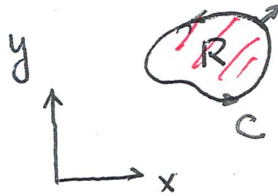
# Lesson 36

## §17.8 Divergence Theorem

$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$




$\vec{n}$ unit outward normal

$C = \partial R$

$$\boxed{\iint_R \nabla \cdot \vec{F}\, dA} = \oint_C \vec{F} \cdot \vec{n}\, ds$$

Green's Thrm

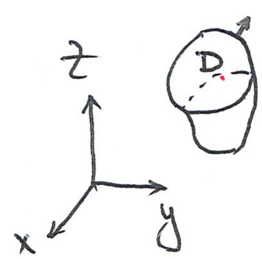
$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

$$\iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{(\nabla \times \vec{F} \cdot \vec{n})} dA = \oint_{\partial R} \vec{F} \cdot d\vec{r}$$

$$(\nabla \times \vec{F} \cdot \vec{n})\, dA$$


$\partial S = C$

$\vec{F}(x,y,z) = \langle f, g, h \rangle$




$\vec{n}$ unit outward normal

$S = \partial D$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_{\partial D = S} \left(\vec{F} \cdot \vec{n}\right) dS$$

Gauss' Divergence Thrm

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

Ex. 1 $\vec{F} = \langle x, y, z \rangle$, S: the sphere $x^2+y^2+z^2=a^2$ pointing outward

$\nabla\cdot\vec{F} = 1+1+1=3$ D: the solid region inside of S.

$$\begin{cases} x = \rho\sin\varphi\cos\theta & 0\le\rho\le a \\ y = \rho\sin\varphi\sin\theta & 0\le\varphi\le\pi \\ z = \rho\cos\varphi & 0\le\theta\le 2\pi \end{cases}$$

$$\iiint_D \nabla\cdot\vec{F}\,dV = 3\iiint_D 1\,dV = 3\cdot\frac{4}{3}\pi a^3$$

$$= 3\int_0^a\int_0^\pi\int_0^{2\pi} \underbrace{\rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho}_{dV} = 3\cdot 2\pi\cdot\left[-\cos\varphi\right]_0^\pi\cdot\left[\frac{1}{3}\rho^3\right]_0^a = 3\cdot 2\pi\cdot 2\cdot\frac{1}{3}a^3 = 4\pi a^3$$

S: $\vec{r}(\varphi,\theta) = \langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi\rangle$

R: $0\le\varphi\le\pi$, $0\le\theta\le 2\pi$

$= \langle x, y, z\rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S (\vec{F}\cdot\vec{n})\,dS = \iint_R \langle x,y,z\rangle\cdot(a\sin\varphi)\langle x,y,z\rangle\,dA$$

$$= \iint_R a\sin\varphi\,(x^2+y^2+z^2)\,dA$$

$$= a^3\int_0^\pi\int_0^{2\pi}\sin\varphi\,d\theta\,d\varphi$$

$$= 2\pi a^3\int_0^\pi \sin\varphi\,d\varphi = 2\pi a^3\left[-\cos\varphi\right]_0^\pi = 4\pi a^3$$

$$\vec{r}_\varphi\times\vec{r}_\theta = a\sin\varphi\,\vec{r}(\varphi,\theta)$$

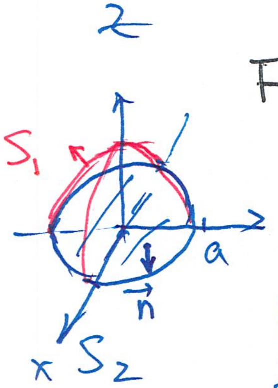
$$|\vec{r}_\varphi\times\vec{r}_\theta| = a\sin\varphi\,|\vec{r}(\varphi,\theta)|$$

$$= a^2\sin\varphi$$

**Ex. 2** $\vec F = \vec a \times \vec r = \langle 1,0,1\rangle \times \langle x,y,z\rangle = \langle -y,\ x-z,\ y\rangle$ a rotation field

$\nabla\cdot\vec F = \frac{\partial}{\partial x}(-y)+\frac{\partial}{\partial y}(x-z)+\frac{\partial}{\partial z}(y) = 0+0+0=0$

S: the hemisphere $x^2+y^2+z^2=a^2$, for $z\ge 0$, together with its base in the xy-plane.




Find the net outward flux across S.

$$\iint_S (\vec F\cdot\vec n)\,dS = \iiint_D (\nabla\cdot\vec F)\,dV = 0$$

$$\parallel$$

$$\iint_{S_1} + \iint_{S_2}(\vec F\cdot \vec n)\,dS = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \langle -y,x-z,y\rangle\cdot a\sin\varphi\langle x,y,z\rangle\, d\theta\, d\varphi = \underbrace{\int_0^{\frac{\pi}{2}}\int_0^{2\pi} a\sin\varphi\,[-xy + y(x-z)+yz]\, d\theta\, d\varphi}_{=0}$$

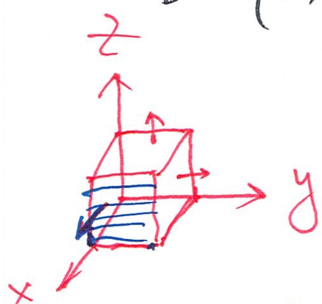
$$+ \iint_R \langle -y,x-z,y\rangle\cdot\langle 0,0,-1\rangle\, dx\,dy \quad + \iint_R -y\,dx\,dy = -\int_0^a\int_0^{2\pi} r\sin\theta\, d\theta\, r\,dr$$

$$= -\left[\tfrac13 r^3\right]_0^a\left[-\cos\theta\right]_0^{2\pi}=0$$

**Ex. 3** Find the net outward flux of $\vec F = xyz\langle1,1,1\rangle$ across the boundary of the cube

$D = \{(x,y,z):\ 0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1\}$.

