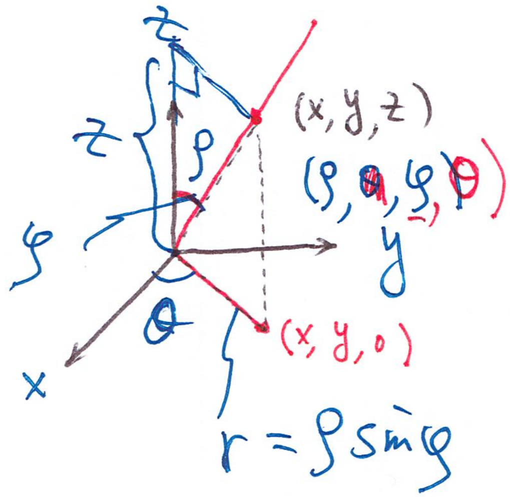
Lesson 23

• <u>spherical coordinates</u>



$$\begin{cases} x = \rho\sin\varphi\cos\theta = r\cos\theta \\ y = \rho\sin\varphi\sin\theta = r\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$r = \rho\sin\varphi$

$x^2 + y^2 = r^2$

$$\boxed{x^2 + y^2 + z^2 = \rho^2}$$

$r^2 + z^2 = \rho^2$

$$\boxed{\rho = 1}$$

Ex. 5 Express the following sets in rectangular coordinates and identify the set. $(a > 0)$

(a) $\{(\rho, \varphi, \theta) : \rho = 2a\cos\varphi, \ 0 \le \varphi \le \frac{\pi}{2}, \ 0 \le \theta \le 2\pi\}$.

(b) $\{(\rho, \varphi, \theta) : \rho = 4\sec\varphi, \ 0 \le \varphi < \frac{\pi}{2}, \ 0 \le \theta \le 2\pi\}$

$\rho^2 = 2a\,\rho\cos\varphi = 2az$

$\parallel$

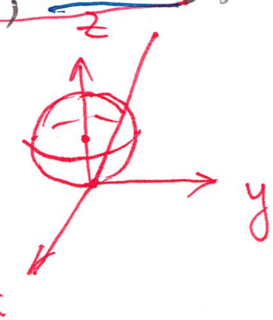
$x^2 + y^2 + z^2 = 2az$

$x^2 + y^2 + (z^2 - 2az + a^2) = a^2$

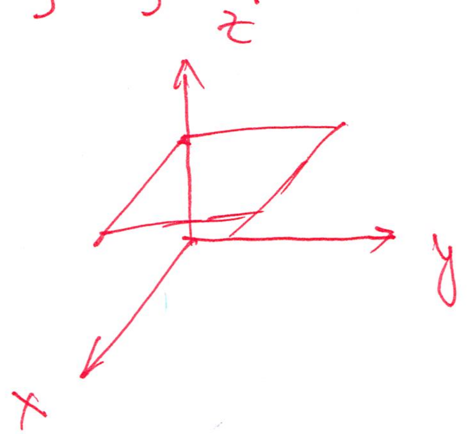
$\parallel$

$x^2 + y^2 + (z - a)^2 = a^2$ sphere

at $(0, 0, a)$, radius a



$z = \rho\cos\varphi = 4$



- simple descriptions of some sets

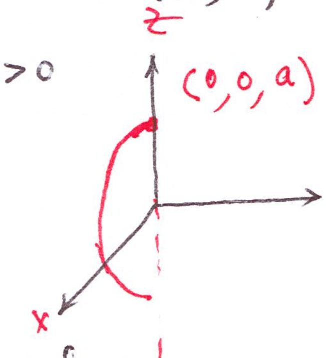
(1) Sphere, radius a, center $(0,0,0)$

$\{(\rho,\varphi,\theta):\ \rho = a\}$, $a>0$

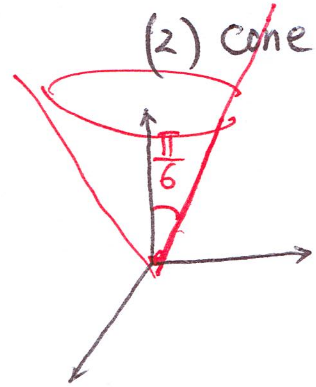
$\theta = 0$

$\varphi: 0 \to \pi$

$\theta: 0 \to 2\pi$



(2) cone $\{(\rho,\varphi,\theta):\ \boxed{\varphi=\varphi_0}\}$, $\varphi_0 \neq 0, \frac{\pi}{2}, \pi$

