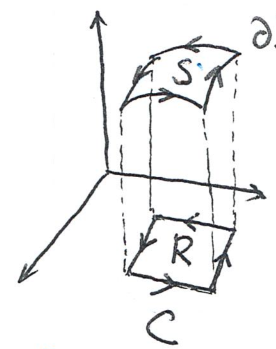
Proof of Stoke's Thrm (a special case)

$\vec{F} = \langle f, g, h \rangle$, $\quad S: z = s(x,y)$, $(x,y) \in R$



$$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_S \nabla\times\vec{F}\cdot\vec{n}\, dS = \iint_R \nabla\times\vec{F}\cdot\left\langle -\frac{\partial s}{\partial x}, -\frac{\partial s}{\partial y}, 1\right\rangle dA$$

$$\|$$

$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \oint_{\partial S} f\,dx + g\,dy + h\,dz$$

$$= \oint_{\partial S} (f + s_x)\,dx + (g + s_y)\,dy$$

$S: \vec{r}(x,y) = \langle x, y, s(x,y)\rangle$, $(x,y)\in R$

$$\vec{r}_x \times \vec{r}_y = \left\langle -\frac{\partial s}{\partial x}, -\frac{\partial s}{\partial y}, 1\right\rangle$$

$\partial S: \vec{R}(t) = \langle x(t), y(t), \underbrace{s(x(t), y(t))}_{z(t)}\rangle$

$a \le t \le b$

$\partial R = C: \vec{r}(t) = \langle x(t), y(t), 0\rangle$

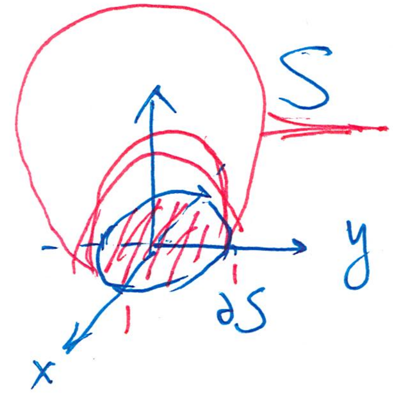
$a \le t \le b$

$$\frac{dz}{dt} = \frac{\partial s}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial s}{\partial y}\cdot\frac{dy}{dt}$$

$$dz = s_x\,dx + s_y\,dy$$

- $C = \partial S_1 = \partial S_2$ with the same orientation

$$\iint_{S_1} \nabla\times\vec{F}\cdot d\vec{S} = \iint_{S_2} \nabla\times\vec{F}\cdot d\vec{S}$$



$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \iint_{S} \left(\nabla\times\vec{F}\cdot\vec{n}\right) dS$$

$\vec{F}$ is conservative $\iff \exists\, \varphi$ s.t. $\vec{F} = \nabla\varphi$

$$= \iint_{S_0} \left(\nabla\times\vec{F}\cdot\vec{n}\right) dS$$

$S_0 : \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$, $0\le r\le 1$, $0\le\theta\le 2\pi$

- $\nabla\times\vec{F} = \vec{0} \Longrightarrow \vec{F} = \nabla\varphi$

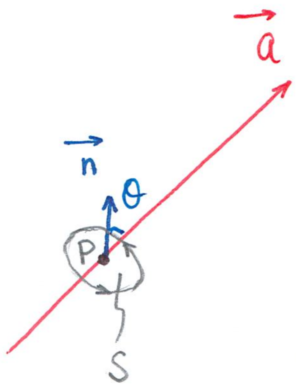
$\Uparrow$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \left(\nabla\times\vec{F}\cdot\vec{n}\right) dS = 0$$

$\vec{r}_r \times r_\theta =$

- interpreting the curl

average circulation $\quad \dfrac{1}{A(S)} \oint_{C=\partial S} \vec{F} \cdot d\vec{r} = \dfrac{1}{A(S)} \iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS$



Example

$\vec{F} = \vec{a} \times \vec{r}$, where $\vec{r} = \langle x, y, z \rangle \implies \nabla \times \vec{F} = 2\vec{a}$

$\omega = \frac{1}{2} |\nabla \times \vec{F}| = |\vec{a}|$

$$\frac{1}{A(S)} \oint_{\partial S} \vec{F} \cdot d\vec{r} = \frac{1}{A(S)} \iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS$$

$$= \frac{1}{A(S)} \iint_S (2\vec{a} \cdot \vec{n})\, dS = \frac{2\vec{a} \cdot \vec{n}}{A(S)} \iint_S dS$$

$$= 2\vec{a} \cdot \vec{n} = \nabla \times \vec{F} \cdot \vec{n}$$

$$= 2|\vec{a}| \cos\theta$$

Ex. 4 Consider the velocity field $\vec{v} = \langle 0, 1-x^2, 0 \rangle$ for $|x| \le 1$ and $|z| \le 1$, which represents a horizontal flow in the y-direction.