$\nabla\cdot\vec F = \frac{\partial}{\partial x}(xyz)+\frac{\partial}{\partial y}(xyz)+\frac{\partial}{\partial z}(xyz) = yz+xz+xy$



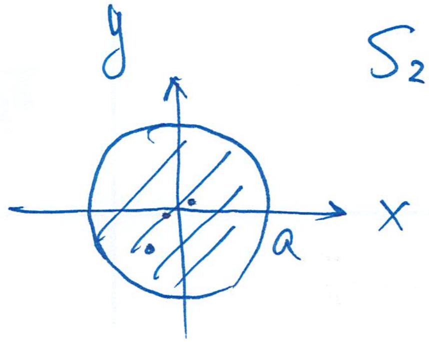

$$\iint_S \vec F\cdot\vec n\, dS = \iiint_D \nabla\cdot\vec F\, dV = \int_0^1\int_0^1\int_0^1 (yz+xz+xy)\, dz\, dy\, dx$$

$S_1$: $x=1$, $(y,z)\in R_1 = \{(y,z)\mid 0\le y\le 1,\ 0\le z\le 1\}$

$\vec r_1(y,z) = \langle 1,y,z\rangle$

$\vec r_y\times\vec r_z = \langle 1, -\frac{\partial x}{\partial y}, -\frac{\partial x}{\partial z}\rangle = \langle1,0,0\rangle$

$$\iint_{S_1}\vec F\cdot\vec n\, dS = \int_0^1\int_0^1 yz\langle1,1,1\rangle\cdot\langle1,0,0\rangle\, dy\,dz = \int_0^1\int_0^1 yz\,dy\,dz$$




$S_2: \; z = 0$

$(x, y) \in R = \{(x,y) \mid x^2 + y^2 \leq a^2\}$

$\vec{r}(x,y) = \langle x, y, 0 \rangle, \; (x, y) \in R$

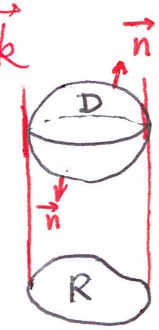
$\vec{r}_x \times \vec{r}_y = \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \rangle = \langle 0, 0, 1 \rangle$

# Proof of Divergence Thrm (a special case)

$\vec{F} = \langle f, g, h \rangle = f\vec{i} + g\vec{j} + h\vec{k}$

$D: p(x,y) \le z \le q(x,y)$

$(x,y) \in R$



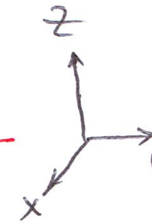

$$\iint_{\partial D} \vec{F} \cdot d\vec{S} = \iint_{\partial D} \vec{F} \cdot \vec{n}\, dS = \iint_{\partial D} f\, \vec{i} \cdot \vec{n}\, dS + \iint_{\partial D} g\, \vec{j} \cdot \vec{n}\, dS + \iint_{\partial D} h\, \vec{k} \cdot \vec{n}\, dS$$

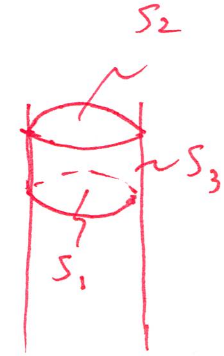
$$\|$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D \frac{\partial f}{\partial x}\, dV + \iiint_D \frac{\partial g}{\partial y}\, dV + \iiint_D \frac{\partial h}{\partial z}\, dV$$

$S_1: z = p(x,y),\ (x,y) \in R$

$S_2: z = q(x,y),\ (x,y) \in R$




$$\iiint_D \frac{\partial h}{\partial z}\, dV = \iint_R \int_{p(x,y)}^{q(x,y)} \frac{\partial h}{\partial z}\, dz\, dA = \iint_R h(x,y,z)\Big|_{z=p(x,y)}^{z=q(x,y)} dA = \iint_R \Big[ h(x,y,q(x,y)) - h(x,y,p(x,y)) \Big] dA$$

$$\iint_{\partial D} h\, \vec{k} \cdot \vec{n}\, dS = \iint_{S_1} + \iint_{S_2} h\, \vec{k} \cdot \vec{n}\, dS = \iint_R h(x,y,p(x,y))\, \overbrace{\langle 0,0,1 \rangle \cdot \langle *, *, -1 \rangle}^{-1}\, dA + \iint_R h(x,y,q(x,y))\, \langle 0,0,1 \rangle \cdot \langle *, *, 1 \rangle\, dA$$

$S_3$ : ?

$\vec{k} \cdot \vec{n} = 0$

$S_1: z = p(x,y)$

$(x,y) \in R$

$-\vec{r}_x \times \vec{r}_y = \langle +\frac{\partial p}{\partial x}, +\frac{\partial p}{\partial y}, -1 \rangle$

$S_2: z = q(x,y)$

$(x,y) \in R$

$\vec{r}_x \times \vec{r}_y = \langle -\frac{\partial q}{\partial x}, -\frac{\partial q}{\partial y}, 1 \rangle$