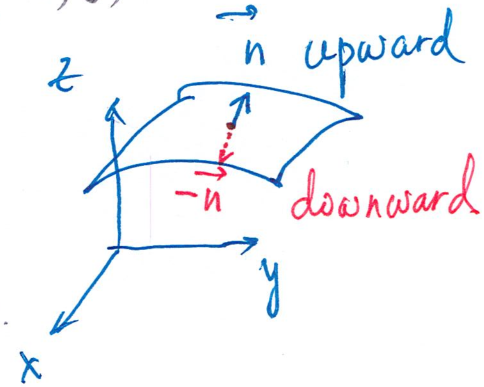
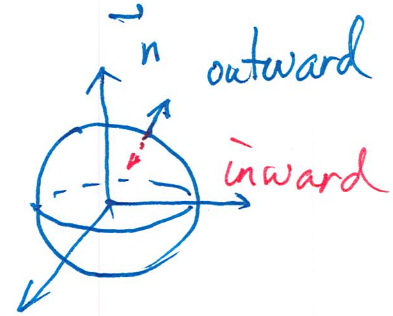
Lesson 33

- surface integrals of vector fields $\vec{F}(x,y,z) = \langle f, g, h \rangle$
  - two-sided / orientable surface





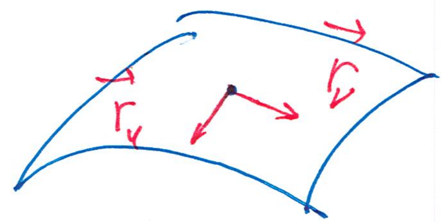
non-orientabale surface: the Möbius strip.

S: a smooth oriented surface with parametrization $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, for $(u,v) \in R$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \left(\vec{F} \cdot \vec{n}\right) dS$$

$$= \iint_R \vec{F}\left(\vec{r}(u,v)\right) \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \underbrace{\cancel{|\vec{r}_u \times \vec{r}_v|}\, du\, dv}_{dS}$$

$$= \iint_R \vec{F}\left(\vec{r}(u,v)\right) \cdot \left(\vec{r}_u \times \vec{r}_v\right) du\, dv$$



$\vec{r}(u,v)$

$\frac{\partial \vec{r}}{\partial u}$, $\frac{\partial \vec{r}}{\partial v}$

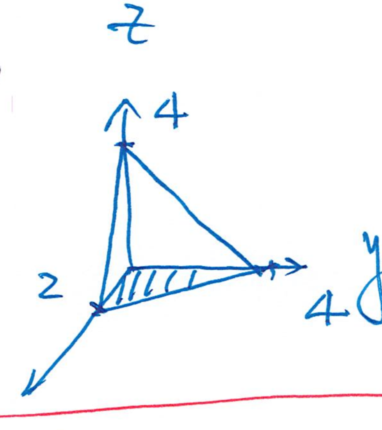
$\vec{r}_u \times \vec{r}_v \perp S$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\left(\vec{r}(t)\right) \cdot \vec{r}\,'(t)\, dt$$

Ex. 7 $\vec{F} = \langle 0, 0, -1 \rangle$, corresponding to a constant downward flow.

Find the flux in the downward direction across the surface S, which is the plane $z = 4 - 2x - y$ in the 1^{st} octant.

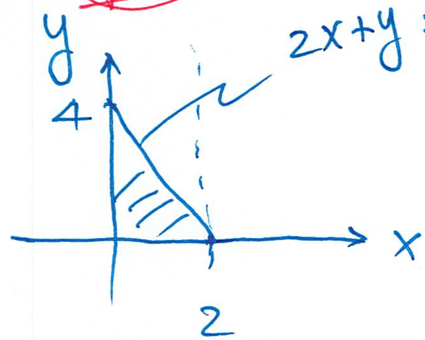


$S: \ \vec{r}(x,y) = \langle x, y, 4-2x-y \rangle$

$\vec{r}_x \times \vec{r}_y = \langle 2, \overset{1}{\cancel{y}}, 1 \rangle$ upward

downward

$\iint_S (\vec{F} \cdot \vec{n})\, dS$



$= \iint_R \langle 0,0,-1 \rangle \cdot \left[- \langle 2, 1, 1 \rangle \right] dx\,dy \qquad \begin{cases} 0 \le x \le 2 \\ 0 \le y \le 4 - 2x \end{cases}$

$= \iint_R 1 \, dx\,dy = A(R) = \frac{1}{2} \cdot 2 \cdot 4 = 4$

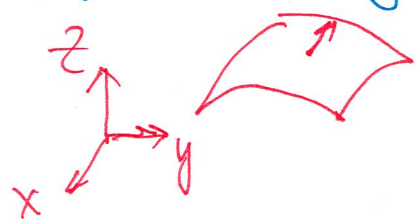
$S: \ z = f(x,y), \ (x,y) \in R$

$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

$\vec{r}_x = \langle 1, 0, \frac{\partial f}{\partial x} \rangle$

$\vec{r}_y = \langle 0, 1, \frac{\partial f}{\partial y} \rangle$

$\vec{r}_x \times \vec{r}_y = \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \rangle$ upward



Ex. 8 Radial vector field $\vec{F} = \langle x, y, z \rangle$. Is the upward flux of the field greater across the hemisphere $x^2+y^2+z^2=1$, for $z \geq 0$, or across the paraboloid $z = 1-x^2-y^2$, for $z \geq 0$?

