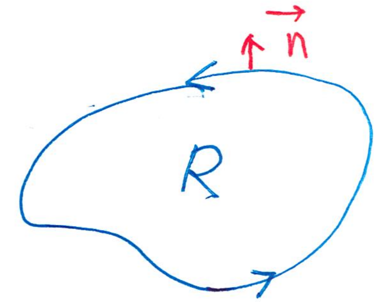
Green's Theorem



$$\vec{F} = \langle f, g \rangle$$

$$\iint_R \nabla \times \vec{F} \, dA = \oint_{\partial R} \left(\vec{F} \cdot \vec{T} \right) ds$$

$$\nabla \times \vec{F} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$$

$$\iint_R \nabla \cdot \vec{F} \, dA = \oint_{\partial R} \left(\vec{F} \cdot \vec{n} \right) ds$$

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$$\partial R : \ \vec{r}(t) = \langle x(t), y(t) \rangle$$

$$t : a \to b$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle x'(t), y'(t) \rangle}{|\vec{r}\,'(t)|}$$

$$\vec{n} = \vec{T} \cdot \vec{k} = \frac{1}{|\vec{r}\,'(t)|} \langle x'(t), y'(t), 0 \rangle \cdot \langle 0, 0, 1 \rangle$$

$$= \frac{1}{|\vec{r}\,'(t)|} \langle y'(t), -x'(t) \rangle$$

Lesson 30

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

§17.5 Divergence and Curl

- divergence of a vector field $\vec{F} = \langle f, g, h \rangle$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Ex.1 Computing $\nabla \cdot \vec{F}$

(a) $\vec{F} = \langle x, y, z \rangle$ (a radial field) $\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$

(b) $\vec{F} = \langle -y, x-z, y \rangle$ (a rotation field) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x-z) + \frac{\partial}{\partial z}(y) = 0 + 0 + 0 = 0$

(c) $\vec{F} = \langle -y, x, z \rangle$ (a spiral field) $\nabla \cdot \vec{F} = 0 + 0 + 1 = 1$

Ex.2 $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$, $p = 1$ $\quad \vec{F} = \frac{\langle x, y, z \rangle}{|\vec{r}|} = \left\langle \frac{x}{(x^2+y^2+z^2)^{\frac{1}{2}}}, \frac{y}{(\quad)^{1/2}}, \frac{z}{(\quad)^{1/2}} \right\rangle$

$$\frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-1/2}\right]$$

$$= (\quad)^{-\frac{1}{2}} + x\left(-\tfrac{1}{2}\right)(\quad)^{-\frac{1}{2}-1} \cdot 2x$$

$$= (x^2+y^2+z^2)^{-\frac{3}{2}}\left[(\cancel{x^2}+y^2+z^2) - \cancel{x^2}\right]$$

$$\nabla \cdot \vec{F} = \frac{1}{|\vec{r}|^3}\left[(y^2+z^2) + (x^2+z^2) + (x^2+y^2)\right]$$

$$= 2\frac{|\vec{r}|^2}{|\vec{r}|^3} = \frac{2}{|\vec{r}|}$$

Ex. 3 $\vec{F} = \langle f, g \rangle = \langle x^2, y \rangle$ and $C: x^2 + y^2 = 2^2$.

(a) Without computing it, whether $\nabla \cdot \vec{F}(1,1)$ is positive or negative. why?

(b) Confirm your conjecture in part (a) by computing $\nabla \cdot \vec{F}(1,1)$.

(c) Based on (b), over what regions within the circle $\nabla \cdot \vec{F} > 0$ and $\nabla \cdot \vec{F} < 0$?

(d) By inspection of the figure, on what part of the circle is the flux across the boundary outward? Is the net flux out of circle positive or negative?

