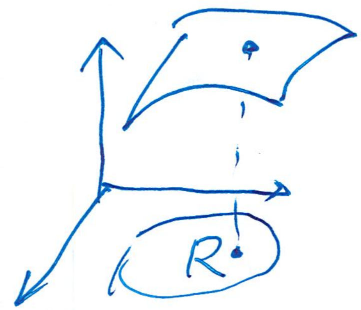
Lesson 31

§17.6 Surface Integrals (parametrized surface, scalar function, vector field)

- representations of surfaces $\{(x, y, f(x,y)) \mid (x,y) \in R\}$



$z = f(x,y)$

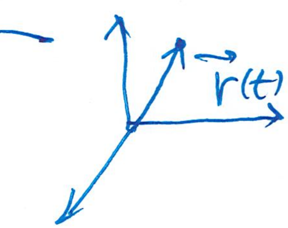
$(x, y, f(x,y))$

- graph $z = f(x,y)$

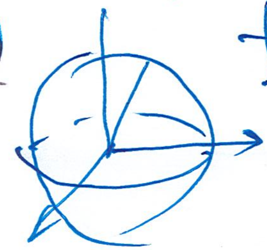
  $(x,y) \in R$

$\boxed{\vec{r}(x,y) = \langle x, y, f(x,y)\rangle, (x,y) \in R}$

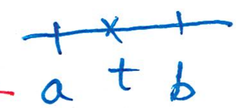
$\vec{r}(t)$



- level surfaces



$f(x,y,z) = x^2 + y^2 + z^2 = 1$

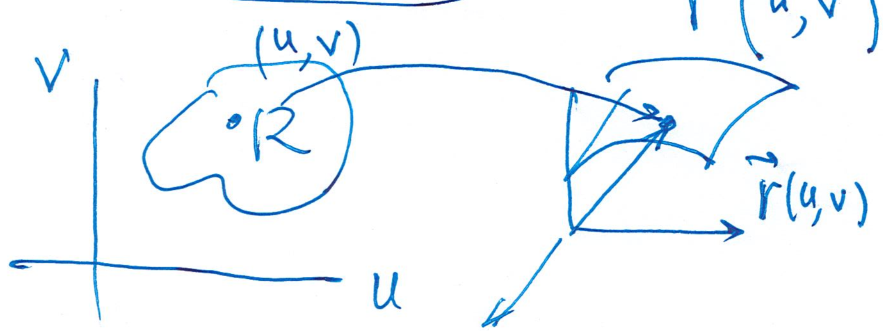


$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle, a \le t \le b$

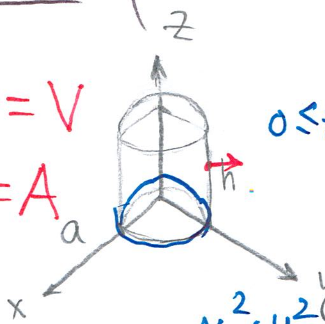
- parametrized surfaces

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

$(u,v) \in R$



cylinder (radius a and height h)



$\pi r^2 h = V$

$2\pi r h = A$

$0 \le z \le h$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \xRightarrow{r=a} \begin{cases} x = a\cos\theta \\ y = a\sin\theta \\ z = z \end{cases} \quad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le z \le h \end{matrix}$$

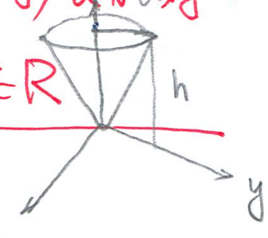
$$\vec{r}(u,v) = \langle a\cos u,\ a\sin u,\ v\rangle,\ \begin{cases} 0 \le u \le 2\pi \\ 0 \le v \le h \end{cases} \quad R$$

$$x^2+y^2=a^2 \Rightarrow r^2=a^2 \Rightarrow \boxed{r=a} \qquad u=\theta,\ v=z$$

cone (height h and radius a)

$\vec{r}(x,y) = \langle x, y, \frac{h}{a}\sqrt{x^2+y^2}\rangle$

$(x,y)\in R$



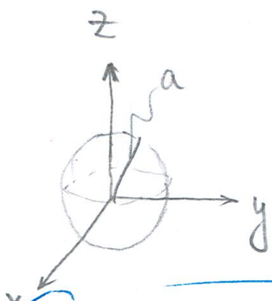
$$z = \frac{h}{a}\sqrt{x^2+y^2} \rightarrow z = h = \underline{\frac{h}{a}\sqrt{x^2+y^2}} \Longrightarrow x^2+y^2=a^2$$

$$\boxed{z = \frac{h}{a} r} \Rightarrow r = \frac{a}{h} z \quad \begin{cases} x = \frac{a}{h} z\cos\theta \\ y = \frac{a}{h} z\sin\theta \\ z = z \end{cases} \quad \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le z \le h \end{cases} \quad \underline{R}$$

$$\vec{r}(u,v) = \langle \frac{a}{h} v\cos u,\ \frac{a}{h} v\sin u,\ v\rangle \quad R: \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le z \le h \end{matrix}$$

sphere (radius a)

