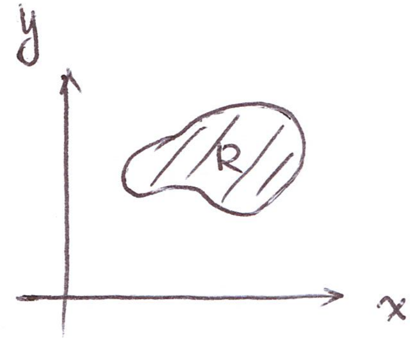
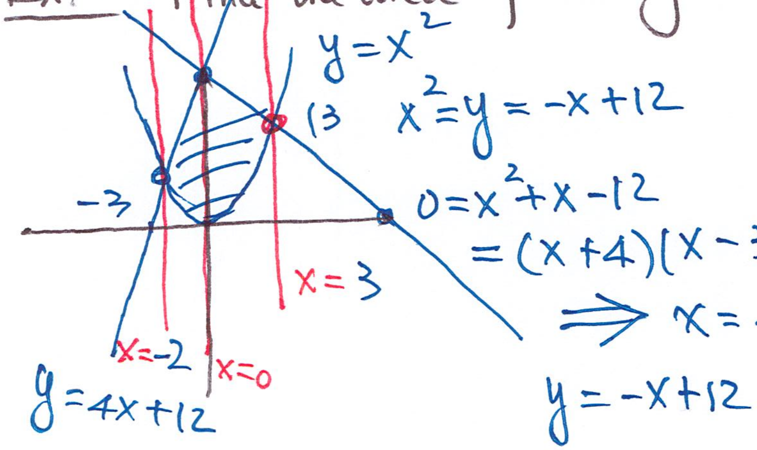
- Finding area by double integrals

$$\text{area of } R = \iint_{R} 1 \, dA$$



Ex. 6 Find the area of the region R bounded by $y=x^2$, $y=-x+12$, and $y=4x+12$.



$x^2 = y = -x+12$

$0 = x^2 + x - 12$

$= (x+4)(x-3)$

$\Rightarrow x = -4, \text{ or } 3$

$x^2 = 4x+12$

$0 = (x^2 - 4x + 4) - 4 - 12$

$= (x-2)^2 - 16$

$x - 2 = \pm 4$

$x = 2 \pm 4 = 6, \text{ or } -2$

$$\begin{cases} -2 \le x \le 0 \\ x^2 \le y \le 4x+12 \end{cases}$$

$$\begin{cases} 0 \le x \le 3 \\ x^2 \le y \le -x+12 \end{cases}$$

$0 = 4x+12$

$\Rightarrow x = -3$

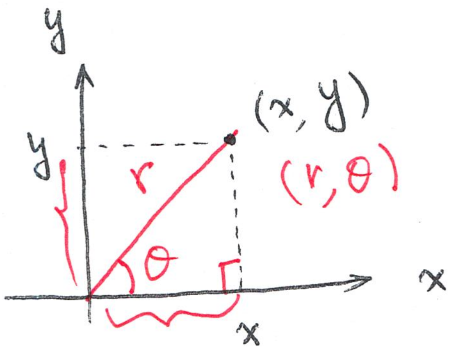
$$A = \int_{-2}^{0}\int_{x^2}^{4x+12} dy\,dx + \int_{0}^{3}\int_{x^2}^{-x+12} dy\,dx$$

$$= \int_{-2}^{0}\left(4x+12-x^2\right)dx + \int_{0}^{3}\left(-x+12-x^2\right)dx$$

Lesson 20

$dx = u'(t)\,dt$

§16.3 Double Integrals in Polar Coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

$x = u(t)$

- Change of Variables $\quad T: (x, y) \longrightarrow (g(u,v), h(u,v))$ $\quad$ one-to-one transformation

$$R \longrightarrow S = T(R)$$

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v), h(u,v))\,\left|J(u,v)\right|\,du\,dv,$$

Jacobian

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- polar coordinates

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

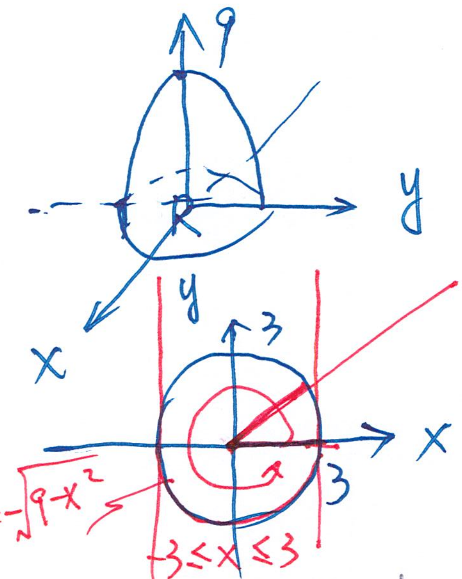
$$= r\cos^2\theta + r\sin^2\theta = r$$

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Ex. 1 Find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

$$\Downarrow$$

$$z = 0$$



$$0 = z = 9 - x^2 - y^2$$

$$\Rightarrow x^2 + y^2 = 9$$

$$V = \iint_R \left[9 - (x^2+y^2)\right] dA$$

$$\begin{cases} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{cases}$$

$$= \int_0^3 \int_0^{2\pi} (9 - r^2)\, d\theta\, r\, dr$$

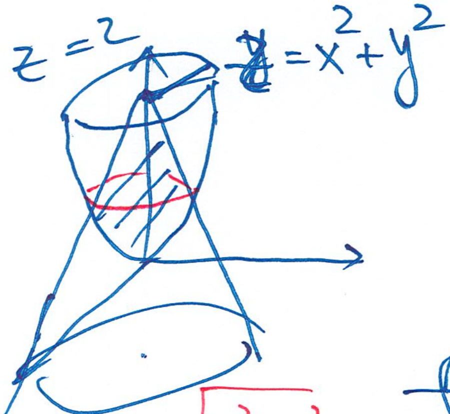
$$= 2\pi \int_0^3 (9r - r^3)\, dr$$

$$= 2\pi \left[\frac{9}{2} r^2 - \frac{1}{4} r^4\right]_0^3 = 2\pi \left[\frac{81}{2} - \frac{3^4}{4}\right]$$

