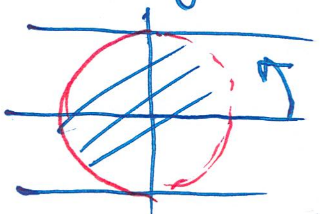
#40 Evaluate $I=\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}(x^2z+y^2z+z^3)\,dV$ by using spherical coordinates.

$$-\sqrt{a^2-x^2-y^2}\le z\le\sqrt{a^2-x^2-y^2}$$

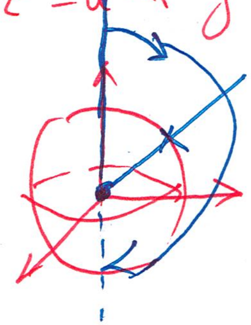
$$\left.\begin{array}{l}-\sqrt{a^2-y^2}\le x\le\sqrt{a^2-y^2}\\ -a\le y\le a\end{array}\right\}R$$

$$z=\sqrt{a^2-x^2-y^2}\Rightarrow z^2=a^2-x^2-y^2$$



$y=a$

$y=-a$



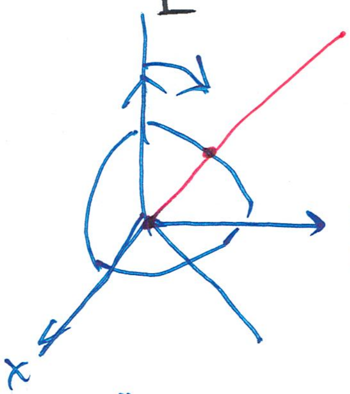
$0\le\rho\le a$

$0\le\varphi\le\pi$

$0\le\theta\le 2\pi$

$$I=\int_0^a d\rho\int_0^{\pi}d\varphi\int_0^{2\pi}d\theta\ \rho\cdot\rho\cos\varphi\cdot\rho^2\sin\varphi$$

#25 $I=\iiint_E x\,e^{x^2+y^2+z^2}\,dV$, E: the portion of the unit ball $x^2+y^2+z^2\le 1$ that lies in the 1st octant.

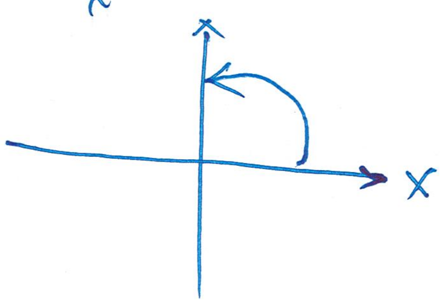


$0\le\rho\le 1$

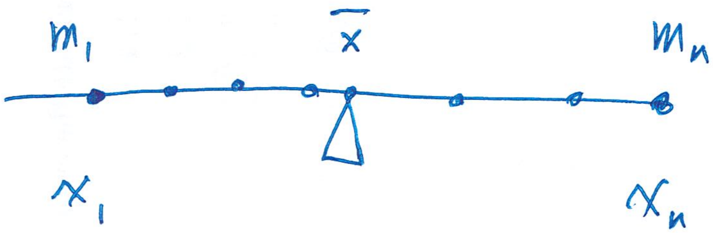
$0\le\varphi\le\frac{\pi}{2}$

$0\le\theta\le\frac{\pi}{2}$

$$I=\int_0^1 d\rho\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}}d\theta\ \rho\sin\varphi\cos\theta\cdot e^{\rho^2}\cdot\rho^2\sin\varphi$$



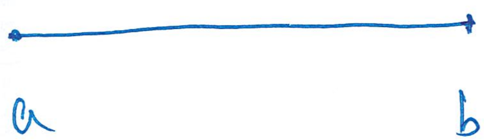
center of mass



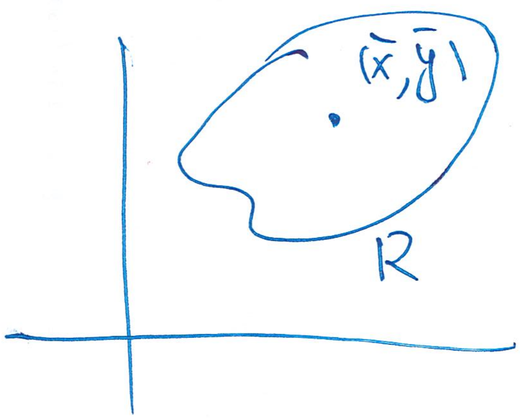
$$\bar{x} = \frac{m_1 x_1 + \cdots + m_n x_n}{m_1 + \cdots + m_n}$$

total moment

total mass



$$\bar{x} = \frac{\int_a^b x\,\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$$



$$\bar{x} = \frac{\iint_R x\,\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

