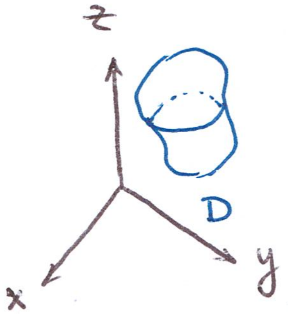
Lesson 21

§16.4 Triple Integrals



$$\iiint_D f(x,y,z)\,dV = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \sum_{i=1}^{p}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\,\underline{\Delta x\Delta y\Delta z}$$

$$\parallel$$

$$\iint_R \left(\int_{g(x,y)}^{h(x,y)} f\,dz\right) dA$$

- elementary regions

$$D.\begin{cases} (x,y)\in R \\ g(x,y)\le z\le h(x,y)\end{cases}$$

$$\parallel$$

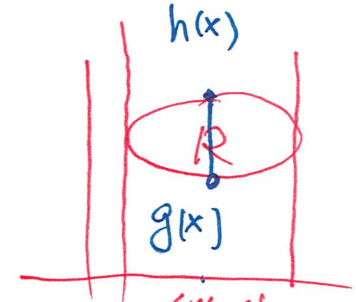
$$\iint_R \left(\int_{g(x,z)}^{h(x,z)} f\,dy\right) dA$$

$$D:\begin{cases} (x,z)\in R \\ g(x,z)\le y\le h(x,z)\end{cases}$$

$$\parallel$$

$$\iint_R \left(\int_{g(y,z)}^{h(y,z)} f\,dx\right) dA$$

$$D;\begin{cases} (y,z)\in R \\ g(y,z)\le x\le h(y,z)\end{cases}$$

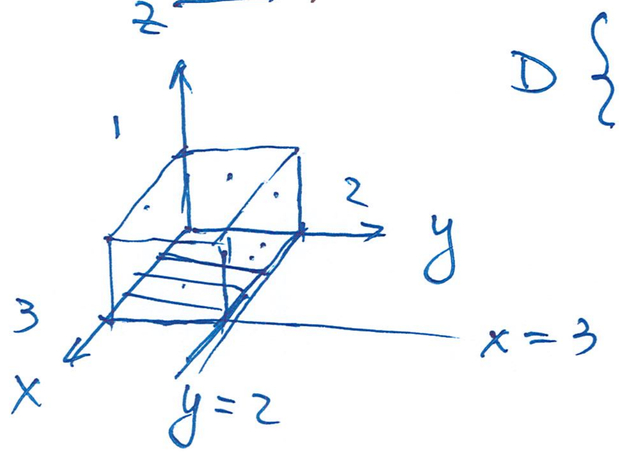


$a\le x\le b$

$$\iint_R f(x,y)\,dA$$

$$=\int_a^b\left(\int_{g(x)}^{h(x)} f(x,y)\,dy\right)dx$$

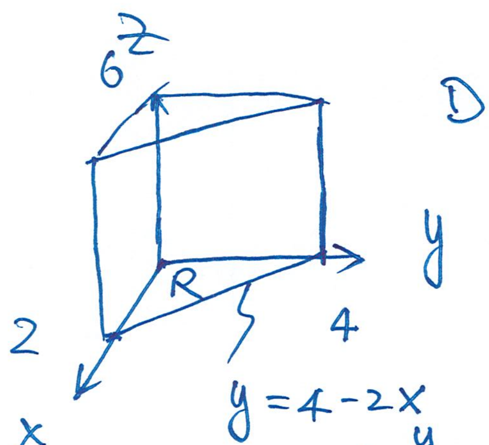
Ex. 1 A solid box D is bounded by the planes $x=0$, $x=3$, $y=0$, $y=2$, $z=0$, and $z=1$. The density of the box decreases linearly in the positive z-direction is given by $f(x,y,z)=2-z$. Find the mass of the box.



$$D \begin{cases} (x,y)\in R=[0,3]\times[0,2] \\ 0\le z\le 1 \end{cases}$$

$$m=\iiint_D f\,dV=\iint_R\left(\int_0^1 (2-z)\,dz\right)dA=\int_0^3 dx\int_0^2\int_0^1 (2-z)\,dz\,d$$