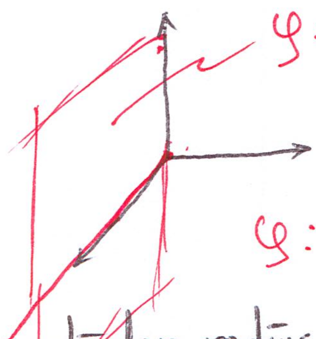


$\varphi = \frac{\pi}{6}$

$\rho: 0 \to +\infty$

$\theta: 0 \to 2\pi$

(3) vertical half-plane $\{(\rho,\varphi,\theta):\ \theta=\theta_0\}$



$\varphi: 0 \to \frac{\pi}{2}$

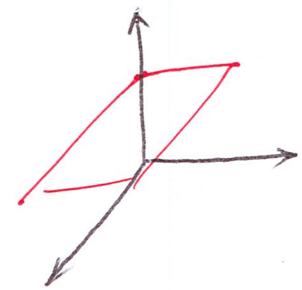
$\theta = 0$

$\varphi: \frac{\pi}{2} \to \pi$

(4) horizontal plane, $z = a$

$a>0$: $\{(\rho,\varphi,\theta):\ \rho = a\sec\varphi,\ 0 \le \varphi < \frac{\pi}{2}\}$

$a<0$: $\{(\rho,\varphi,\theta):\ \rho = a\sec\varphi,\ \frac{\pi}{2} < \varphi \le \pi\}$



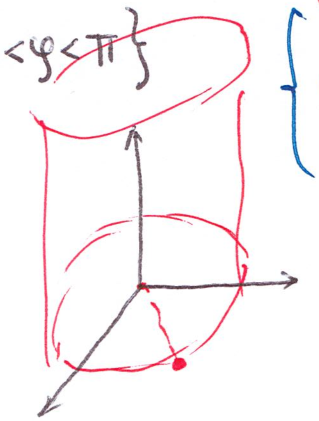
$\rho\cos\varphi = a$

$\rho = a\frac{1}{\cos\varphi}$

(5) cylinder, radius $a>0$

$\{(\rho,\varphi,\theta):\ \rho = a\csc\varphi,\ 0<\varphi<\pi\}$

$r = \rho\sin\varphi = a$

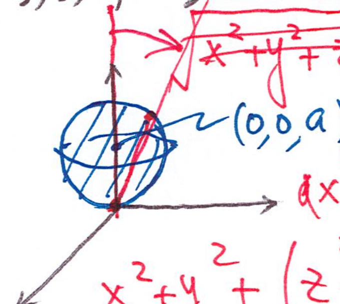


$$\begin{cases} 0 \le \rho \le 2a\cos\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{cases}$$

(6) sphere, radius $a>0$, center $(0,0,a)$

$\{(\rho,\varphi,\theta):\ \rho = 2a\cos\varphi,\ 0 \le \varphi \le \frac{\pi}{2}\}$

$\cancel{x^2+y^2+z^2 = 2a}$



$\rho^2 = 2a\rho\cos\varphi = 2az$

$x^2+y^2+z^2 = 2az$

$x^2+y^2+(z^2-2az+a^2) = a^2$

$x^2+y^2+(z-a)^2 = a^2$

$$\iiint_D f\,dV = \iiint f(\rho\sin\varphi \longrightarrow)\ \underbrace{\left|\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}\right|}_{=\ \rho^2\sin\varphi}\ d\rho\, d\varphi\, d\theta$$

