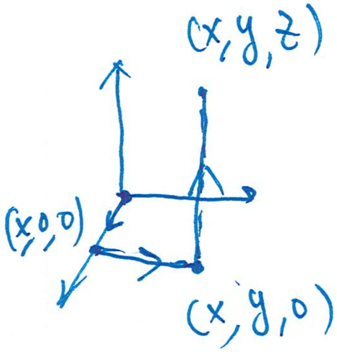
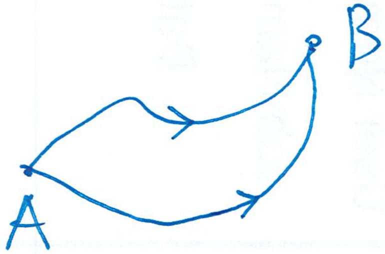
conservative $\qquad$ potential $\qquad\qquad\qquad\qquad$ irrotational

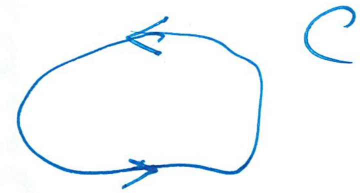
$$\vec{F} = \nabla \varphi \quad \Longrightarrow \quad \nabla \times \vec{F} = \vec{0}$$



$$\Uparrow \qquad \underline{\varphi(x,y,z)} = \varphi(0,0,0) + \int_0^x f(x,0,0)\,dx + \int_0^y g(x,y,0)\,dy + \int_0^z h(x,y,z)\,dz \qquad \Downarrow$$

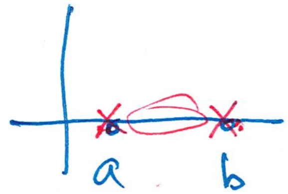
$\int_C \vec{F}\cdot d\vec{r}$ is path indept. $\quad \Longleftarrow \quad \oint_C \vec{F}\cdot d\vec{r} = 0 \quad \forall$ simple, closed oriented curve





§17.4 Green's Theorem

- fundamental thrms



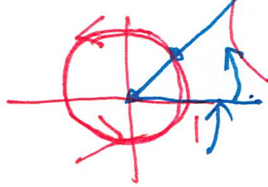
$$\int_a^b f'(x)\,dx = f(b) - f(a) \quad \text{and} \quad \int_C \nabla\varphi \cdot d\vec{r} = \varphi(B) - \varphi(A)$$

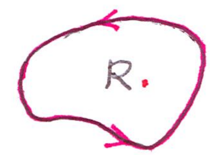
- circulation form of Green's Thrm

$\vec{F} = \langle f(x,y), g(x,y)\rangle$, $\quad \vec{\nabla\times F} = \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$

R — a simply connected region, $C = \partial R$ counterclockwise

$$\iint_R \vec{\nabla\times F}\, dA = \oint_C \vec{F}\cdot d\vec{r}$$





$\partial R = C$

$\vec{r}(t) = \langle \overset{x}{\underset{\parallel}{}}\cos t, \overset{y}{\underset{\parallel}{}}\sin t\rangle \quad t: 0 \to 2\pi$

Ex. 1 $\vec{F} = \langle -y, x\rangle$, $R = \{(x,y) : x^2+y^2 \le 1\}$. Verify Green's Thrm.

$$\vec{\nabla\times F} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 1+1 = 2$$

$$\iint_R \vec{\nabla\times F}\,dA = \iint_R 2\,dA = 2\,A(R) = 2\pi$$

$$= \int_0^1 dr \int_0^{2\pi} d\theta\; 2\,\underline{r} = 2\pi \int_0^1 2r\,dr = 2\pi\, r^2\Big|_0^1 = 2\pi$$

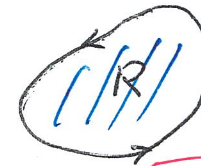
$$\oint_C \langle -y, x\rangle \cdot d\vec{r} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = 2\pi$$

• calculating area by Green's Thrm

$$A(R) = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C \left(-y\,dx + x\,dy\right)$$



$\vec{F} = \langle f, g \rangle$

$$A(R) = \iint_R dA = \iint_R \nabla\times\vec{F}\, dA = \oint_C \vec{F}\cdot d\vec{r}$$

If $\nabla\times\vec{F} = 1$

$\nabla\times\vec{F} = \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$

$$= \oint_C \cancel{0}\,dx + x\,dy$$

(1) $g = x,\ f = 0$

$\vec{F} = \langle 0, x \rangle$

(2) $g = 0,\ f = -y$

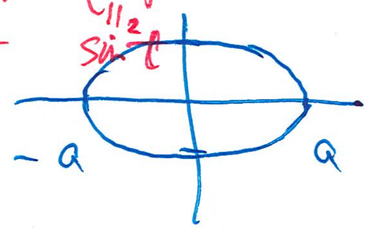
$\vec{F} = \langle -y, 0 \rangle$

$\nabla\times\vec{F} = 1$

$\cos 2t = \cos^2 t - \sin^2 t$

Ex. 2 Find the area of the ellipse $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = 1$.