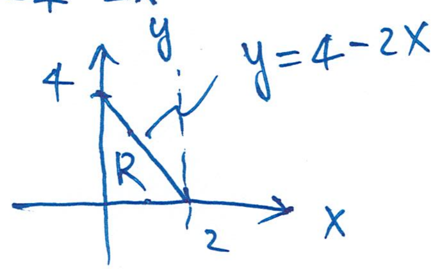
Ex. 2 Find the volume of the prism D in the first octant bounded by the planes $y=4-2x$ and $z=6$.



$$D \begin{cases} 0\le z\le 6 \\ (x,y)\in R \end{cases}$$

$$V=\iiint_D 1\,dV$$

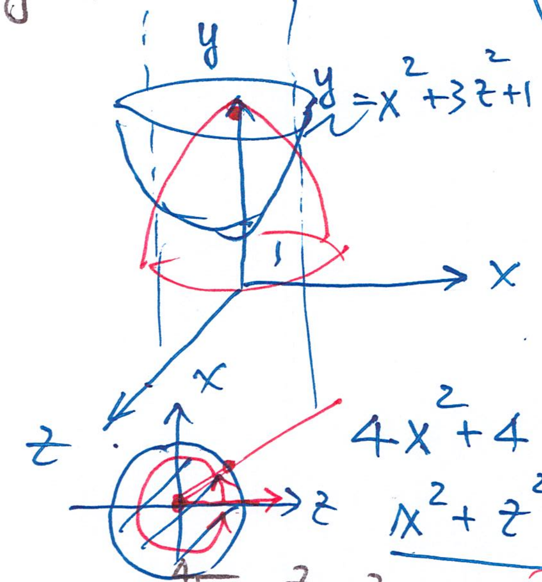
$$A=\iint_R 1\,dA$$



$$=\iint_R\left(\int_0^6 1\,dz\right)dA$$

$$=\int_0^2 dx\int_0^{4-2x} dy\int_0^6 1\,dz$$

Ex. 3 Find the volume of the solid D bounded by the paraboloids $y=x^2+3z^2+1$ and $y=5-3x^2-z^2$.



$$V=\iint_R\left(\int_{x^2+3z^2+1}^{5-3x^2-z^2} 1\,dy\right)dA=\iint_R\left[(5-3x^2-z^2)-(x^2+3z^2+1)\right]dA$$

$$x^2+3z^2+1=y=5-3x^2-z^2$$

$$4x^2+4z^2=4$$

$$x^2+z^2=1$$

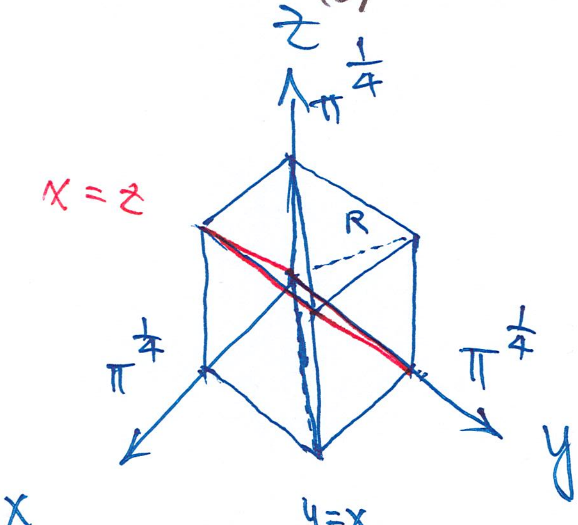
$$=\iint_R 4\left(1-(x^2+z^2)\right)dA$$

$$=\int_0^{2\pi}d\theta\int_0^1 r\,dr\,4(1-r^2)=2\pi\cdot 4\int_0^1 r(1-r^2)\,dr$$

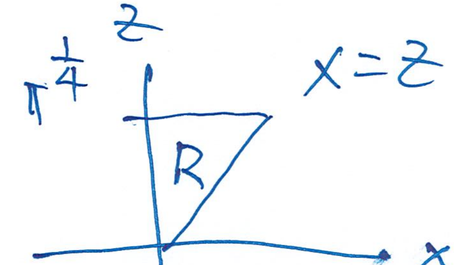
Ex. 4 $I=\int_0^{\sqrt[4]{\pi}}\int_0^z\int_y^z 12y^2z^3\sin x^4\,dx\,dy\,dz$. (a) Sketch the region of integration D

(b) Evaluate the integral.

