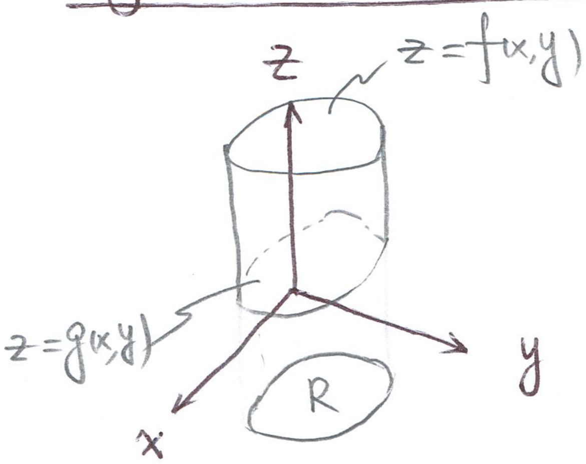
$$R: \begin{cases} y \le x \le \sqrt{\pi} \\ 0 \le y \le \sqrt{\pi} \end{cases} \qquad \begin{cases} 0 \le x \le \sqrt{\pi} \\ 0 \le y \le x \end{cases}$$

$$= \int_0^{\sqrt{\pi}} x \sin x^2\,dx$$



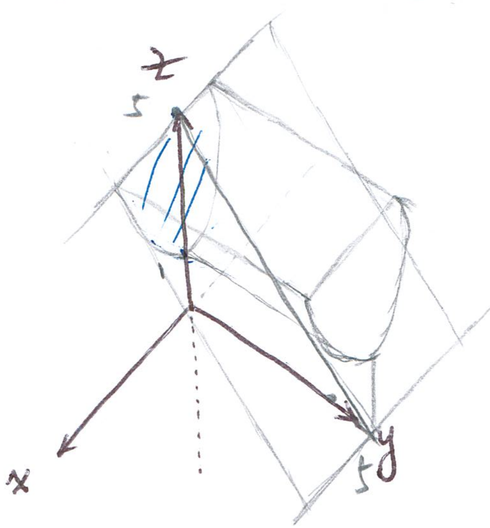
$$= -\frac{1}{2}\cos x^2 \Big|_0^{\sqrt{\pi}} = -\frac{1}{2}\left[\cos\pi - \cos 0\right] = 1$$

• Regions between two surfaces



$$V = \iint_R \left(f(x,y) - g(x,y) \right) dA$$

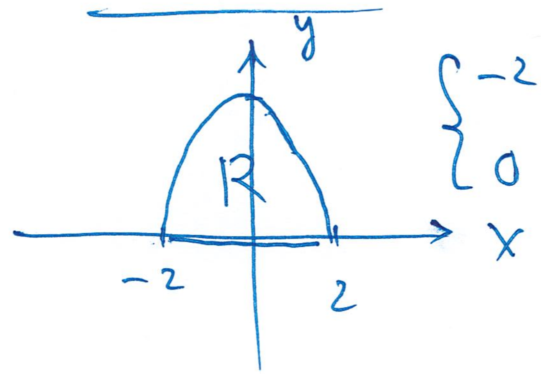
Ex. 5 Find the volume of the solid bounded by the parabolic cylinder $z = 1 + x^2$ and the planes $z = 5 - y$ and $y = 0$



$$V = \iint_R \left((5-y) - (1+x^2) \right) dA$$

$$= \int_{-2}^{2} dx \int_0^{4-x^2} \left(4 - y - x^2\right) dy$$

$$1 + x^2 = 5 - y \Rightarrow y = 4 - x^2$$



$$\begin{cases} -2 \le x \le 2 \\ 0 \le y \le 4 - x^2 \end{cases}$$