$$\vec{F}(x,y) = \langle f(x,y),\, g(x,y) \rangle : R^2 \to R^2$$

$$\vec{F}(x,y,z) = \langle f(x,y,z),\, g(x,y,z),\, h(x,y,z) \rangle : R^3 \to R^3$$

$\vec{r}(t) = \langle x(t),\, y(t) \rangle$ — flow curves of $\vec{F}$

$$\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$$

$$\langle x'(t),\, y'(t) \rangle = \langle f(x(t), y(t)),\, g(x(t), y(t))$$

$$\begin{cases} \dfrac{dx}{dt} = f(x(t), y(t)) \\[2ex] \dfrac{dy}{dt} = g(x(t), y(t)) \end{cases}$$

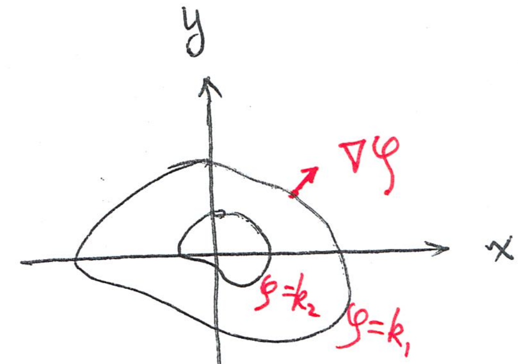
$x(t) = ?$

$y(t) = ?$

- ## Gradient Fields and Potential Functions

gradient field $\vec{F} = \nabla \varphi$

φ is a potential function of the vector field $\vec{F}$.



Level curves $\varphi(x,y)=k$

Ex. 4 (a) Sketch and interpret the gradient field associated with the temperature function $T = 200 - x^2 - y^2$ on the circular plate $R = \{(x,y) : x^2+y^2 \le 25\}$. $x^2+y^2=5^2$

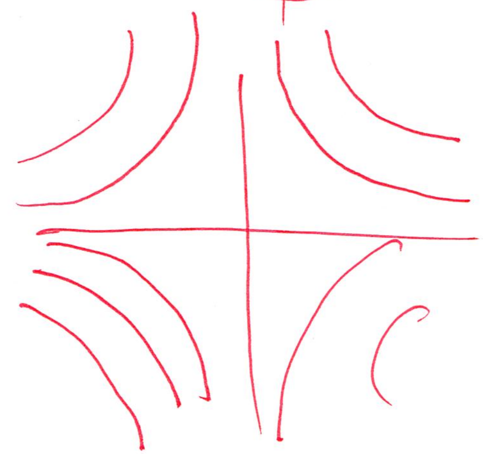
(b) Sketch and interpret the gradient field assoc. with the velocity potential $\varphi = \tan^{-1}(xy)$.

(a) $\nabla T = \langle T_x, T_y \rangle = \langle -2x, -2y \rangle = -2\langle x, y\rangle$

$x^2+y^2 = C+200 \Leftarrow 200-x^2-y^2 = T = C$

(b) $\nabla \varphi = \langle \varphi_x, \varphi_y \rangle = \left\langle \dfrac{y}{1+(xy)^2}, \dfrac{x}{1+(xy)^2} \right\rangle$

$C = \varphi = \tan^{-1}(xy) \Rightarrow xy = \tan C = C_1$



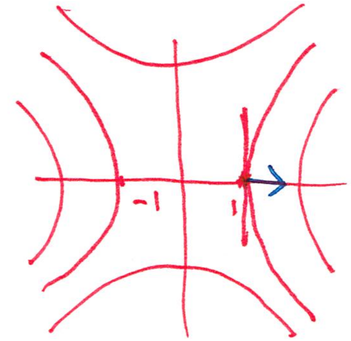
- Equipotential curves and surfaces

$\vec{F} = \nabla \varphi$,

$x^2 - y^2 = c$

$c = 1$: $x^2 = y^2 + 1$

$c = -1$: $y^2 = x^2 + 1$



$\nabla \varphi(1,0) = \langle 1, 0 \rangle$

- equipotential curves : $\varphi = k$ (level curves)
- flow curves or streamlines : $\vec{r}(t) = \langle x(t), y(t) \rangle$ such that $\vec{r}\,'(t) = \vec{F}(x(t), y(t))$

Ex. 5 (Equipotential curves)

The equipotential curves for the potential function $\varphi(x,y) = \frac{1}{2}(x^2 - y^2) = c$ are shown in Fig. 17.15.

(a) Find the gradiet field assoc. with φ and verify that the gradient field is orthogonal to the equipotential curve at (2, 1).

(b) Verify that the vector field $\vec{F} = \nabla \varphi$ is orthogonal to the equipotential curves at all pts (x, y).