$\boxed{\rho = a}$



$$\begin{cases} x = \rho\,\overset{\sin\varphi}{\cancel{\cos\varphi}}\,\cancel{\sin\theta}\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases} \xRightarrow{\rho=a} \begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases} \quad R: \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{matrix}$$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

$x^2+y^2+z^2=a^2$

$$\vec{r}(u,v) = \langle a\sin u\cos v,\ a\sin u\,\overset{\sin}{\cancel{\cos}}\,v,\ a\cos u\rangle$$

$$R: \begin{matrix} 0 \le u \le \pi \\ 0 \le v \le 2\pi \end{matrix}$$

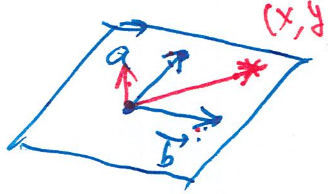
Ex. 1 Find parametric descriptions for the following surfaces

(a) the plane $3x - 2y + z = 2$;

$z = 2 - 3x + 2y$

$\langle x - x_0, y - y_0, z - z_0 \rangle$

$\vec{r}(x,y) = \langle x, y, 2 - 3x + 2y \rangle = u\vec{a} + v\vec{b} = \vec{r}(u,v)$



(b) the paraboloid $z = x^2 + y^2$, for $\boxed{0 \le z \le 9}$

$z = r^2 \Rightarrow r = \sqrt{z}$

$\vec{r}(x,y) = \langle x, y, x^2 + y^2 \rangle$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \xrightarrow{r = \sqrt{z}} \begin{cases} x = \sqrt{z}\cos\theta \\ y = \sqrt{z}\sin\theta \\ z = z \end{cases}$

$\boxed{\vec{r}(u,v) = \langle \sqrt{v}\cos u, \sqrt{v}\sin u, v \rangle, \quad 0 \le u \le 2\pi, \ 0 \le z \le 9}$

- surface integrals of scalar-valued functions

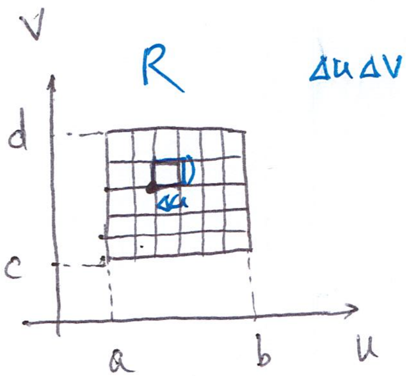
$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$(u,v) \in R$

$\iint_S f(x,y,z)\,dS =$

$\frac{\partial \vec{r}}{\partial u} \cdot \Delta u \qquad \frac{\partial \vec{r}}{\partial v} \Delta v$

$|\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$



$\Delta u \Delta v$

$\iint_S f(x,y,z)\,dS$

$= \iint_R f(\vec{r}(u,v)) \underbrace{|\vec{r}_u \times \vec{r}_v|\,du\,dv}_{dS}$

$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t)) \underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$

Ex. 2 Find the surface area of the following surfaces $\quad V=\frac{4}{3}\pi a^3, \quad S=4\pi a^2$

(a) A cylinder with radius $a>0$ and height h (excluding the circular ends);

$$\begin{array}{cccc} & \vec{i} & \vec{j} & \vec{k} \\ \vec{r}_u = & \langle -a\sin u, & a\cos u, & 0\rangle \\ \vec{r}_v = & \langle 0, & 0, & 1\rangle \end{array}$$

$$\vec{r}_u \times \vec{r}_v = \langle a\cos u,\ a\sin u,\ 0\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = a$$

$$A(S) = \iint_S 1\, dS = \iint_R 1\, \underbrace{a\, du\, dv}_{dS}$$

$$= a\iint_R du\, dv = a\int_0^{2\pi} du \int_0^h dv$$

$$= a \cdot 2\pi h$$

(b) a sphere of a radius a.

$$\cancel{\vec{r}_u = \langle a\cos u\cos v,\ a\cos u\cos v,\ -a\sin u\rangle}$$

$$\cancel{\vec{r}_v = \langle -a\sin u\sin v,\ a\sin u\cos v,\ 0\rangle}$$

$$\vec{r}(u,v) = \langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle$$

$$\cancel{\vec{r}_u \times \vec{r}_v = \langle}$$

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \langle a\cos u\cos v,\ a\cos u\sin v,\ -a\sin u\rangle$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \langle -a\sin u\sin v,\ a\sin u\cos v,\ 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,\ a^2\cos u\sin u\rangle$$

$$= a\sin u\, \vec{r}(u,v)$$

$$|\vec{r}_u \times \vec{r}_v| = a\sin u\, |\vec{r}(u,v)| = a^2\sin u$$

$$A(S) = \iint_S dS = \int_0^{2\pi}\int_0^{\pi} a^2\sin u\, du\, dv$$

$$= 2\pi a^2 \left[-\cos u\right]_0^{\pi} = 4\pi a^2$$