(a) Suppose you place a paddle wheel at the pt $P(\frac{1}{2}, 0, 0)$. Using physical arguments, in which of the coordinate directions should the axis of the wheel point in order for the wheel to spin? In which direction does it spin? What happens if you place the wheel at $Q(-\frac{1}{2}, 0, 0)$?

(b) Compute and graph curl of $\vec{v}$ and provide an interpretation.

$$\nabla \times \vec{V} = \begin{vmatrix} \bar{i} & \bar{j} & k \\ \partial_x & \partial_y & \partial_z \\ 0 & 1-x^2 & 0 \end{vmatrix} = \langle 0, 0, \underline{\underline{-2x}} \rangle$$

$$\nabla \times \vec{V} \cdot \vec{k} = \langle 0, 0, -2x \rangle \cdot \langle 0, 0, 1 \rangle = -2x \begin{cases} \underline{\underline{< 0}} & x > 0 \\ > 0 & x < 0 \end{cases}$$

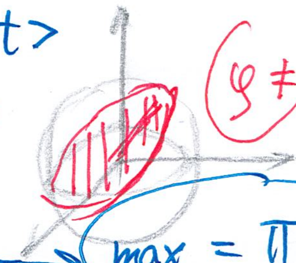
#37 Let S be the disk enclosed by the curve $C: \vec{r}(t) = \langle \cos\varphi \cos t, \sin t, \sin\varphi \cos t \rangle$

for $0 \le t \le 2\pi$, where $0 \le \varphi \le \frac{\pi}{2}$ is a fixed angle.

Use Stoke's Thrm and a surface integral to find the circulation on C of the vector field $\vec{F} = \langle -y, -z, x \rangle$ as a function of φ? For what a value of φ is the circulation a maximum?

$S: \vec{R}(r,t) = \langle r\cos\varphi\cos t, r\sin t, r\sin\varphi\cos t \rangle$

$0 \le \varphi \le \pi$



$\varphi \ne 0$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F}\cdot\vec{n})\,dS$$

$R: \left.\begin{matrix} 0 \le t \le 2\pi \\ 0 \le r \le 1 \end{matrix}\right\}$

$$= \iint_R \langle 1,-1,1\rangle \cdot r\langle -\sin\varphi, 0, \cos\varphi\rangle dA$$

$$= \iint_R r(\cos\varphi - \sin\varphi)\,dA = (\cos\varphi - \sin\varphi)\iint_R r\,dA = \pi(\cos\varphi - \sin\varphi) \longrightarrow \max = \pi$$

$$\iint_R r\,dA = \int_0^{2\pi}\int_0^1 r\,dr\,dt = \pi$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & -z & x \end{vmatrix} = \langle 1, -1, 1\rangle$$

$\vec{R}_r = \langle \cos\varphi\cos t, \sin t, \sin\varphi\cos t \rangle$

$\vec{R}_t = \langle -r\cos\varphi\sin t, r\cos t, -r\sin\varphi\sin t \rangle$

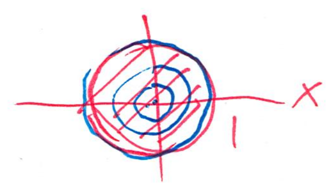
$\vec{R}_r \times \vec{R}_t = \langle -r\sin\varphi, 0, r\cos\varphi \rangle$

$C: \vec{r}(t) = \langle \underbrace{\cos\varphi\cos t}_{x(t)}, \underbrace{\sin t}_{y(t)}, \underbrace{\sin\varphi\cos t}_{z(t)} \rangle$ $\quad 0 \le t \le 2\pi$

$$x^2 + z^2 = (\cos\varphi\cos t)^2 + (\sin\varphi\cos t)^2 = \cos^2 t$$

$y = \sin t$

$$x^2 + z^2 + y^2 = 1$$



$\varphi = 0$ $\quad \vec{r}(t) = \langle \cos t, \sin t, 0 \rangle \quad 0 \le t \le 2\pi$

$\vec{r}(r,t) = \langle r\cos t, r\sin t, 0 \rangle$