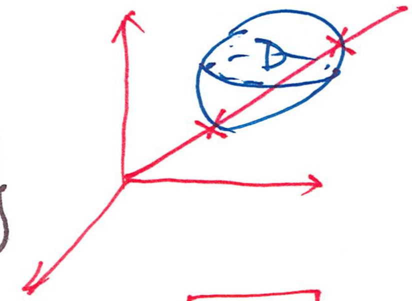
$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\varphi}, & \frac{\partial z}{\partial\theta} \end{vmatrix} = \rho^2\sin\varphi$$

$$= \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \cos\varphi\ \rho^2\cos\varphi\sin\varphi + \rho\sin\varphi\,\rho\sin^2\varphi = \rho^2\sin\varphi$$

- triple integrals in spherical coordinates

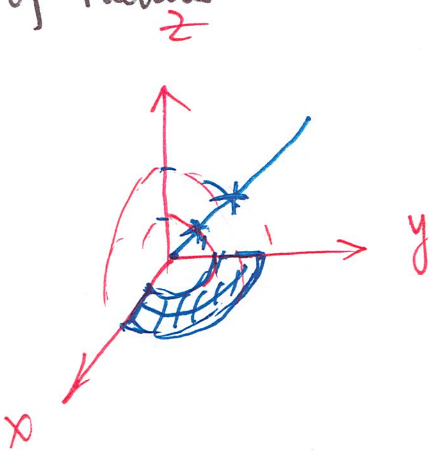


$$D = \left\{ (\rho, \varphi, \theta) : \quad 0 \le g(\varphi,\theta) \le \rho \le h(\varphi,\theta), \quad a \le \varphi \le b, \quad \alpha \le \theta \le \beta \right\}$$

$$\iiint_D f(x,y,z)\, dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi)\, \boxed{\rho^2 \sin\varphi}\, d\rho\, d\varphi\, d\theta .$$

$$\text{abs}\ \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}$$

Ex. 6 $I = \iiint_D \left(x^2+y^2+z^2 \right)^{-\frac{3}{2}} dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin.



$$1 \le \rho \le 2$$

$$0 \le \varphi \le \frac{\pi}{2}$$

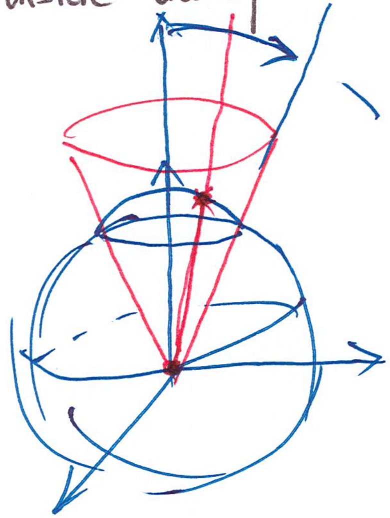
$$0 \le \theta \le \frac{\pi}{2}$$

$$I = \int_1^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^{-3} \underline{\rho^2 \sin\varphi}\, d\theta\, d\varphi\, d\rho$$

$$= \int_0^{\frac{\pi}{2}} 1\, d\theta \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \int_1^2 \rho^{-1}\, d\rho$$

$$= \frac{\pi}{2} \cdot \left[-\cos\varphi \right]_0^{\frac{\pi}{2}} \cdot \left[\ln\rho \right]_1^2$$

Ex. 7 Find the volume of the solid region D that lies inside the cone $\varphi = \frac{\pi}{6}$ and inside the sphere $\rho = 4$.



