Green BDACC BEDAD E
E

Orange BDEDA CEBAD C
D

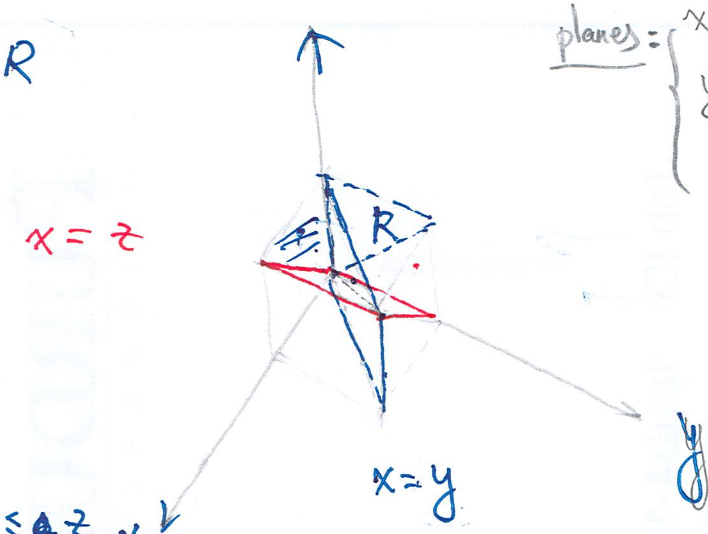
Alternative DABEC CDABA E
D

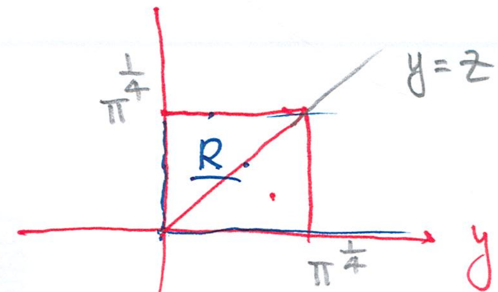
Ex. 4

$$\int_0^{\pi^{\frac14}}\int_0^{z}\int_y^{z} 12\,y^2 z^3 \sin x^4 \, dx\, dy\, dz$$

$$\left.\begin{cases} y \le x \le z \\ 0 \le y \le z \\ 0 \le z \le \pi^{\frac14} \end{cases}\right\} R$$

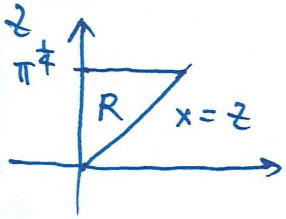
planes: $\begin{cases} x = y, \; x = z \\ y = 0, \; y = z \\ z = 0, \; z = \pi^{\frac14} \end{cases}$





$R: \begin{cases} 0 \le y \le z \\ 0 \le z \le \pi^{\frac14} \end{cases}$

$$\begin{cases} 0 \le y \le x \\ (x, z) \in R \end{cases}$$



$\begin{cases} 0 \le x \le z \\ 0 \le z \le \pi^{\frac14} \end{cases}$

$$\begin{cases} 0 \le x \le y \\ (y, z) \in R \end{cases}$$

$$\int_0^{\pi^{\frac14}}\int_0^{z}\int_y^{z} 12\,y^2 z^3 \sin x^4 \, dx\, dy\, dz$$

$$= \iint_R \left(\int_0^{x} \left(12 z^3 \sin^4\right) y^2\, dy \right) dA = \iint_R \left(4 z^3 \sin x^4\right) y^3 \Big|_0^{x} dA = 4 \iint_R z^3 \left(x^3 \sin x^4\right) dA$$

$$= 4 \int_0^{\pi^{\frac14}} \int_0^{z} z^3 \left(x^3 \sin x^4\right) dx\, dz = \int_0^{\pi^{\frac14}} z^3 \left(-\cos x^4\right)\Big|_{x=0}^{x=z} dz = \int_0^{\pi^{1/4}} z^3 \left(1 - \cos z^4\right) dz$$

$$= \frac14 \left(z^4 - \sin z^4\right)\Big|_{z=0}^{z=\pi^{\frac14}} = \frac14 \pi$$

Lesson 22

§16.5 Triple Integrals in Cylindrical and Spherical Coordinates

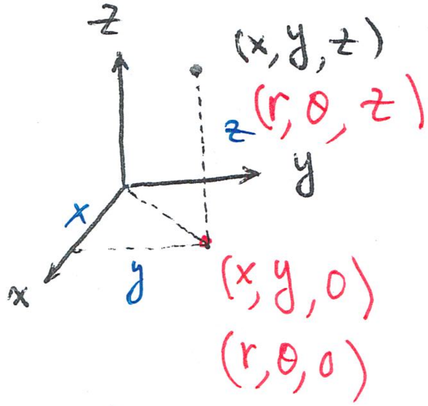
• cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad r^2 = x^2 + y^2, \quad \tan\theta = y/x, \quad z = z$$

