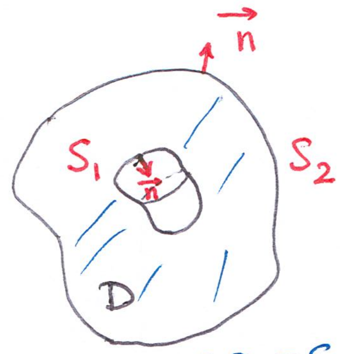
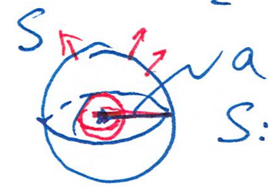
- Divergence Thrm for Hollow Regions




$$\iiint_D \nabla\cdot\vec F\,dV = \iint_{\partial D}\left(\vec F\cdot\vec n\right)dS = \iint_{S_2}\vec F\cdot\vec n\,dS + \iint_{S_1}\vec F\cdot\vec n\,dS$$

$\partial D = S_1 \cup S_2$




$$S: \begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases} \quad \vec r_\varphi \times \vec r_\theta = a\sin\varphi\,\vec r(\varphi,\theta)$$

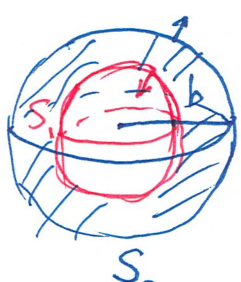
Ex. 4 $\vec F = \dfrac{\vec r}{|\vec r|^3} = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$

(b) Find the outward flux of $\vec F$ across any sphere that encloses the origin.

(a) Find the net outward flux of $\vec F$ across the surface of the region $D = \{(x,y,z) : a^2 \le x^2+y^2+z^2 \le b^2\}$

$$\frac{\partial}{\partial x}\left(\frac{x}{|\vec r|^3}\right) = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-3/2}\right] = (x^2+y^2+z^2)^{-3/2} + x\cdot\left(-\frac{3}{2}\right)\cdot(x^2+y^2+z^2)^{-5/2}\cdot 2x$$

$$= (x^2+y^2+z^2)^{-\frac{5}{2}}\left[(x^2+y^2+z^2) - 3x^2\right] = \frac{1}{|\vec r|^5}(y^2+z^2-2x^2)$$




$$\iint_{\substack{S_1\cup S_2 \\ \| \\ \partial D}}\left(\vec F\cdot\vec n\right)dS = \iiint_D \nabla\cdot\vec F\,dV = 0$$

$$\nabla\cdot\vec F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \frac{1}{|\vec r|^5}\left\{(y^2+z^2-2x^2) + (x^2+z^2-2y^2) + (y^2+x^2-2z^2)\right\}$$

$= 0$

$$\iint_S\left(\vec F\cdot\vec n\right)dS \ne \iiint_D\left(\nabla\cdot\vec F\right)dV$$

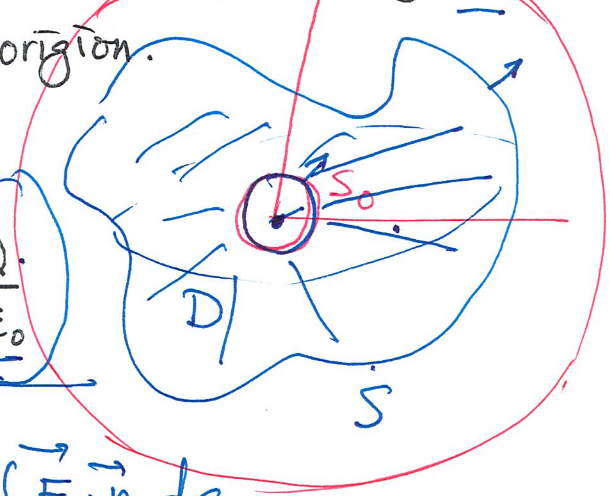
$$= \int_0^{\pi}\int_0^{2\pi}\frac{\langle x, y, z\rangle}{|\vec r|^3}\cdot a\sin\varphi\,\langle x, y, z\rangle\,d\theta\,d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi}\frac{x^2+y^2+z^2 = \cancel{a^2}}{\cancel{|\vec r|^3}\ \cancel{a^3}}\,\cancel{a}\sin\varphi\,d\theta\,d\varphi = 2\pi\left[-\cos\varphi\right]_0^{\pi} = 4\pi$$

• Gauss' Law $\quad E(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$ the electric field due to a point charge Q located at the origion.

S : any surface enclosing the origion.

the flux of the electric field across S : $\iint_S \vec{E}\cdot d\vec{S} \overset{?}{=} \frac{Q}{\varepsilon_0}$




$$\iint_S \vec{E}\cdot\vec{n}\,dS = \iint_{S_0^-} \vec{E}\cdot\vec{n}\,dS$$

$$= \frac{Q}{\cancel{4\pi}\varepsilon_0}\boxed{\iint_{S_0^-} \frac{\vec{r}}{|\vec{r}|^3}\cdot\vec{n}\,dS}$$

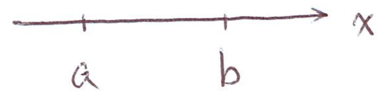
$$= \frac{Q}{\varepsilon_0}$$

$$0 = \iiint_D \nabla\cdot\vec{E}\,dV = \iint_{\partial D} \vec{E}\cdot\vec{n}\,dS$$

$$= \iint_S \vec{E}\cdot\vec{n}\,dS \;\cancel{+}\; - \iint_{S_0^-} \vec{E}\cdot\vec{n}\,dS$$

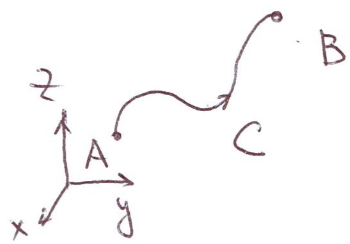
# A Final Perspective

- fundamental Thrm of Calculus




$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

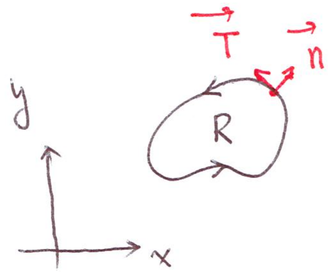
- fundamental Thrm of line integral




$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

- Green's Thrm

$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$



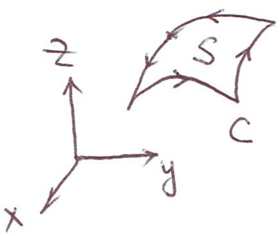

$(\nabla \times \vec{F} \cdot \vec{n})$

$$\iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \oint_{\partial R} \left( \vec{F} \cdot \vec{T} \right) ds$$

$$\iint_R \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA = \oint_{\partial R} \left( \vec{F} \cdot \vec{n} \right) ds$$

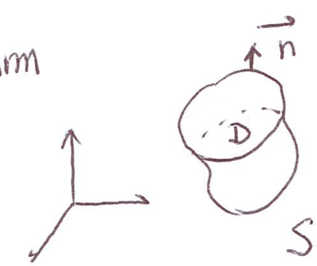
- Stoke's Thrm

$\vec{F}(x,y,z) = \langle f, g, h \rangle$




$$\iint_S \left( \nabla \times \vec{F} \right) \cdot \vec{n}\, dS = \oint_{\partial S} \left( \vec{F} \cdot \vec{T} \right) ds$$

- Gauss Divergence Thrm




$$\iiint_D \left( \nabla \cdot \vec{F} \right) dV = \iint_{\partial D} \left( \vec{F} \cdot \vec{n} \right) ds$$