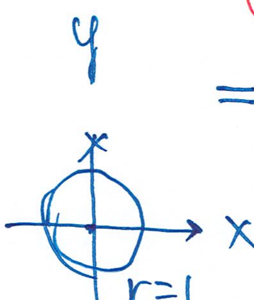
$$y = -\sqrt{9-x^2}$$

$$-3 \le x \le 3$$

Ex. 2 Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the cone $z = 2 - \sqrt{x^2+y^2}$.



$$V = \iint_R \left[\left(2 - \sqrt{x^2+y^2}\right) - \left(x^2+y^2\right)\right] dA$$



$$= \int_0^1 \int_0^{2\pi} \left[(2 - r) - r^2\right] d\theta\, r\, dr$$

$$= 2\pi \int_0^1 (-r^3 - r^2 + 2)\, dr$$

$$x^2 + y^2 = z = 2 - \sqrt{x^2+y^2}$$

$$r^2 = 2 - r \Rightarrow 0 = r^2 + r - 2$$

$$= (r+2)(r-1)$$

$$\Rightarrow r = -2, \text{ or } \boxed{r = 1}$$

Ex. 3 Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \{(r,\theta): 2 \le r \le 4,\ 0 \le \theta \le 2\pi\}$.

$$V = \int_2^4 \int_0^{2\pi} \left(\underline{r^2 \cos\theta \sin\theta} + 10 \right) d\theta \; r\,dr$$

$$= 2\pi \int_2^4 10\, r\,dr = 10\pi r^2 \Big|_2^4$$

$$\int_0^{2\pi} \cos\theta \sin\theta \, d\theta$$

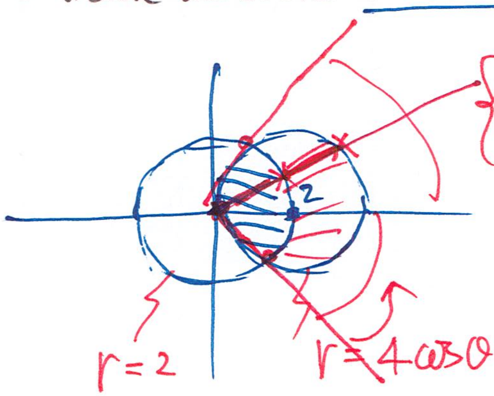
$$= \frac{1}{2} \int_0^{2\pi} \sin 2\theta \, d\theta$$

$$= -\frac{1}{2} \cdot \frac{1}{2} \cos 2\theta \Big|_0^{2\pi} = 0$$

Ex. 4 Write an iterated integral in polar coordinates for $\iint_R g(r,\theta)\, dA$: $= r\,dr\,d\theta$

(a) The region outside the circle $r=2$ and inside the circle $r = 4\cos\theta$ ($r=2$, centered at $(2,0)$).

(b) region inside both circles of (a).



$$\begin{cases} 2 \le r \le 4\cos\theta \\ -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \end{cases}$$

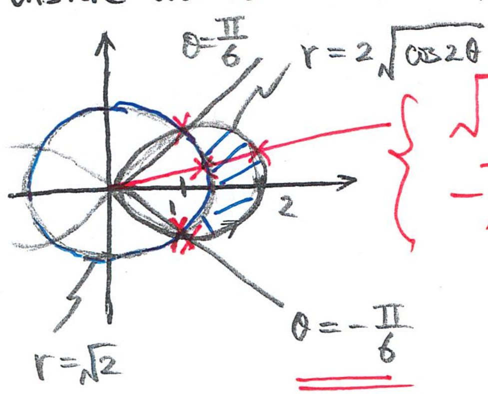
$$2 = r = 4\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

$$\begin{cases} 0 \le r \le 2 \\ -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \end{cases}$$

Ex. 5 Compute the area of the region in the first and fourth quadrants outside the circle $r=\sqrt{2}$ and inside the lemniscate $r^2 = 4\cos 2\theta$.



$$\begin{cases} \sqrt{2} \le r \le 2\sqrt{\cos 2\theta} \\ -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \end{cases}$$

$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_{\sqrt{2}}^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} r^2 \Big|_{\sqrt{2}}^{2\sqrt{\cos 2\theta}} d\theta = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (4\cos 2\theta - 2)\,d\theta$$

$$2 = r^2 = 4\cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$$

~~Ex. 6~~ Find the average value of the y-coordinates of the points in the semicircular disk of radius a given by $R = \{(r,\theta) : 0 \le r \le a,\ 0 \le \theta \le \pi\}$.

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \iint_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,r\,dr\,d\theta = 2\pi \int_0^{\infty} r e^{-r^2}\,dr$$

$$= 2\pi \left(-\frac{1}{2} e^{-r^2}\right)\Big|_0^{\infty} = -\pi\left(e^{-\infty} - e^{0}\right) = \pi$$