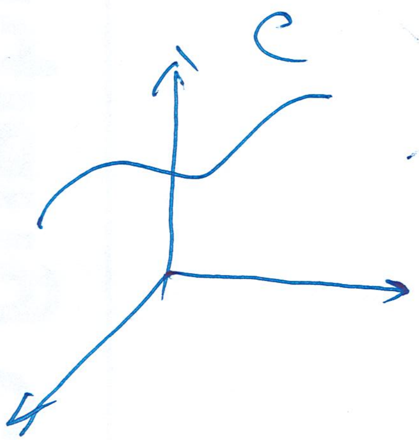
$$\int_C f(x,y,z)\,\underline{ds} = \int_a^b f\big(x(t), y(t), z(t)\big)\,\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

$C: \ \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$?

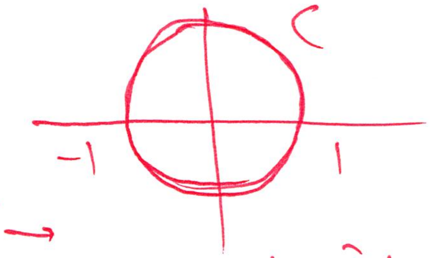
$a \le t \le b$

$t: a \to b$



Ex. 1 The temperature of the circular plate $R = \{(x,y) : x^2+y^2 \le 1\}$ is $T(x,y) = 100(x^2+2y^2)$.
Find the average temperature along the edge of the plate.

$$a(T) = \int_C T\,ds \Big/ \int_C ds = \frac{\int_0^{2\pi} 100(\cos^2 t + 2\sin^2 t)\cdot 1 \cdot dt}{2\pi}$$

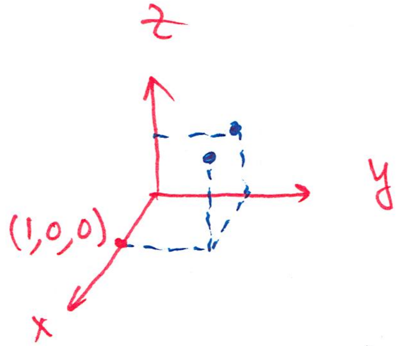


$\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

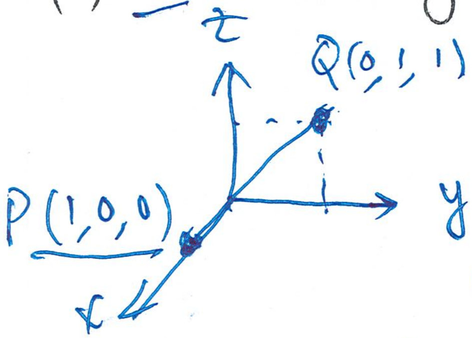
$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$$= \frac{100}{2\pi}\int_0^{2\pi} (1 + \underbrace{\sin^2 t}_{\frac{1-\cos 2t}{2}})\,dt$$



Ex. 2 Evaluate $\int_C (xy + 2z)\,ds$

(a) C: the line segment from $P(1,0,0)$ to $Q(0,1,1)$;



$\vec{v} = \langle -1,1,1 \rangle = \overrightarrow{PQ}$

$\vec{r}(t) = \langle 1,0,0 \rangle + t\langle -1,1,1 \rangle = \langle 1-t, t, t \rangle$

$0 \le t \le 1$

$\vec{r}\,'(t) = \langle -1,1,1 \rangle$

$|\vec{r}\,'(t)| = \sqrt{3}$

$$\int_C (xy+2z)\,ds = \int_0^1 [(1-t)t + 2t]\sqrt{1+1+1}\,dt$$

$$= \sqrt{3}\int_0^1 (t - t^2 + 2t)\,dt$$

$$= \sqrt{3}\int_0^1 (3t - t^2)\,dt = \sqrt{3}\left[\tfrac{3}{2}t^2 - \tfrac{1}{3}t^3\right]_0^1 = \tfrac{7}{6}\sqrt{3}$$

(b) C: the line segment from $Q(0,1,1)$ to $P(1,0,0)$

(b) $\vec{v} = \langle 1,0,0 \rangle - \langle 0,1,1 \rangle = \langle 1,-1,-1 \rangle = \overrightarrow{QP}$

$\vec{r}(t) = \langle 0,1,1 \rangle + t\langle 1,-1,-1 \rangle = \langle t, 1-t, 1-t \rangle$, $0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1,-1,-1 \rangle$

$|\vec{r}\,'(t)| = \sqrt{3}$

$$\int_C (xy+2z)\,ds = \int_0^1 [t(1-t) + 2(1-t)]\cdot\sqrt{3}\,dt$$

$$= \sqrt{3}\int_0^1 [2 - t^2 - t]\,dt = \tfrac{7}{6}\sqrt{3}$$

Lesson 27

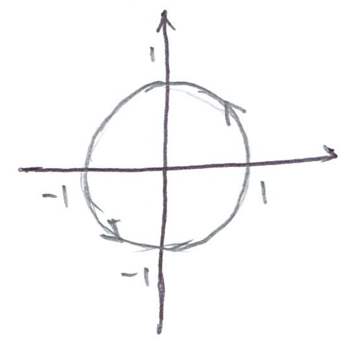
- line integrals of vector fields $\vec{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$

