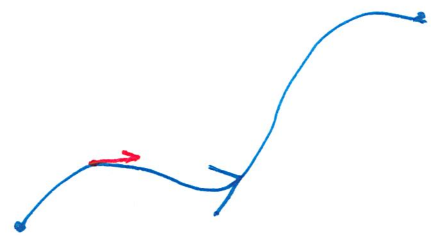
C : parametrization : $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

$$\vec{F} = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$$



oriented curve

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (\vec{F}\cdot\vec{T})\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}\,\cancel{|\vec{r}\,'(t)|}\,dt$$

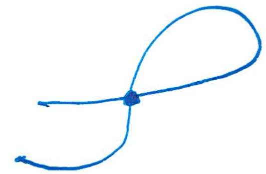
$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\underbrace{\vec{r}\,'(t)\,dt}_{d\vec{r}}$$

$\parallel$

$$\int_C f\,dx + g\,dy + h\,dz$$

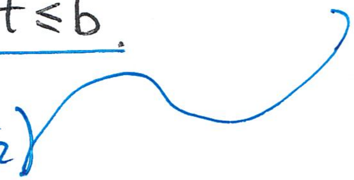
Lesson 28

§17.3 Conservative Vector Fields

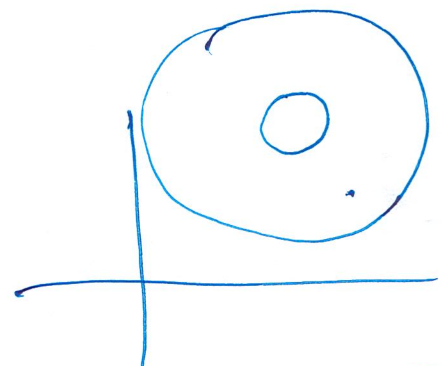


- simple and closed curves

  a curve C in $\mathbb{R}^2$ or $\mathbb{R}^3$ is parametrized by $\vec{r}(t)$, where $a \le t \le b$.

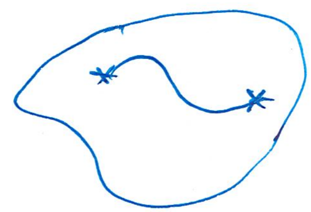


  - C is simple $\iff$ $\forall\, t_1 \neq t_2 \in [a,b]$, $\vec{r}(t_1) \neq \vec{r}(t_2)$
  - C is closed $\iff$ $\vec{r}(a) = \vec{r}(b)$

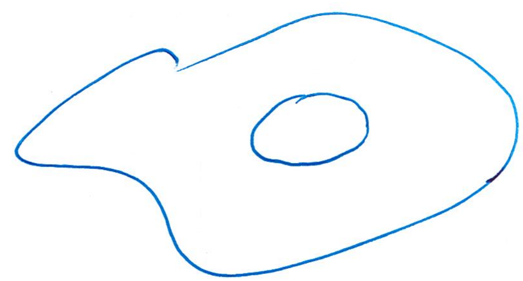


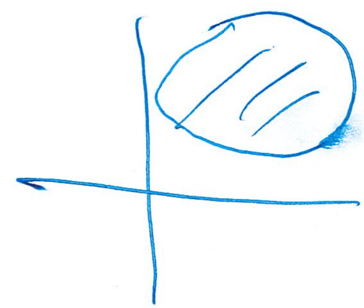
- connected and simple connected regions

  R is an open region in $\mathbb{R}^2$ or $\mathbb{R}^3$



  - R is connected $\iff$
  - R is simple connected $\iff$





- Conservative Vector Field

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

a vector field $\vec{F}$ is conservative $\Longleftrightarrow$ $\vec{F} = \nabla \varphi$

$\vec{F} = \langle f(x,y,z), g, h \rangle$, $\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix}$

$$= \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

Theorem 17.3 (test for conservative field)

Assumptions: D is a simply connected region in R^2 or R^3.