$\dfrac{x}{a} = \cos t, \quad \dfrac{y}{b} = \sin t$



$$A(R) = \oint_C x\,dy$$

$$= \int_0^{2\pi} a\cos t\, d(b\sin t)$$

$$= \int_0^{2\pi} ba\cos^2 t\, dt$$

$$= ab\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$\vec{r} = \langle a\cos t, b\sin t \rangle$

$0 \le t \le 2\pi$

$$\boxed{\begin{aligned} &\int_C f\,dx + g\,dy + h\,dz \qquad \vec{r} = \langle x(t), y(t), z(t)\rangle,\ t: a\to b \\ &= \int_a^b f(x(t),\cdots)\,dx(t) + \cdots \end{aligned}}$$

- flux of a vector field $\vec{F} = \langle f(x,y), g(x,y) \rangle$ across C

$C: \vec{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$, a smooth oriented curve and simple

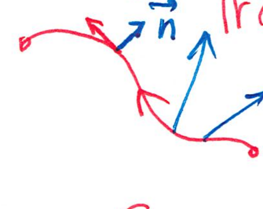
the flux of $\vec{F}$ across C $\quad \displaystyle\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\langle y', -x'\rangle\,dt$ $\quad \vec{n}$ — outward normal $= \vec{T}\times\vec{k}$

$\vec{k} = \langle 0, 0, 1\rangle$

$\vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$$\begin{vmatrix} i & j & k \\ \frac{x'}{|\vec{r}'(t)|} & \frac{y'}{|\vec{r}'(t)|} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \int_C f\,dy - g\,dx$$



Ex. 8 Find outward flux across the unit circle with counterclockwise orientation for

(a) $\vec{F} = \langle x, y\rangle$; (b) $\vec{F} = \langle -y, x\rangle$

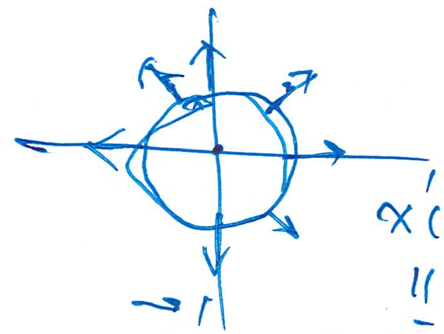
$\vec{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$

$\vec{n} = \left\langle \dfrac{y'}{|\vec{r}'(t)|}, -\dfrac{x'}{|\vec{r}'(t)|}, 0 \right\rangle$

(a) $\displaystyle\int_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} x(t)\,dy(t) - y(t)\,dx(t)$

$$= \int_0^{2\pi} \cos t\,d\sin t - \sin t\,d\cos t$$

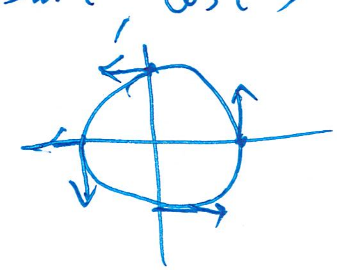
$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$$



$\vec{r}'(t) = \langle \underset{x'(t)}{-\sin t}, \underset{y'(t)}{\cos t}\rangle$

(b) $\displaystyle\int_C \langle -y, x\rangle\cdot\vec{n}\,ds = \int_0^{2\pi} \langle -\sin t, \cos t\rangle\cdot\langle \cos t, \sin t\rangle\,dt$

$$= \int_0^{2\pi} 0\,dt = 0$$



• flux form of Green's Thrm $\vec{F} = \langle f, g \rangle$, $\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$.