C is an oriented curve

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a parametrization of C with the same orientation

$\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ $\quad a \le t \le b \quad t: a \to b$

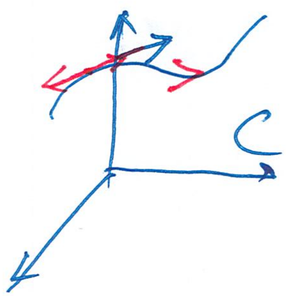
e.g., $C: x^2 + y^2 = 1$ counterclockwise



$$\boxed{\int_C \vec{F} \cdot d\vec{r}} = \int_C \left(\vec{F} \cdot \vec{T}\right) ds = \int_a^b \vec{F}\left(\vec{r}(t)\right) \cdot \frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}} \underbrace{\cancel{|\vec{r}\,'(t)|}\, dt}_{ds}$$

$$\boxed{= \int_a^b \vec{F}\left(\vec{r}(t)\right) \cdot \underbrace{\vec{r}\,'(t)\, dt}_{d\vec{r}}}$$

$\vec{r}(t) = \langle \qquad \rangle$



$$= \int_a^b \langle f(\vec{r}(t)), g(\vec{r}(t)), h(\vec{r}(t)) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

$$= \int_a^b f\left(\vec{r}(t)\right) \underbrace{x'(t)\, dt}_{dx} + g\left(\vec{r}(t)\right) \underbrace{y'\, dt}_{dy} + h\left(\vec{r}(t)\right) \underbrace{z'(t)\, dt}_{dz}$$

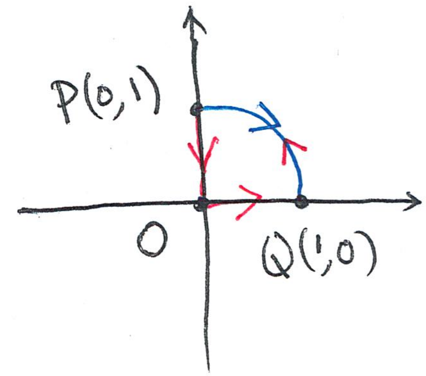
$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C f\,dx + g\,dy + h\,dz}$$

Ex. 4 Evaluate $\int_C (\vec{F}\cdot\vec{T})\,ds$ with $\vec{F} = \langle y-x,\, x\rangle$

(a) C_1 : the quarter circle from $P(0,1)$ to $Q(1,0)$

(b) $-C_1$: the quarter circle from $Q(1,0)$ to $P(0,1)$

(c) C_2 : from $P(0,1)$ to $Q(1,0)$ via two line segments through $O(0,0)$



$C_1 : \vec{r}(t) = \langle \cos t, \sin t\rangle$

$t : \frac{\pi}{2} \to 0$

$\vec{r}\,'(t) = \langle -\sin t, \cos t\rangle$

$-C_1 : \vec{r}(t) = \langle \cos t, \sin t\rangle$

$t : 0 \to \frac{\pi}{2}$

$C_2 : \vec{r}_1(t) = \langle 0, t\rangle$

$t : 1 \to 0$

$\vec{r}_2(t) = \langle t, 0\rangle$

$t : 0 \to 1$

(a) $$\int_C \vec{F}\cdot d\vec{r} = \int_{\frac{\pi}{2}}^{0} \overset{\vec{F}(\vec{r}(t))}{\langle \sin t - \cos t,\, \cos t\rangle} \cdot \overset{\vec{r}\,'(t)}{\langle -\sin t, \cos t\rangle}\, dt$$

$$= \int_{\frac{\pi}{2}}^{0} \left[\left(-\sin^2 t + \cos t \sin t\right) + \cos^2 t\right] dt$$

$$= \int_{\frac{\pi}{2}}^{0} \left(2\cos^2 t - 1 + \cos t \sin t\right) dt$$

(b) $$= \int_C \vec{F}\cdot d\vec{r} = \int_{0}^{\frac{\pi}{2}} \left(2\cos^2 t - 1 + \cos t \sin t\right) dt$$