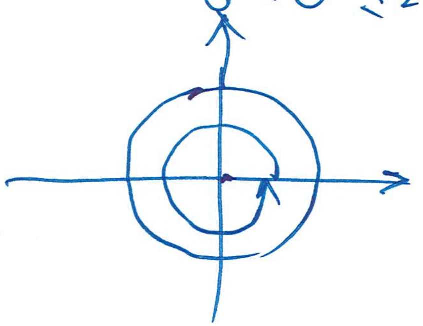
$$V = \iiint_{D} dV = \int_0^4 d\rho \int_0^{\frac{\pi}{6}} d\varphi \int_0^{2\pi} d\theta \; \rho^2 \sin\varphi$$

$$= \int_0^4 \rho^2 \, d\rho \cdot \int_0^{\frac{\pi}{6}} \sin\varphi \, d\varphi \int_0^{2\pi} d\theta$$

$0 \leq \rho \leq 4$

$0 \leq \varphi \leq \frac{\pi}{6}$

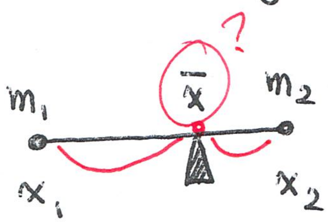
$0 \leq \theta \leq 2\pi$



Lesson 24

§16.6 Integrals for Mass Calculations (center of mass)

- discrete objects



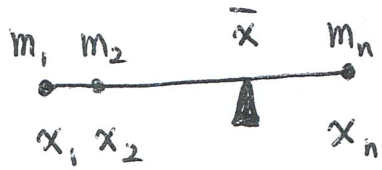
$$(x_2-\bar{x})m_2 = (\bar{x}-x_1)m_1$$

$$m_1\bar{x}+m_2\bar{x} = x_2m_2+x_1m_1$$

$$\bar{x} = \frac{m_1x_1+m_2x_2}{m_1+m_2}$$

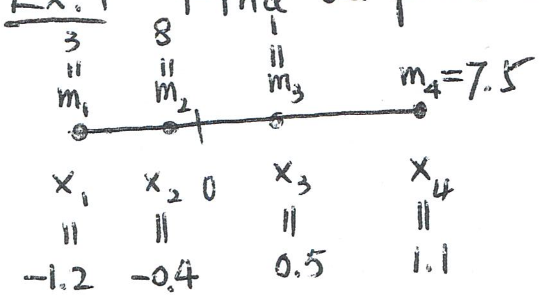
$$(x_1-\bar{x})m_1 + (x_2-\bar{x})m_2 = 0$$

$$\boxed{(x_1-\bar{x})m_1 + (x_2-\bar{x})m_2 + \cdots + (x_n-\bar{x})m_n = 0}$$



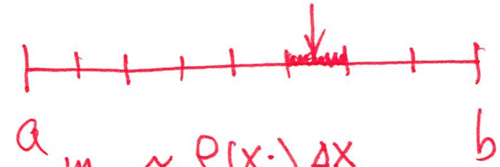
$$\bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1+\cdots+m_n}$$

Ex. 1 Find the point at which the system shown in Figure balances.



- continuous objects in one dimension

a thin rod with density ρ



$m_i \approx \rho(x_i)\,\Delta x$

uniform partition : $a = x_0 < x_1 < \cdots < x_n = b$, $\quad x_i = x_0 + i\,\Delta x$, $\quad \Delta x = \dfrac{b-a}{n}$

$$\bar{x} = \lim_{\Delta x \to 0} \frac{m_1 x_1 + \cdots + m_n x_n}{m_1 + \cdots + m_n} = \lim_{\Delta x \to 0} \frac{\sum_{i=1}^{n} \left(\rho(x_i)\Delta x\right) x_i}{\sum_{i=1}^{n} \rho(x_i)\,\Delta x}$$

$$\boxed{\bar{x} = \int_a^b x\,\rho(x)\,dx \Big/ \int_a^b \rho(x)\,dx}$$

Ex. 2 Suppose a thin 2-m bar is made of an alloy whose density in kg/m is $\rho(x) = 1 + x^2$, where $0 \le x \le 2$. Find the center of mass of the bar.

$$\bar{x} = \int_0^2 x(1+x^2)\,dx \Big/ \int_0^2 (1+x^2)\,dx$$

$$= \left[\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4\right]_0^2 \Big/ \left[x + \tfrac{1}{3}x^3\right]_0^2$$