$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle x', y' \rangle}{|\vec{r}'(t)|}$

$\vec{n}$ unit | outward normal

R — a simply connected region, $C = \partial R$ counterclockwise

$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \cdot \langle f, g \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$

$$\iint_R \nabla \cdot \vec{F} \, dA = \boxed{\oint_C \vec{F} \cdot \vec{n} \, ds}$$

$$\parallel$$

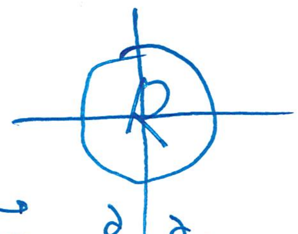
$$\boxed{\oint_C f\,dy - g\,dx} = \boxed{\oint_C \langle f, g \rangle \cdot \frac{\langle y', -x' \rangle}{|\vec{r}'(t)|}} \, |\vec{r}'(t)| \, dt$$

$C = \partial R$

$\vec{n} = ? = \vec{T} \times \vec{k} = \begin{vmatrix} i & j & k \\ x' & y' & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \langle y'(t), -x'(t) \rangle / |\vec{r}'(t)|$

Ex.3 Use Green's Thrm to compute the outward flux of the radial field $\vec{F} = \langle x, y \rangle$ across the unit circle $C = \{(x,y) : x^2 + y^2 = 1\}$.



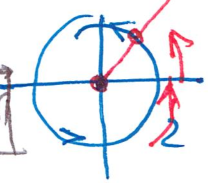
$$\oint_C \vec{F} \cdot \vec{n} \, ds = \iint_R \nabla \cdot \vec{F} \, dA$$

$\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \cdot \langle x, y \rangle$

$$= \iint_R 2 \, dA = 2 \cdot \pi \cdot 1^2 = 2\pi$$

$= 1 + 1 = 2$

Ex.4 C is the boundary of the disk $R = \{(x,y) : x^2 + y^2 \le 4\}$ oriented counterclockwise



$$\cancel{\oint_C \langle -(4y^3 + \cos x^2), 4x^3 + \sin y^2 \rangle \cdot d\vec{r}}$$

$$\oint_C (4x^3 + \sin y^2)\,dy - (4y^3 + \cos x^2)\,dx$$

$d\vec{r} = \langle dx, dy \rangle$

$d\vec{n} = \langle dy, -dx \rangle$

$$= \oint_C \langle 4x^3 + \sin y^2, 4y^3 + \cos x^2 \rangle \cdot \vec{n} \, ds$$

$$= \iint_R \nabla \cdot \langle 4x^3 + \sin y^2, 4y^3 + \cos x^2 \rangle \, dA$$

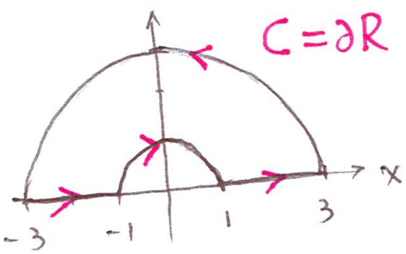
$$= \iint_R (12x^2 + 12y^2) \, dA$$

$$= \int_0^2 dr \int_0^{2\pi} d\theta \, \underbrace{12r^2 \cdot r}_{12r^3}$$

$$2\pi \cdot 3r^4 \Big|_0^2 = 6\pi 2^4$$

- more general regions

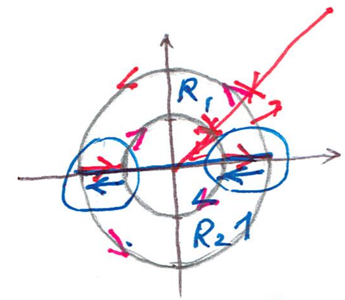
Ex. 5 $\vec{F} = \langle y^2, x^2 \rangle$, $R = \{(x,y) : 1 \le x^2+y^2 \le 9,\ y \ge 0\}$



Find the circulation of $\vec{F}$ on C.

$$\oint_{C=\partial R} (\vec{F} \cdot \vec{n})\, ds$$

Ex. 6 $\vec{F} = \langle xy^2, x^2 y \rangle$, $R = \{(x,y) : 1 \le x^2+y^2 \le 4\}$



Find the outward flux of $\vec{F}$ across ∂R.

$$\boxed{\iint_R \nabla \cdot \vec{F}\, dA} = \iint_{R_1} + \iint_{R_2} \nabla \cdot \vec{F}\, dA$$

$$= \oint_{\partial R_1} + \int_{\partial R_2} \vec{F} \cdot \vec{n}\, dA$$

$$\iint_R \nabla \cdot \langle xy^2, x^2 y \rangle\, dA = \iint_{R_2} (y^2 + x^2)\, dA$$

$$= \int_1^2 dr \int_0^{2\pi} d\theta\, r^2 \cdot r = 2\pi \int_1^2 r^3\, dr$$

$$= 2\pi \cdot \frac{1}{4} r^4 \Big|_1^2$$