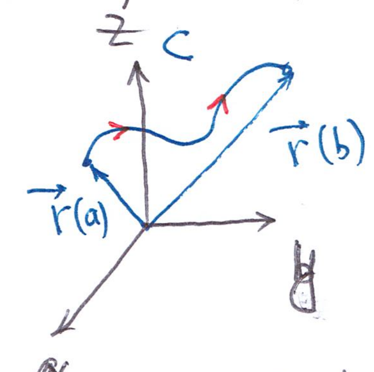
(c) $$\int_C \vec{F}\cdot d\vec{r} = \int_1^0 \overset{\vec{F}\cdot\vec{r}_1}{\langle t, 0\rangle \cdot \langle 0, 1\rangle}\, dt + \int_0^1 \overset{\vec{F}\cdot\vec{r}_2}{\langle -t, t\rangle \cdot \langle 1, 0\rangle}\, dt$$

$$= 0 + \int_0^1 (-t)\, dt = -\frac{1}{2}$$

- work done in a force field $\vec{F}$ along path C

$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \le t \le b$ with the orientation from $\vec{r}(a)$ to $\vec{r}(b)$

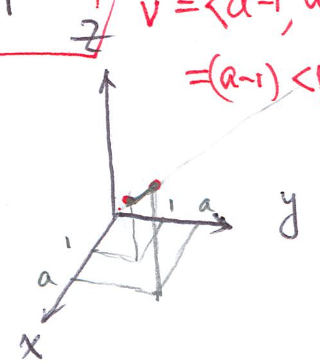
$$W = \int_C \left(\vec{F} \cdot \vec{T} \right) ds$$



Ex. 5 Gravitational and electrical forces between point masses and point charges obey inverse square laws: they act along the line joining the centers and they vary as $\frac{1}{r^2}$, r is the distance between the centers.
The force of attraction (or repulsion) of an inverse square force field is given by $\boxed{\vec{F} = k \frac{\langle x, y, z \rangle}{(x^2+y^2+z^2)^{3/2}} = k \, \vec{r} / |\vec{r}|^3}$, where k is a phy. const.
Find the work done in moving an object along the following paths:
(a) C_1: the line segment from $(1,1,1)$ to (a,a,a), where $a > 1$;
(b) C_2: the extension of C_1 produced by letting $a \to \infty$.

$(1,1,1)$

$\vec{v} = \langle a-1, a-1, a-1 \rangle = (a-1)\langle 1,1,1 \rangle$



$C: \vec{r}(t) = \underline{\langle t, t, t \rangle}$

$t: 1 \to a$

$\vec{r}(t) = \langle 1, 1, 1 \rangle + t(a-1) \underline{\underline{\langle 1,1,1 \rangle}}$

$t: 0 \to 1$

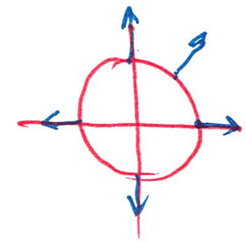
$$\int_{C_1} \vec{F} \cdot \vec{T} \, ds = \int_1^a k \cdot \frac{\langle t, t, t \rangle}{(t^2+t^2+t^2)^{3/2}} \cdot \langle 1,1,1 \rangle \, dt = \int_1^a k \frac{3t}{t^3 \cdot 3^{3/2}} \, dt$$

$$= \frac{k}{\sqrt{3}} \int_1^a t^{-2} \, dt = \frac{k}{\sqrt{3}} \left(-t^{-1} \right) \Big|_1^a = \frac{k}{\sqrt{3}} \left(1 - \frac{1}{a} \right)$$

(b) $$\int_{C_2} \vec{F} \cdot \vec{T} \, ds = \int_1^\infty \text{———} = \lim_{a \to \infty} \frac{k}{\sqrt{3}} \left(1 - \frac{1}{a} \right) = \frac{k}{\sqrt{3}}.$$

• circulation of a vector field

C : a closed smooth oriented curve in $D \subset \mathbb{R}^3$



$$\begin{cases} \vec{r}(t) = \langle \cos t, \sin t \rangle \\ t : 0 \to 2\pi \end{cases}$$

circulation of $\vec{F}$ on C $\quad I = \int_C \vec{F} \cdot \vec{T}\, ds$

Ex. 6 Let C be the unit circle with counterclockwise orientation. Find the circulation on C of the following vector fields:

(a) the radial vector field $\vec{F} = \langle x, y \rangle$;

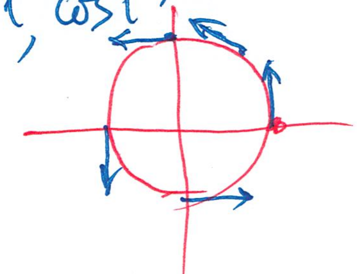
$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_0^{2\pi} \langle \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0$$

(b) the rotation vector field $\vec{F} = \langle -y, x \rangle$.

$$I = \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi$$



$\vec{F}(1,0) = \langle 0, 1 \rangle$

$\vec{F}(0,1) = \langle -1, 0 \rangle$

Ex. 7 Circulation of $\vec{F} = \langle z, x, -y \rangle$ on the tilted ellipse $\vec{r}(t) = \langle \cos t, \sin t, \cos t \rangle$ for $0 \le t \le 2\pi$.