$0 \le r < +\infty$

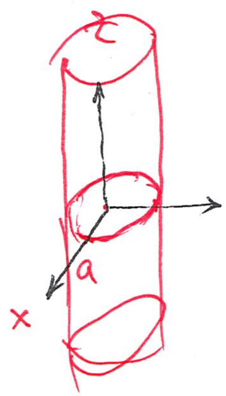
$0 \le \theta \le 2\pi$

$-\infty < z < +\infty$



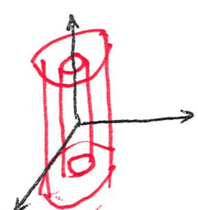
(1) cylinder

$\{(r,\theta,z): r = a > 0\}$



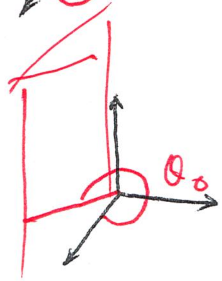
(2) cylindrical shell

$\{(r,\theta,z): 0 < a \le r \le b\}$



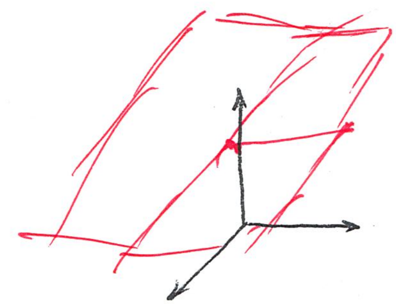
(3) vertical half-plane

$\{(r,\theta,z): \theta = \theta_0\}$



(4) horizontal plane

$\{(r,\theta,z): z = a\}$



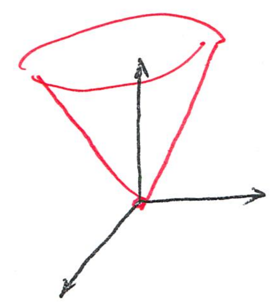
(5) cone

$\{(r,\theta,z): z = ar\}$

$a \ne 0$

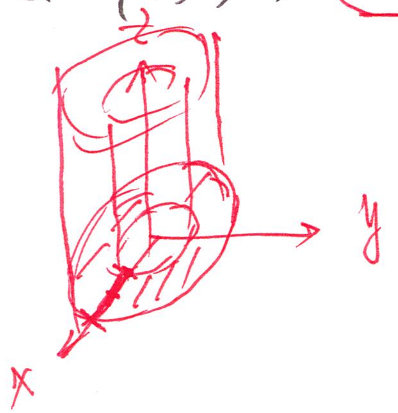
$z = ar = a\sqrt{x^2+y^2}$

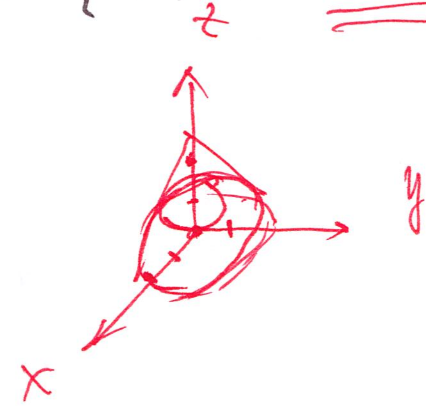
$z = 1 = a\sqrt{x^2+y^2}$



Ex. 1 Identify and sketch the following sets in cylindrical coordinates

(a) $Q = \{(r,\theta,z) : 1 \le r \le 3, z \ge 0\}$. (b) $S = \{(r,\theta,z) : z = 1-r, 0 \le r \le 1\}$





$r = \frac{1}{2}$

$z = 1 - \frac{1}{2} = \frac{1}{2}$

- triple integrals in cylindrical coordinates

$$D = \left\{(r,\theta,z) : \begin{matrix} \alpha \le \theta \le \beta \\ 0 \le g(\theta) \le r \le h(\theta) \end{matrix},\ G(x,y) \le z \le H(x,y)\right\}.$$

$$\iiint_D f(x,y,z)\,dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f\,dz\right) dA$$

$$= \iint_R \int_{g(r,\theta)}^{h(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\,r\,dr\,d\theta$$

Ex. 2 $\displaystyle\int_0^{2\sqrt{2}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \left(\int_{-1}^{2} \sqrt{1+x^2+y^2}\,dz\right) dy\,dx$

$$= \int_0^{2\sqrt{2}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} 3\sqrt{1+x^2+y^2}\,dy\,dx$$

R

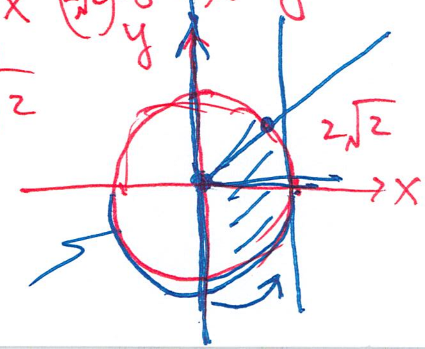
$$R \begin{cases} -\sqrt{8-x^2} \le y \le \sqrt{8-x^2} \\ 0 \le x \le 2\sqrt{2} \end{cases}$$