$$D\begin{cases} y\le x\le z\\ 0\le y\le z\\ 0\le z\le \pi^{\frac14}\end{cases}\Big\}R$$



$$\begin{cases}0\le y\le x\\ (x,z)\in R\end{cases}$$

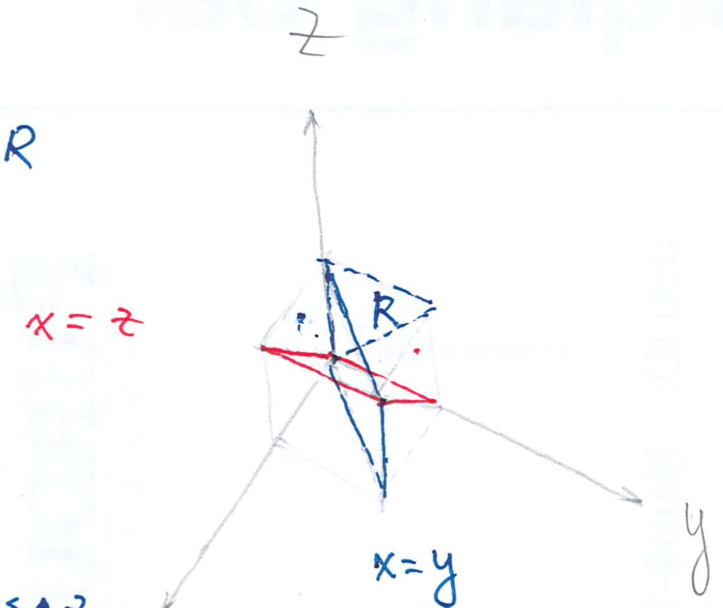


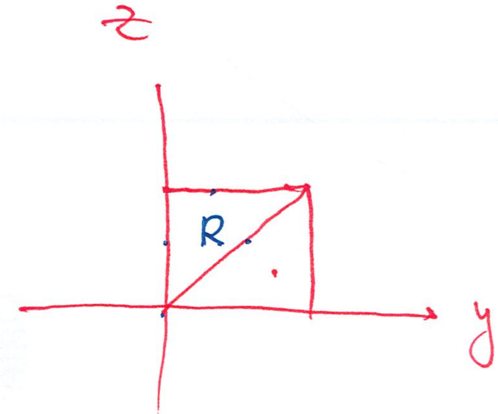
$y=x$
$x=z$
$y=0$
$y=z$
$z=0$
$z=\pi^{\frac14}$

$$I=\iint_R\int_0^x\left(12z^3\sin x^4\right)y^2\,dy\,dA=\iint_R 4z^3\sin x^4\,y^3\Big|_0^x dA=\iint_R 4z^3x^3\sin x^4\,dA$$

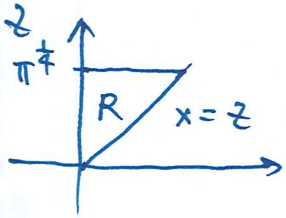
Ex. 4

$$\left\{\begin{array}{l} y \le x \le z \\ 0 \le y \le z \\ 0 \le z \le \pi^{\frac{1}{4}} \end{array}\right.\Bigg\}R$$





$$\left\{\begin{array}{l} 0 \le y \le x \\ (x, z) \in R \end{array}\right.$$



$$\left\{\begin{array}{l} 0 \le x \le z \\ 0 \le z \le \pi^{\frac{1}{4}} \end{array}\right.$$

$$\left\{\begin{array}{l} 0 \le x \le y \\ (y, z) \in R \end{array}\right.$$

$$\int_0^{\pi^{\frac{1}{4}}} \int_0^{z} \int_y^{z} 12 y^2 z^3 \sin x^4 \, dx\, dy\, dz$$

$$= \iint_R \left(\int_0^{x} \left(12 z^3 \sin^4\right) y^2\, dy \right) dA = \iint_R \left(4 z^3 \sin x^4\right) y^3 \Big|_0^{x} dA = 4 \iint_R z^3 \left(x^3 \sin x^4\right) dA$$

$$= 4 \int_0^{\pi^{\frac{1}{4}}} \int_0^{z} z^3 \left(x^3 \sin x^4\right) dx\, dz = \int_0^{\pi^{\frac{1}{4}}} z^3 \left(-\cos x^4\right) \Big|_{x=0}^{x=z} dz = \int_0^{\pi^{1/4}} z^3 \left(1 - \cos z^4\right) dz$$

$$= \frac{1}{4} \left(z^4 - \sin z^4\right) \Big|_{z=0}^{z=\pi^{\frac{1}{4}}} = \frac{1}{4}\pi$$