(a) $\nabla \varphi = \langle \varphi_x, \varphi_y \rangle = \langle x, -y \rangle$, $\nabla \varphi(2,1) = \langle 2, -1 \rangle$

$\nabla \varphi(1,0) = \langle 1, 0 \rangle$

$\frac{d}{dx}\left(x^2 = y^2 + 1\right)$

$2x = 2y\,y' \Rightarrow y' = \frac{x}{y}$

vector tangent to C $\langle y, x \rangle$

$\nabla \varphi \cdot \langle y, x \rangle = \langle x, -y \rangle \cdot \langle y, x \rangle$

$= xy - yx = 0$

equipotential curve $\perp$ flow curve

Lesson 26

§17.2 Line Integrals

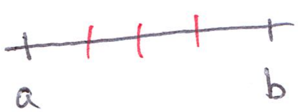
- scalar function $f(x,y,z)$
- vector field $\vec{F}(x,y,z)$

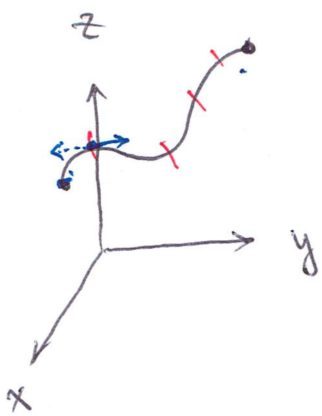
$$\int_C f\,ds \quad \text{and} \quad \int_C \vec{F}\cdot d\vec{r} = \int_C (\vec{F}\cdot\vec{T})\,ds \qquad \vec{T} \text{ — unit tangent}$$

C: a smooth curve parametrized by $\boxed{\vec{r}(t) = \langle x(t), y(t), z(t)\rangle, \ \forall t \in [a,b]}$

f: a function defined on C

$\vec{F}$: a vector field defined on C

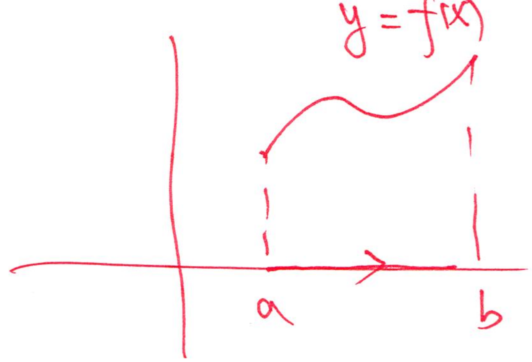




$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b f(x(t), y(t), z(t))\,|\vec{r}\,'(t)|\,dt$$

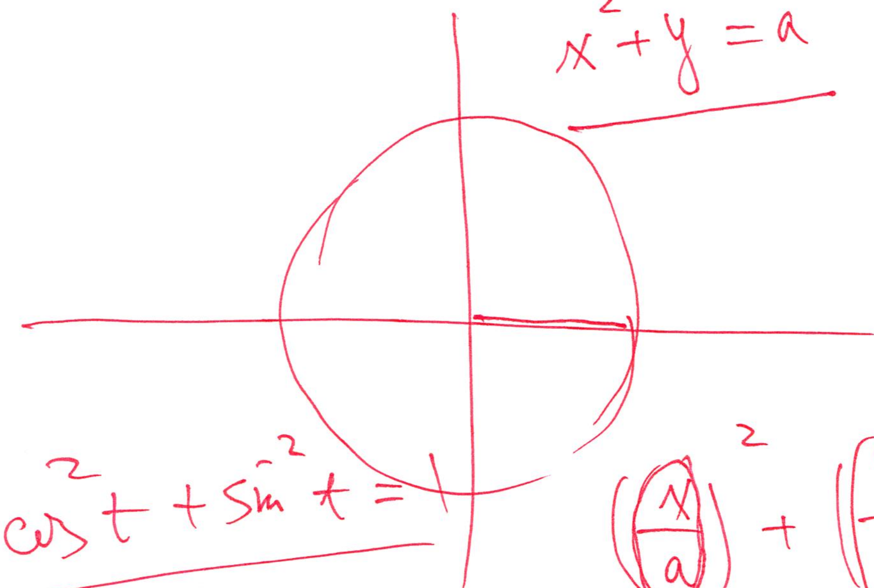
$$L = \int_C 1\,ds = \int_a^b \underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$



$$\vec{r}(t) = \langle t, f(t) \rangle, \quad a \le t \le b$$

$$\vec{r}(x) = \langle x, f(x) \rangle, \quad a \le x \le b$$

$$x^2 + y^2 = a^2$$



$$\vec{r}(t) = \langle \overbrace{a\cos t}^{x(t)}, \overbrace{a\sin t}^{y(t)} \rangle$$

$$0 \le t \le 2\pi$$

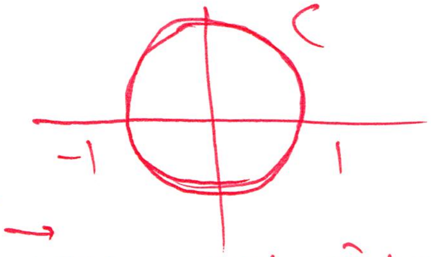
$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\vec{r}(t) = \langle a\cos t, b\sin t \rangle$$

Ex. 1 The temperature of the circular plate $R = \{(x,y) : x^2+y^2 \le 1\}$ is $T(x,y) = 100(x^2+2y^2)$.