$y = \sqrt{8-x^2}$

$y^2 = 8 - x^2$

$(2\sqrt{2})^2 = 8 = x^2 + y^2$

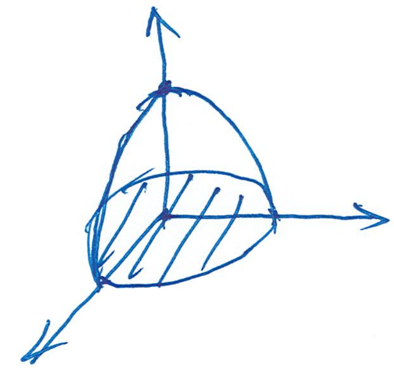
$$= \int_0^{2\sqrt{2}} r\,dr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\ 3\sqrt{1+r^2} = 3\pi \int_0^{2\sqrt{2}} r\sqrt{1+r^2}\,dr$$



$r = 2\sqrt{2}$

Ex.3 Find the mass of the solid D bounded by the paraboloid $z=4-r^2$ and the plane $z=0$, where the density of the solid given in cylindrical coordinates is $f(r,\theta,z)=5-z$.

$$m=\iiint_D (5-z)\,dV=\iint_R\left(\int_0^{4-r^2}(5-z)\,dz\right)dA$$



$$=\iint_R\left[5(4-r^2)-\frac{1}{2}(4-r^2)^2\right]dA$$

$$=\int_0^2 r\,dr\int_0^{2\pi}d\theta\left[12-r^2-\frac{1}{2}r^4\right]r\,d\theta\,dr=2\pi\int_0^2\left(12r-r^3-\frac{1}{2}r^5\right)dr$$

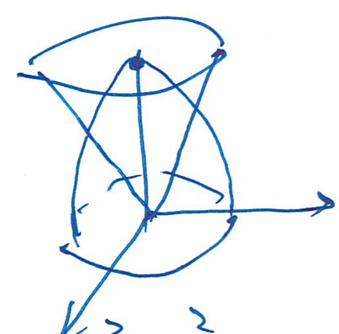
$$=2\pi\left[6r^2-\frac{1}{4}r^4-\frac{1}{12}r^6\right]_0^2$$

$z=0:\ r^2=4\Rightarrow r=2$

Ex.4 Find the volume of the solid D between the cone $z=\sqrt{x^2+y^2}$ and the inverted paraboloid $z=12-x^2-y^2$

$$V=\iiint_D 1\,dV=\iint_R\left(\int_{\sqrt{x^2+y^2}}^{12-x^2-y^2}dz\right)dA=\iint_R\left[(12-x^2-y^2)-\sqrt{x^2+y^2}\right]dA$$

?



$$=\int_0^3 r\,dr\int_0^{2\pi}\left[12-r^2-r\right]d\theta$$

$$=2\pi\int_0^3\left(12r-r^3-r^2\right)dr$$

$$=2\pi\left[6r^2-\frac{1}{4}r^4-\frac{1}{3}r^3\right]_0^3$$

$\sqrt{x^2+y^2}=z=12-x^2-y^2$

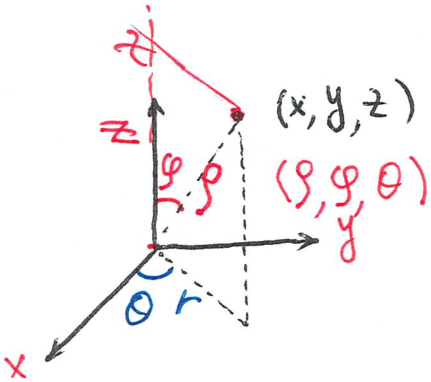
$(x^2+y^2)+\sqrt{x^2+y^2}-12=0$

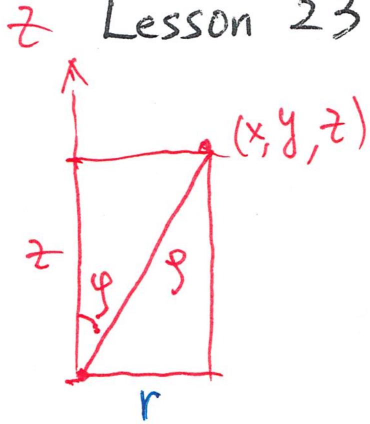
$0=r^2+r-12=(r+4)(r-3)$

$\Rightarrow \cancel{r=-4},\ r=3$

Lesson 23

- <u>spherical coordinates</u>





$r = \rho \sin\varphi$

$$\begin{cases} x = \boxed{r\cos\theta =} \rho\sin\varphi\cos\theta \\ y = \boxed{r\sin\theta =} \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$\left.\begin{array}{l} x^2+y^2 = r^2 = \rho^2\sin^2\varphi \\ z^2 = \rho^2\cos^2\varphi \end{array}\right\} \Rightarrow x^2+y^2+z^2 = \rho^2$$

$\boxed{\rho = a}$

<u>Ex. 5</u> Express the following sets in rectangular coordinates and identify the set. $(a > 0)$

(a) $\{(\rho,\varphi,\theta) : \rho = 2a\cos\varphi,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\}$.

(b) $\{(\rho,\varphi,\theta) : \rho = 4\sec\varphi,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\}$