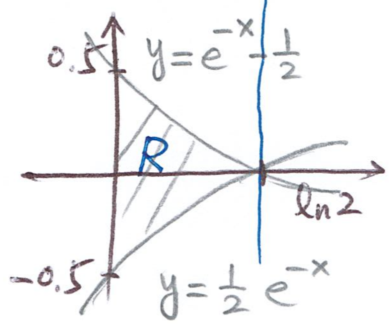
$$\bar{y} = \frac{\iint_R y\,\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

- two dimensions

Let ρ be an area density over a closed bounded region R in 2 dimensions.
The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\,\rho(x,y)\,dA\,, \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\,\rho(x,y)\,dA\,, \quad m = \iint_R \rho\,dA \text{ — mass}$$

Ex. 3 Find $\boxed{\text{the centroid}}$ of the unit-density of the region bounded by the y-axis and the curves $y = e^{-x} - \frac{1}{2}$ and $y = \frac{1}{2} - e^{-x}$.



$$m = \iint_R 1\,dA = \int_0^{\ln 2}\int_{\frac{1}{2}e^{-x}}^{e^{-x}-\frac{1}{2}} 1\,dy\,dx$$

$$\bar{x} = \iint_R x\,dA \Big/ m = \frac{1}{m}\int_0^{\ln 2}\int_{\frac{1}{2}e^{-x}}^{\cancel{\frac{1}{2}}e^{-x}-\frac{1}{2}} x\,dy\,dx$$

~~Ex.4~~ Find the center of mass of $R = \{(x,y) : -1 \le x \le 1,\ 0 \le y \le 1\}$ with a density of $\rho(x,y) = 2-y$.

• <u>three dimensions</u> region D in R^3, $\rho(x,y,z)$ — density

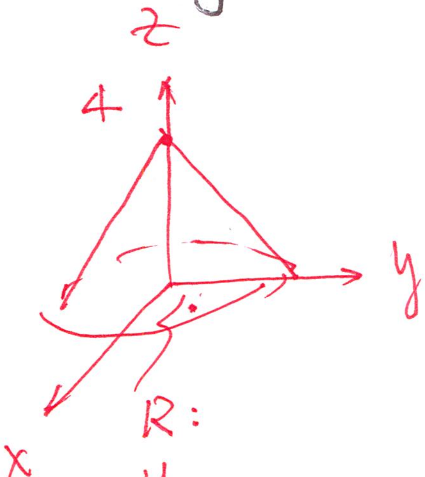
<u>mass</u> $m = \iiint_D \rho \, dV$, <u>moments</u>: $M_{yz} = \iiint_D x\rho \, dV$, $M_{xz} = \iiint_D y\rho \, dV$, $M_{xy} = \iiint_D z\rho \, dV$

<u>the center of mass</u> $(\bar{x}, \bar{y}, \bar{z})$: $\bar{x} = M_{yz}/m$, $\bar{y} = M_{xz}/m$, $\bar{z} = M_{xy}/m$

<u>Ex. 5</u> Find the center of mass of the constant-density solid cone D bounded by the surface $z = 4 - \sqrt{x^2+y^2}$ and $z = 0$.