(b) $\nabla \cdot \vec{F} = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 2x + 1$ $\qquad \nabla \cdot \vec{F}(1,1) = (2x+1)\Big|_{(1,1)} = 3 > 0$

(c) $\nabla \cdot \vec{F} = 2x + 1 > 0 \Longrightarrow x > -\frac{1}{2}$ positive

$\nabla \cdot \vec{F} = 2x + 1 < 0 \Longrightarrow x < -\frac{1}{2}$ neg.

$$\oint_C (\vec{F} \cdot \vec{n})\, ds = \iint_R (\nabla \cdot \vec{F})\, dA = \iint_R (2x+1)\, dA > 0$$

- <u>curl of a vector field</u> $\vec{F} = \langle f, g, h\rangle$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

- <u>general rotation vector field</u>

$\vec{F} = \vec{a}\times\vec{r}$ with $\vec{r} = \langle x, y, z\rangle$, $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\nabla\times\vec{F} = ?$ $\nabla\cdot\vec{F} = ?$

$|\nabla\times\vec{F}| = ?$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \langle a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x\rangle$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix} = \langle a_1 + a_1, a_2 + a_2, a_3 + a_3\rangle = 2\langle a_1, a_2, a_3\rangle$$

$\vec{a}$ – angular velocity field

$$|\nabla\times\vec{F}| = |2\vec{a}| = 2|\vec{a}| \qquad \omega = |\vec{a}| = \frac{1}{2}|\nabla\times\vec{F}|$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x) = 0$$

- properties

$$\nabla \cdot_{\times} (\vec{F} + \vec{G}) = \nabla \cdot_{\times} \vec{F} + \nabla \cdot_{\times} \vec{G}$$

$$(fg)' = f'g + fg'$$

$$\nabla \cdot_{\times} (c\vec{F}) = c(\nabla \cdot_{\times} \vec{F}), \quad \nabla \cdot (u\vec{F}) = \nabla u \cdot \vec{F} + u(\nabla \cdot \vec{F})$$

$$\| $$

$$\nabla u \cdot \vec{F} + u(\nabla \cdot \vec{F})$$

$$\boxed{\nabla \times (\nabla \varphi) = \vec{0}} \quad \text{and} \quad \boxed{\nabla \cdot (\nabla \times \vec{F}) = 0}$$

Ex. 5 $\vec{r} = \langle x, y, z \rangle$, $\varphi = \frac{1}{|\vec{r}|} = (x^2+y^2+z^2)^{-\frac{1}{2}}$.

(a) compute $\vec{F} = \nabla\left(\frac{1}{|\vec{r}|}\right)$;

$$\frac{\partial \varphi}{\partial x} = \left(-\frac{1}{2}\right)(\quad)^{-\frac{1}{2}-1} \cdot 2x$$

$$= \frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{x}{|\vec{r}|^3} \qquad |\vec{r}| = (x^2+y^2+z^2)^{\frac{1}{2}}$$

$$\vec{F} = \nabla \varphi = \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right\rangle$$

$$= \frac{1}{|\vec{r}|^3} \langle x, y, z \rangle = \frac{\vec{r}}{|\vec{r}|^3}$$

(b) compute $\nabla \cdot \vec{F}$

y

$$\cancel{\nabla \cdot \vec{F}} \quad \frac{\partial}{\partial x}\left(\frac{x}{|\vec{r}|^3}\right) = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$

$$= (x^2+y^2+z^2)^{-\frac{3}{2}} + x\left(-\frac{3}{2}\right)(\quad)^{-\frac{5}{2}} \cdot 2x$$

y

$$= (x^2+y^2+z^2)^{-\frac{5}{2}}\left[(x^2+y^2+z^2) - 3x^2\right]$$

$$= \frac{1}{|\vec{r}|^5}(y^2+z^2-2x^2)$$

$$\nabla \cdot \vec{F} = \frac{1}{|\vec{r}|^5}\left[(y^2+z^2-2x^2) + (x^2+z^2-2y^2) + (x^2+y^2-2z^2)\right]$$

$$= 0$$