Section 17.6 (Part 2) (Katy Yochman on Nov. 11)

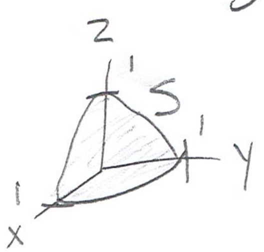
Surface S can be parameterized by $\vec{r}(u,v)$, $(u,v) \in R$

Then surface **area** of S is $\iint_S dS = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA$

The surface integral of function f over S is

$$\iint_S f(x,y,z)\, dS = \iint_R f(\vec{r}(u,v)) \underbrace{|\vec{r}_u \times \vec{r}_v| \, dA}_{dS}$$

Ex. 1 Evaluate $\boxed{\iint_S f \, dS}$ where $f(\rho, \varphi, \theta) = \cos\varphi$ and S is the part of the unit sphere in the first octant.



Parameterize $S = \{(\rho, \varphi, \theta) \mid \rho = 1,\ 0 \le \underset{u}{\underset{\|}{\varphi}} \le \frac{\pi}{2},\ 0 \le \underset{v}{\underset{\|}{\theta}} \le \frac{\pi}{2}\}$

$x = \rho \sin\varphi \cos\theta$ $\qquad u = \varphi$

$\;\;= \sin\varphi \cos\theta$ $\qquad v = \theta$

$y = \sin\varphi \sin\theta$ $\qquad \vec{r}(u,v) = \langle \sin u \cos v,\ \sin u \sin v,\ \cos u \rangle$

$z = \cos\varphi$

$\vec{r}_u = \langle \cos u \cos v,\ \cos u \sin v,\ -\sin u \rangle$

$\vec{r}_v = \langle -\sin u \sin v,\ \sin u \cos v,\ 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle \sin^2 u \cos v,\ -(-\sin^2 u \sin v),\ \sin u \cos u \cos^2 v + \sin u \cos u \sin^2 v \rangle$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \left(\sin u \cos u \underbrace{(\cos^2 v + \sin^2 v)}_{=1}\right)^2}$$

$$= \sqrt{\sin^4 u \underbrace{(\cos^2 v + \sin^2 v)}_{=1} + \sin^2 u \cos^2 u}$$

$$= \sqrt{\sin^4 u + \sin^2 u \cos^2 u}$$

$$= \sqrt{\sin^2 u \underbrace{(\sin^2 u + \cos^2 u)}_{=1}}$$

$$= \sin u$$

$$\iint_S f \, dS = \iint_R \cos u \, |\vec{r}_u \times \vec{r}_v| \, dA = \iint_R \cos u \sin u \, dA$$

$f = \cos \varphi = \cos u$, $\varphi = u$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \cos u \sin u \, du \, dv \qquad x = \sin u, \quad dx = \cos u \, du$$

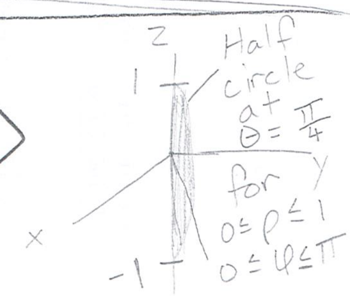
$$= \frac{\pi}{2} \left[\frac{1}{2} \sin^2 u \right]_0^{\pi/2}$$

$$= \frac{\pi}{2} \left[\frac{1}{2} (1 - 0) \right] = \boxed{\frac{\pi}{4}}$$

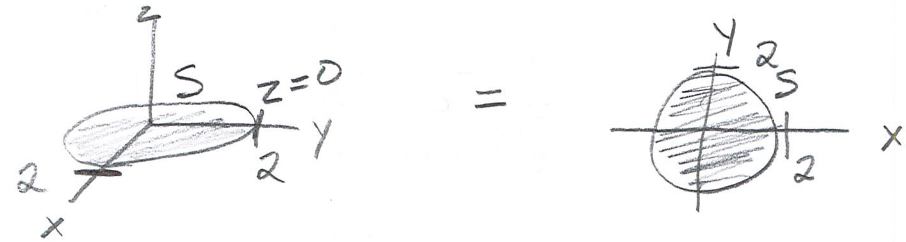
Sphere of radius 1 with constant $\theta = \frac{\pi}{4}$

$$\vec{r}(\rho, \varphi) = \left\langle \rho \sin \varphi \left(\cos \tfrac{\pi}{4}\right), \rho \sin \varphi \left(\sin \tfrac{\pi}{4}\right), \rho \cos \varphi \right\rangle$$

$$|\vec{r}_\rho \times \vec{r}_\varphi| = \rho^2 \sin \varphi$$



Polar coordinates:



$$\vec{r}(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 0\rangle$$

$$|\vec{r}_r \times \vec{r}_\theta| = r$$

A surface $z = g(x,y)$ is an explicitly defined surface with parameterization $\vec{r}(x,y) = \langle x, y, g(x,y)\rangle$