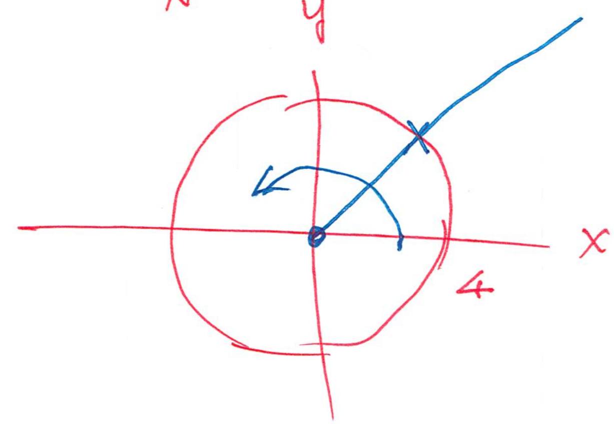
$$0 = z = 4 - \sqrt{x^2+y^2}$$

$$x^2 + y^2 = 4^2$$



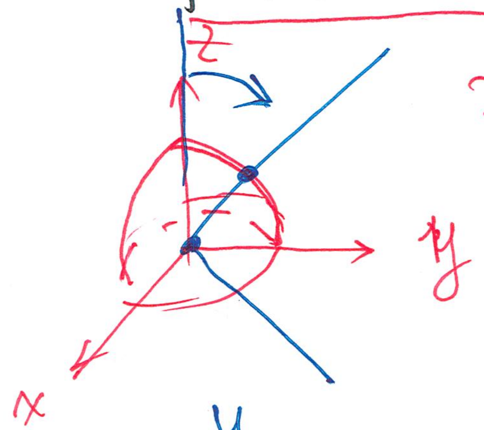
$$m = \iiint_D \rho \, dV = \rho \iint_R \int_0^{4-\sqrt{x^2+y^2}} dz \, dA$$

$$= \rho \iint_R \left(4 - \sqrt{x^2+y^2}\right) dA = \rho \int_0^4 dr \int_0^{2\pi} d\theta \, (4-r) \, r$$



$$M_{yz} = \rho \int_0^4 dr \int_0^{2\pi} d\theta \, (4-r) \cdot r\cos\theta \cdot r$$

Ex. 6 Find the center of mass of the interior of the hemisphere D of radius a with its base on the xy-plane. The density of the object given in spherical coordinates is $f(\rho,\varphi,\theta) = 2 - \frac{\rho}{a}$.



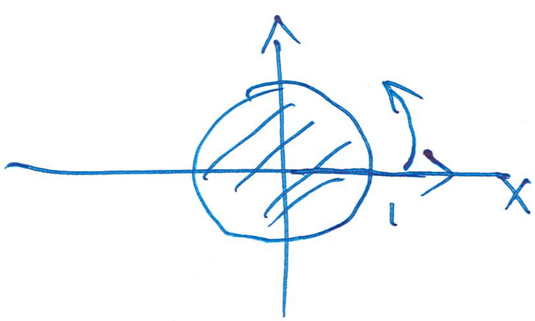
$D:$

$$0 \le \rho \le a$$
$$0 \le \varphi \le \frac{\pi}{2}$$
$$0 \le \theta \le 2\pi$$

$$\frac{\pi}{2} \le \varphi \le \pi$$

$$m = \iiint_D \left(2 - \frac{\rho}{a}\right) dV = \int_0^a d\rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\pi} d\theta \left(2 - \frac{\rho}{a}\right) \cdot \rho^2 \sin\varphi$$

y



Lesson 25

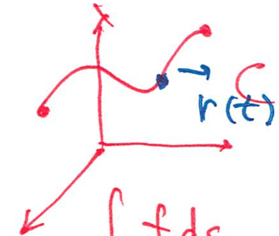
$f(x)$ $\quad \int_a^b f(x)\,dx$

Chapter 17 Vector Calculus

$\iint_R f\,dA$

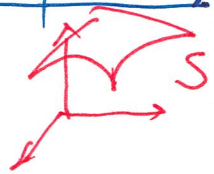
$\vec r(t) = \langle x(t), y(t), z(t)\rangle$

$f(x,y)$; $f(x,y,z)$



§17.1 Vector Fields

$\vec F(x,y)$ $\quad \iint_S f\,dS$ $\quad \int_C f\,ds$



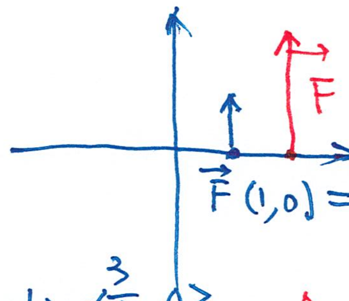
- Vector Fields in Two-Dimensions

$$\vec F(x,y) = \langle f(x,y), g(x,y)\rangle : R^2 \to R^2$$

$$\vec F(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$$

Ex. 1 Sketch the following vector fields

(a) $\vec F(x,y) = \langle 0, x\rangle$ (a shear field)