Find the average temperature along the edge of the plate.



$$a(T) = \int_C T\,ds \Big/ \int_C ds = \frac{\int_0^{2\pi} 100(\cos^2 t + 2\sin^2 t)\cdot 1 \cdot dt}{2\pi}$$

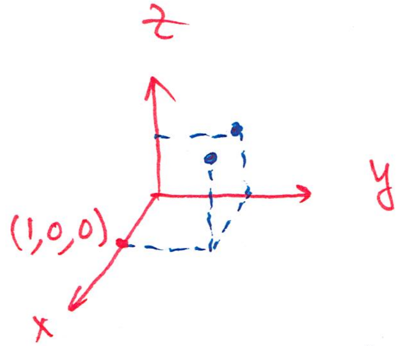
$$= \frac{100}{2\pi}\int_0^{2\pi} (1+\underbrace{\sin^2 t}_{\frac{1-\cos 2t}{2}})\,dt$$

$\vec{r}(t) \underset{@}{=} \langle \cos t, \sin t \rangle$

$0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

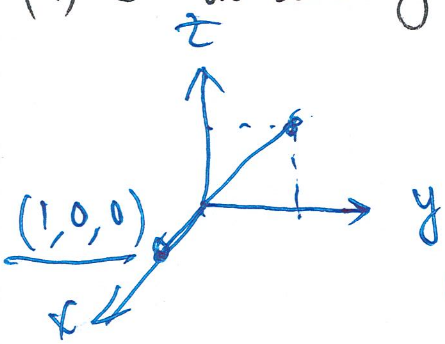
$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$



Ex. 2 Evaluate $\int_C (xy+2z)\,ds$

(a) C : the line segment from P(1,0,0) to Q(0,1,1);

(b) C : the line segment from Q(0,1,1) to P(1,0,0)



$\vec{v} = \langle -1, 1, 1 \rangle$

$\vec{r}(t) = \langle 1,0,0 \rangle + t\langle -1,1,1 \rangle = \langle 1-t, t, t \rangle$

$0 \le t \le 1$ $\quad \vec{r}\,'(t) = \langle -1, 1, 1 \rangle$

$$\int_C (xy+2z)\,ds = \int_0^1 [(1-t)t + 2t]\sqrt{1+1+1}\,dt$$

$$= \sqrt{3}\int_0^1 (t - t^2 + 2t)\,dt$$

$$= \sqrt{3}\int_0^1 (3t - t^2)\,dt = \sqrt{3}\left[\tfrac{3}{2}t^2 - \tfrac{1}{3}t^3\right]_0^1 = \frac{7}{6}\sqrt{3}$$

(b) $\vec{v} = \langle 1,0,0 \rangle - \langle 0,1,1 \rangle = \langle 1,-1,-1 \rangle$

$\vec{r}(t) = \langle 0,1,1 \rangle + t\langle 1,-1,-1 \rangle = \langle t, 1-t, 1-t \rangle, 0 \le t \le$

$\vec{r}\,'(t) = \langle 1,-1,-1 \rangle$

$|\vec{r}\,'(t)| = \sqrt{3}$

$$\int_C (xy+2z)\,ds = \int_0^1 [t(1-t) + 2(1-t)] \cdot \sqrt{3}\,dt$$

$$= \sqrt{3}\int_0^1 [2 - t^2 - t]\,dt = \frac{7}{6}\sqrt{3}$$

Ex. 3 An eagle soars on the ascending spiral path $C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$= \langle 2400 \cos \frac{t}{2}, 2400 \sin \frac{t}{2}, 500t \rangle,$

where x, y, and z are measured in feet and t is measured in minutes. How far does the eagle fly over the time interval $0 \le t \le 10$?

$$\vec{r}\,' = \langle -1200 \sin \frac{t}{2}, 1200 \cos \frac{t}{2}, 500 \rangle$$

$$|\vec{r}\,'| = \sqrt{1200^2 + 500^2}$$

$$L = \int_C ds = \int_0^{10} |\vec{r}\,'(t)| \, dt$$

$$= 10 \sqrt{1200^2 + 500^2}$$