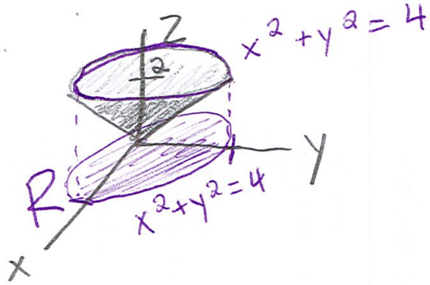
$$\vec{r}_x = \langle 1, 0, g_x(x,y)\rangle \qquad \vec{r}_x \times \vec{r}_y = \langle -g_x, -(g_y), 1\rangle$$

$$\vec{r}_y = \langle 0, 1, g_y(x,y)\rangle \qquad |\vec{r}_x \times \vec{r}_y| = \sqrt{g_x^2 + g_y^2 + 1}$$

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_R f(x,y,g(x,y)) \underbrace{\sqrt{g_x^2 + g_y^2 + 1}\, dA}_{dS}}$$

Ex. 2 Find the average temperature on the cone $z^2 = x^2 + y^2$, $0 \le z \le 2$ where temperature is $T(x,y,z) = 100 - 25z$

$$\text{Avg Temp} = \frac{\text{"Total" Temp over } S}{\text{Surface Area of } S} \quad \begin{array}{l} \longrightarrow \iint_S \mathbf{T}\, dS \\ \longrightarrow \iint_S dS \end{array}$$



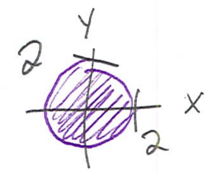
$$\vec{r}(x,y) = \langle x, y, \underbrace{\sqrt{x^2+y^2}}_{g(x,y)} \rangle, \quad (x,y) \in R$$

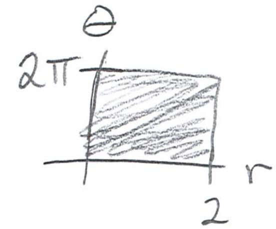
$$|\vec{r}_x \times \vec{r}_y| = \sqrt{g_x^2 + g_y^2 + 1}$$

$$= \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1}$$

$$= \sqrt{\underbrace{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}_{=1} + 1}$$

$$= \sqrt{2}$$





surface area $= \iint_S dS = \iint_R \sqrt{2}\, dA$

$$\left[= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{2}\, dy\, dx\right]$$

To Polar

$$= \int_0^{2\pi} \int_0^2 \sqrt{2}\, r\, dr\, d\theta$$

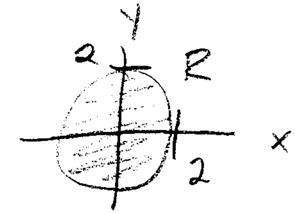
↳ original $\vec{r}(x,y)$ not in terms of r and θ

$$= 2\pi \left[\sqrt{2} \cdot \frac{1}{2} r^2\right]_0^2$$

surface area $= \pi\sqrt{2} \cdot 4$

surface integral of T $= \iint_S \underbrace{T(x,y,\overset{\sqrt{x^2+y^2}}{\overset{\parallel}{z}})}_{100-25\sqrt{x^2+y^2}}\, dS$

$$= \iint_R \underbrace{\left(100-25\sqrt{x^2+y^2}\right)}_{T(\vec{r}(x,y))} \underbrace{\sqrt{2}\, dA}_{=dS}$$



To Polar: $= \int_0^{2\pi}\int_0^{2} (100-25r)\sqrt{2}\; r\,dr\,d\theta$

$$= 2\pi\sqrt{2}\left[\int_0^2 (100r-25r^2)\,dr\right]$$

$$= 2\pi\sqrt{2}\left[50r^2 - \frac{25}{3}r^3\right]_0^2$$

$$= 50\pi\sqrt{2}\left[2r^2-\frac{1}{3}r^3\right]_0^2$$

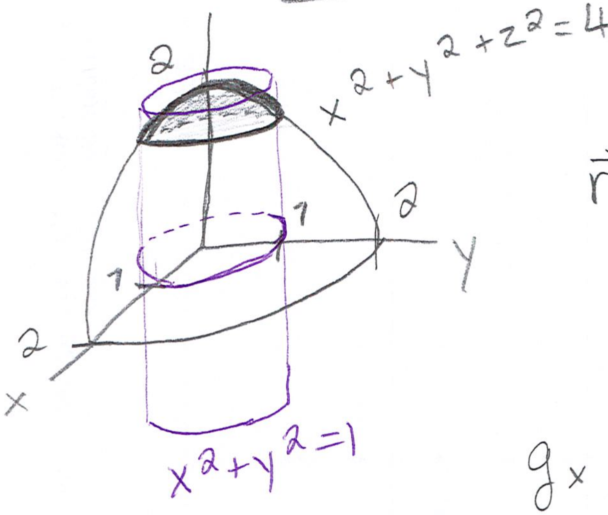
$$= 50\pi\sqrt{2}\left[8-\frac{8}{3}\right]$$