- f, g, h have continuous partial derivatives on D

$\vec{F} = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$ is conservative $\Longleftrightarrow$ $\nabla \times \vec{F} = \vec{0}$

$\vec{F} = \langle f(x,y), g(x,y) \rangle$ is conservative $\Longleftrightarrow$

$$\nabla \times \langle f(x,y), g(x,y), 0 \rangle$$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f(x,y) & g(x,y) & 0 \end{vmatrix} = \left\langle 0, 0, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle = \vec{0}$$

Ex. 1 Determine whether the following vector fields are conservative. If so, find potential functions.

(a) $\vec{F} = \langle \underbrace{e^x \cos y}_{f}, \underbrace{-e^x \sin y}_{g} \rangle$;

$$\nabla \times \vec{F} = \langle 0, 0, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \rangle$$

$$= \langle 0, 0, -e^x \sin y - (-e^x \sin y) \rangle$$

$$= \langle 0, 0, 0 \rangle$$

$$\vec{F} = \langle e^x \cos y, -e^x \sin y \rangle = \nabla \varphi$$

$$= \langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \rangle \quad \boxed{\varphi = ?}$$

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = e^x \cos y \\ \dfrac{\partial \varphi}{\partial y} = -e^x \sin y \end{cases}$$

$$\varphi(x, y) = \int e^x \cos y \, dx = e^x \cos y + C(y)$$

$$\frac{\partial \varphi}{\partial y} = -e^x \sin y + C'(y)$$

$$\Rightarrow C'(y) = 0 \Rightarrow C(y) = C$$

$$\varphi(x, y) = e^x \cos y + C$$

(b) $\vec{F} = \langle \underbrace{2xy - z^2}_{f}, \underbrace{x^2 + 2z}_{g}, \underbrace{2y - 2xz}_{h} \rangle$

$$\nabla \times \vec{F} = \langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \rangle$$

$$= \langle 2 - 2, -2z + 2z, 2x - 2x \rangle = \langle 0, 0, 0 \rangle$$

$$\vec{F} = \nabla \varphi = \langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \rangle \quad \boxed{\varphi = ?}$$

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = 2xy - z^2 \\ \dfrac{\partial \varphi}{\partial y} = x^2 + 2z \\ \dfrac{\partial \varphi}{\partial z} = 2y - 2xz \end{cases}$$

$$\int \left(\frac{\partial \varphi}{\partial x} = 2xy - z^2 \right) dx \Longrightarrow \varphi(x, y, z) = x^2 y - z^2 x + C(y, z)$$

$C(y, z) = ?$

$$\frac{\partial \varphi}{\partial y} = x^2 + 2z = x^2 + \frac{\partial C}{\partial y} \Longrightarrow \frac{\partial C}{\partial y} = 2z$$

$$\Longrightarrow C(y, z) = 2yz + D(z)$$

$$\boxed{\varphi(x, y, z) = x^2 y - z^2 x + 2yz + D(z)}$$

$$2y - 2xz = \frac{\partial \varphi}{\partial z} = -2zx + 2y + D'(z)$$

$$\Longrightarrow D'(z) = 0 \Longrightarrow D(z) = \text{Const.}$$

- Fundamental theorem for line integrals

Fundamental theorem of calculus $\quad \int_a^b F'(x)\,dx = F(b) - F(a)$

Fundamental theorem for line integrals $\quad C$ is a piecewise smooth oriented curve from A to B

Thrm 17.4

$$\boxed{\int_C \nabla\varphi \cdot d\vec{r} = \varphi(B) - \varphi(A)}$$

Proof Let $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $a \le t \le b$ be a parametrization of C, $\vec{r}(a) = A$ and $\vec{r}(b) = B$

$$\int_C \nabla\varphi \cdot d\vec{r} = \int_a^b \nabla\varphi\left(\vec{r}(t)\right) \cdot \vec{r}\,'(t)\,dt$$