$\vec F(2,0) = \langle 0, 2\rangle$

$\vec F(1,0) = \langle 0, 1\rangle$

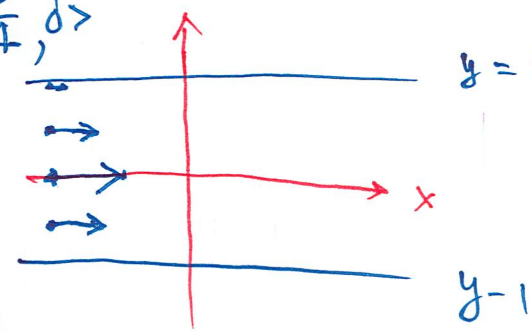
$\vec F(-2, \frac{1}{2}) = \langle \frac{3}{4}, 0\rangle$

(b) $\vec F(x,y) = \langle 1-y^2, 0\rangle$, for $|y| \le 1$ (Channel flow)

$\vec F(-2, 1-\varepsilon)$
$= \langle 2\varepsilon - \varepsilon^2, 0\rangle$

$\vec F(-2,0) = \langle 1, 0\rangle$

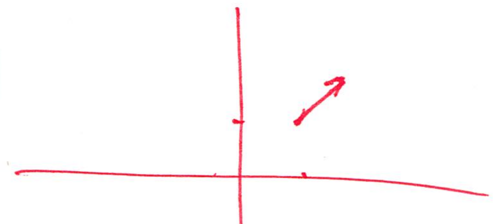
$\vec F(-2,-\frac{1}{2}) = \langle \frac{3}{4}, 0\rangle$



(c) $\vec F(x,y) = \langle -y, x\rangle$, (a rotation field)

- Radial Vector Fields in $\mathbb{R}^2$

$$\vec{F}(x,y) = f(x,y)\,\vec{r}, \quad \text{where } \vec{r} = \langle x, y\rangle$$



$$f(x,y) = |\vec{r}|^{-p}$$

$$\boxed{\vec{F}(x,y) = \frac{\vec{r}}{|\vec{r}|^{p}} = \frac{1}{|\vec{r}|^{p-1}} \cdot \frac{\vec{r}}{|\vec{r}|}}$$

$$|\vec{F}| = \frac{1}{|\vec{r}|^{p-1}}$$

Ex. 2 Let C be the circle $x^2+y^2=a^2$, where $a>0$.

(a) Show that at each point of C, $\vec{F}(x,y) = \dfrac{\vec{r}}{|\vec{r}|}$ is orthogonal to the line tangent to C at that point.

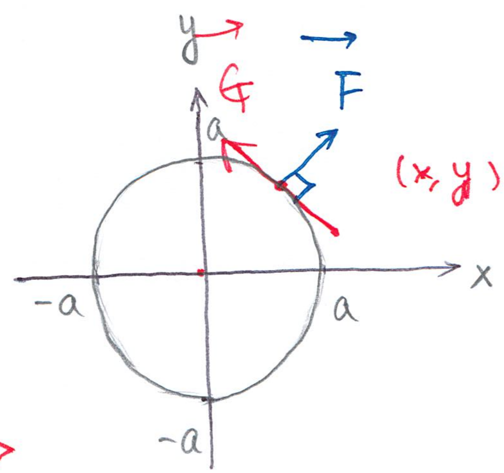
(b) Show that at each point of C, the rotation vector field $\vec{G}(x,y) = \dfrac{\langle -y, x\rangle}{\sqrt{x^2+y^2}}$ is parallel to the line tangent to C.

$$\varphi(x,y) \equiv x^2+y^2 = a^2$$

$$\nabla\varphi = \langle \varphi_x, \varphi_y\rangle = \langle 2x, 2y\rangle$$
$$= 2\langle x, y\rangle$$

$$\vec{F}\Big|_C = \frac{\langle x, y\rangle}{\sqrt{x^2+y^2}}\Bigg|_C = \frac{1}{a}\langle x, y\rangle$$

$$\Rightarrow \vec{F}\Big|_C \,/\!/\, \nabla\varphi, \quad \nabla\varphi \perp C \Rightarrow \vec{F} \perp C$$



$$(x,y)\quad 0 \overset{?}{=} \vec{G}\Big|_C \cdot \nabla\varphi$$
$$= \frac{1}{a}\langle -y, x\rangle \cdot 2\langle x, y\rangle$$
$$= \frac{2}{a}\big((-y)\cdot x + x\cdot y\big)$$
$$= 0$$