Total Temp $= \dfrac{800\pi\sqrt{2}}{3}$

Avg Temp $= \dfrac{\text{Total Temp}}{\text{Surface Area}} = \dfrac{\left(\dfrac{800\cancel{\pi\sqrt{2}}}{3}\right)}{\cancel{\pi\sqrt{2}}\ 4} = \left(\dfrac{800}{3}\right)\dfrac{1}{4} = \boxed{\dfrac{200}{3}}$

Ex. 3 Evaluate $\iint_S y^2\, dS$ where S is the part of the sphere

$x^2+y^2+z^2=4$ that lies <u>inside</u> the cylinder $x^2+y^2=1$ and <u>above</u> the xy-plane.



$$\vec{r}(x,y)=\langle x,y,\underbrace{\overbrace{\sqrt{4-x^2-y^2}}^{=g(x,y)}}_{\text{surface is part of the sphere}}\rangle \text{ with } \underbrace{x^2+y^2\le 1}_{\text{inside the cylinder}}$$

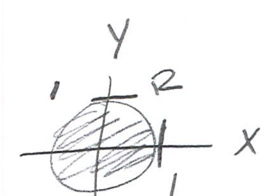
$$g_x=\frac{-x}{\sqrt{4-x^2-y^2}} \qquad g_y=\frac{-y}{\sqrt{4-x^2-y^2}}$$

$$|\vec{r}_x\times\vec{r}_y|=\sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}$$

$$=\sqrt{\frac{x^2+y^2}{4-x^2-y^2}+1}=\sqrt{\frac{\cancel{x^2}+\cancel{y^2}+(4\cancel{-x^2}\cancel{-y^2})}{4-x^2-y^2}}$$

$$=\frac{2}{\sqrt{4-x^2-y^2}}$$

$$\iint_S y^2\,dS=\iint_R y^2\cdot\overbrace{\frac{2}{\sqrt{4-x^2-y^2}}\,dA}^{dS}$$



To Polar $=\int_0^{2\pi}\int_0^1\frac{2r^2\sin^2\theta}{\sqrt{4-r^2}}\cdot r\,dr\,d\theta$

$$= \left[\int_0^{2\pi} \sin^2\theta \, d\theta\right] \cdot \left[\int_0^1 \frac{2 r^2 \cdot \cancel{r}}{\sqrt{4-r^2}} \, dr\right]$$

$$u = 4 - r^2 \Rightarrow r^2 = 4 - u$$
$$du = -2\cancel{r}\, dr$$

$$= \left[\int_0^{2\pi} \tfrac{1}{2}\left(1-\cos(2\theta)\right) d\theta\right] \cdot \left[\int_{r=0}^{r=1} \frac{\cancel{2}(4-u)}{\sqrt{u}} \cdot \left(-\frac{du}{\cancel{2}}\right)\right]$$

$$= \left[\tfrac{1}{2}\left(\theta - \tfrac{1}{2}\sin(2\theta)\right)\right]_0^{2\pi} \left[-\int_{r=0}^{r=1} \frac{4-u}{\sqrt{u}} \, du\right]$$

$$= \left[\tfrac{1}{2}\left(2\pi - 0 - (0-0)\right)\right] \left[-\int_{r=0}^{r=1} \left(4u^{-1/2} - u^{1/2}\right) du\right]$$

$$= [\pi] \left[-\left(8u^{1/2} - \tfrac{2}{3}u^{3/2}\right)\right]_{\substack{\cancel{r=0} \\ u=4}}^{\substack{\cancel{r=1} \\ u=3}}$$

$$u = 4 - 1 = 3$$
$$u = 4 - 0 = 4$$

$$= \pi\left[-\left(8\sqrt{3} - \frac{2}{\cancel{3}} \cdot \cancel{3}\sqrt{3}\right) + \left(8\sqrt{4} - \frac{2}{3} \cdot 4 \cdot \sqrt{4}\right)\right]$$

$$= \pi\left[-8\sqrt{3} + 2\sqrt{3} + 16 - \frac{16}{3}\right]$$

$$= \boxed{\pi\left(\frac{32}{3} - 6\sqrt{3}\right)}$$