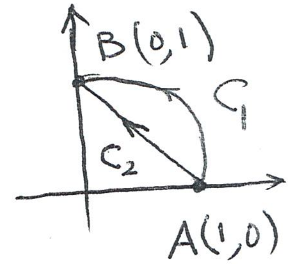
$$= \int_a^b \frac{d}{dt}\varphi\left(\vec{r}(t)\right)\,dt = \varphi\left(\vec{r}(t)\right)\Big|_{t=a}^{t=b}$$

$$= \varphi(B) - \varphi(A)$$

Ex. 3 potential function $\varphi(x,y) = \frac{1}{2}(x^2 - y^2)$, its gradient field $\vec{F} = \langle x, -y \rangle = \nabla\varphi$

(a) C_1: the quarter circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \frac{\pi}{2}$ from $A(1,0)$ to $B(0,1)$

(b) C_2: the line $\vec{r}(t) = \langle 1-t, t \rangle$ for $0 \le t \le 1$ also from A to B.



Verify $\displaystyle\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = \varphi(B) - \varphi(A)$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \langle \cos t, -\sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\frac{\pi}{2}} -\cos t \sin t - \sin t \cos t \, dt$$

$$= -2\int_0^{\frac{\pi}{2}} \cos t \sin t \, dt = -\sin^2 t \Big|_0^{\frac{\pi}{2}} = -1$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 \langle 1-t, -t \rangle \cdot \langle -1, 1 \rangle \, dt$$

$$= \int_0^1 \left[(t-1) - t\right] dt$$

$$= -\int_0^1 dt = -1$$

$$\varphi(0,1) - \varphi(1,0) = \frac{1}{2}(-1) - \frac{1}{2}(1) = -1$$

Ex. 4 $\vec{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle = \nabla\left(x^2y - xz^2 + 2yz\right)$

Evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ where C is a simple curve from $A(-3,-2,-1)$ to $B(1,2,3)$.

$$\int_C \vec{F}\cdot d\vec{r} = \varphi(B) - \varphi(A) = \varphi(1,2,3) - \varphi(-3,-2,-1)$$

$$= (2 - 9 + 12) - (-2\cdot 9 + 3 + 4)$$

$$= 5 + 11 = 16$$

- independence of path

$\int_C \vec{F}\cdot d\vec{r}$ is independent of path $\iff$ $\forall\ C_1, C_2$ with the same starting pt A and terminal pt B

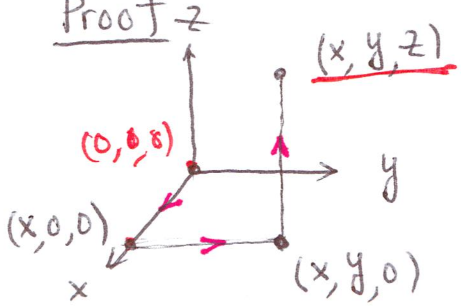
$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$

Thrm 17.5 $\vec{F}$ ~~is a continuous vector~~ field

$\int_C \vec{F}\cdot d\vec{r}$ is indep. of path $\Longrightarrow$ $\vec{F}$ is conservative, i.e., $\cancel{\vec{F}=\nabla\varphi}$ $\exists\, \varphi(x,y,z)$ s.t. $\vec{F}=\nabla\varphi$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel\ ?$

Proof



$\int_C \nabla\varphi\cdot d\vec{r}$ path indep.

$= \varphi(B) - \varphi(A)$

$$\varphi(x,y,z) = \int_C \vec{F}\cdot d\vec{r} = \int_C f\,dx + g\,dy + h\,dz$$

$$= \int_0^x f(x,0,0)\,dx + \int_0^y g(x,y,0)\,dy + \int_0^z h(x,y,z)\,dz$$

Summary

conservative

$\int_C \vec{F}\cdot d\vec{r}$ is path indep $\Longleftrightarrow \boxed{\vec{F} = \nabla\varphi} \Longleftrightarrow \oint_C \vec{F}\cdot d\vec{r} = 0 \Leftrightarrow \nabla\times\vec{F} = \vec{0}$

$$\Downarrow$$

$$\vec{F} = \nabla\varphi$$