S_1: $x^2+y^2+z^2=1$ for $z \geq 0 \Longrightarrow z = \sqrt{1-x^2-y^2}$

$x^2+y^2+z^2=a^2$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$\rho = 1$ a

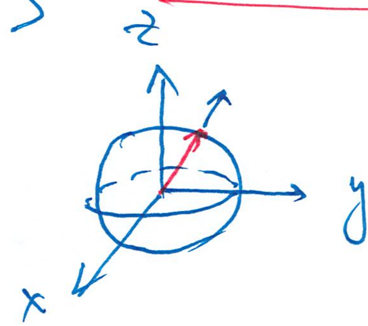
$0 \leq \varphi \leq \frac{\pi}{2}$

$0 \leq \theta \leq 2\pi$

$\vec{r}(\varphi, \theta) = \langle \overset{a}{} \sin\varphi \cos\theta, \overset{a}{} \sin\varphi \sin\theta, \overset{a}{} \cos\varphi \rangle = \langle x, y, z \rangle$

$\vec{r}_\varphi = \langle \overset{a}{} \cos\varphi \cos\theta, \overset{a}{} \cos\varphi \sin\theta, \overset{a}{} -\sin\varphi \rangle$

$\vec{r}_\theta = \langle -\overset{a}{} \sin\varphi \sin\theta, \overset{a}{} \sin\varphi \cos\theta, 0 \rangle$



$\vec{r}_\varphi \times \vec{r}_\theta = \overset{a^2}{} \langle \sin^2\varphi \cos\theta, \sin^2\varphi \sin\theta, \cos\varphi \sin\varphi \rangle$

$= \overset{a^2}{} \sin\varphi \langle \sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi \rangle$

$= \overset{a^2}{} \sin\varphi \, \vec{r}(\varphi, \theta) = \sin\varphi \langle x, y, z \rangle$ outward

$\iint_{S_1} (\vec{F} \cdot \vec{n}) \, dS$

$= \iint_R \langle x, y, z \rangle \cdot \sin\varphi \langle x, y, z \rangle \, d\varphi \, d\theta$

$= \iint_R \sin\varphi (x^2+y^2+z^2) \, d\varphi \, d\theta$

$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin\varphi \, d\theta \, d\varphi = 2\pi \left[-\cos\varphi \right]_0^{\frac{\pi}{2}} = 2\pi$

$\boxed{\vec{r}_\varphi \times \vec{r}_\theta = a^2 \sin\varphi \langle x, y, z \rangle \text{ outward}}$

$S_2: z = 1 - x^2 - y^2$ for $z \geq 0$ $\qquad \vec{r}(x, y) = \langle x, y, 1 - x^2 - y^2 \rangle$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad z = 1 - r^2$$

$\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 1 - r^2 \rangle$

$\vec{r}_r = \langle \cos\theta, \sin\theta, -2r \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\vec{r}_r \times \vec{r}_\theta = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$

$= 2r \langle r\cos\theta, r\sin\theta, \frac{1}{2} \rangle = 2r \langle x, y, \frac{1}{2} \rangle$

$$\iint_{S_2} (\vec{F} \cdot \vec{n})\, dS$$

$$= \int_0^1 \int_0^{2\pi} \langle x, y, z \rangle \cdot 2r \langle x, y, \tfrac{1}{2} \rangle \, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} \left(x^2 + y^2 + \tfrac{1}{2} z\right) 2r \, d\theta\, dr$$

$R: \quad z = 0 = 1 - x^2 - y^2$

$$= \int_0^1 \int_0^{2\pi} \left(r^2 + \tfrac{1}{2}(1 - r^2)\right) \cdot 2r \, d\theta\, dr \qquad \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$= 2\pi \int_0^1 2r \left(\tfrac{1}{2} + \tfrac{1}{2} r^2\right) dr$$

$$= 2\pi \left[\tfrac{1}{2} r^2 + \tfrac{1}{4} r^4\right]_0^1 = 2\pi \cdot \left(\tfrac{1}{2} + \tfrac{1}{4}\right) = \tfrac{3}{2}\pi$$