Lesson 5

Chapter 14. Vector-Valued Functions (4 lectures)

§14.1 Vector-valued functions

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

$$a \le t \le b$$

- aspect of analysis

domain where $\vec{r}(t)$ is well-defined

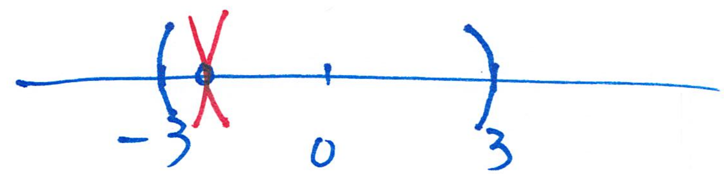
$$\vec{r}(t) = \left\langle \frac{t-2}{t+2},\ \sin t,\ \ln(9-t^2) \right\rangle$$

$f(t) = \dfrac{t-2}{t+2}$ $\quad t+2 \neq 0$ $\quad \boxed{t \neq -2}$

$g(t) = \sin t$ $\quad (-\infty, +\infty)$

$h(t) = \ln(9-t^2)$ $\quad 9 - t^2 > 0$

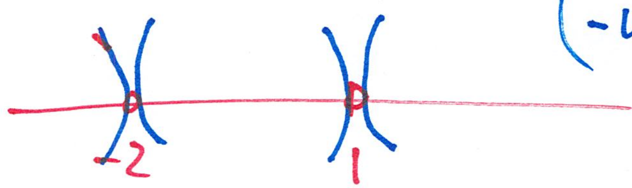
$t^2 < 9 \iff |t| < 3$



$(-3,-2) \cup (-2,3)$

$$\vec{r}(t) = \left\langle \frac{2}{t-1},\ \frac{3}{t+2} \right\rangle$$

$t - 1 \neq 0$ $\qquad t + 2 \neq 0$



$$(-\infty,-2) \cup (-2,1) \cup (1,+\infty) = (-\infty,+\infty) \setminus \{-2, 1\}$$

limit $\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$ $\qquad \frac{0}{0}$

$$\lim_{t\to 0} \vec{r}(t) = \lim_{t\to 0} \langle 1+t^3, te^{-t}, \frac{\sin t}{t} \rangle = \langle \underbrace{\lim_{t\to 0}(1+t^3)}_{=1}, \underbrace{\lim_{t\to 0} te^{-t}}_{=0}, \lim_{t\to 0} \frac{\sin t}{t} \rangle$$

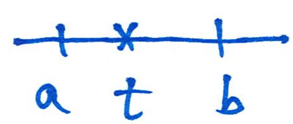
$$\lim_{t\to 0} \frac{\sin t}{t} = \lim_{t\to 0} \frac{(\sin t)'}{(t)'} = \lim_{t\to 0} \frac{\cos t}{1} = 1$$

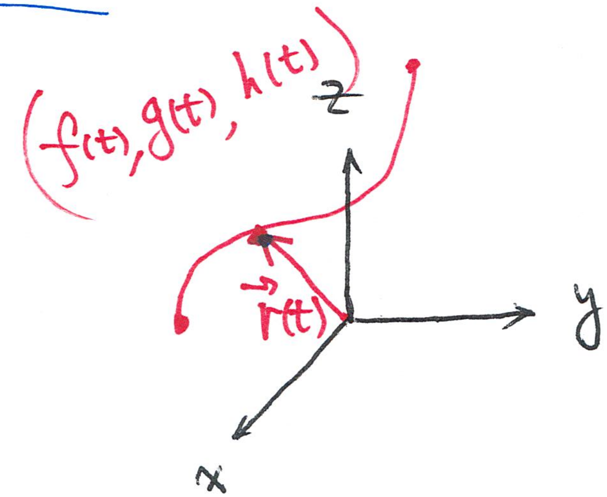
continuity $\vec{r}(t)$ is cont. at $t=t_0$ $\iff \lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$

derivative and integral (§14.2)

- **aspect of geometry** $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ $\quad t \in [a,b]$
  - curves in space

$$C = \{ (\underset{x}{\underset{\parallel}{f(t)}}, \underset{y}{\underset{\parallel}{g(t)}}, \underset{z}{\underset{\parallel}{h(t)}}) \mid t \in [a,b] \}$$





Ex. 1 (Lines as vector-valued functions) line passing through $P(2,-1,4)$ and $Q(3,0,6)$

$\left[\begin{array}{l}\text{a pt on } l \\ \text{a vector} \parallel l\end{array}\right.$ $\quad \vec{v} = \overrightarrow{PQ} = \langle 3-2,\ 0-(-1),\ 6-4\rangle = \langle 1,1,2\rangle$

$$\vec{r}(t) = \langle 2,-1,4\rangle + t\langle 1,1,2\rangle = \langle 2+t,\ -1+t,\ 4+2t\rangle$$

Ex. 2 (a spiral) graph

$$\vec{r}(t) = \left\langle 4\cos t,\ \sin t,\ \frac{t}{2\pi}\right\rangle$$

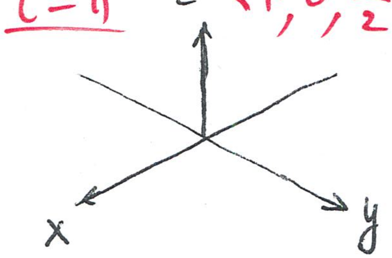
(a) $t\in[0,2\pi]$; (b) $t\in(-\infty,+\infty)$

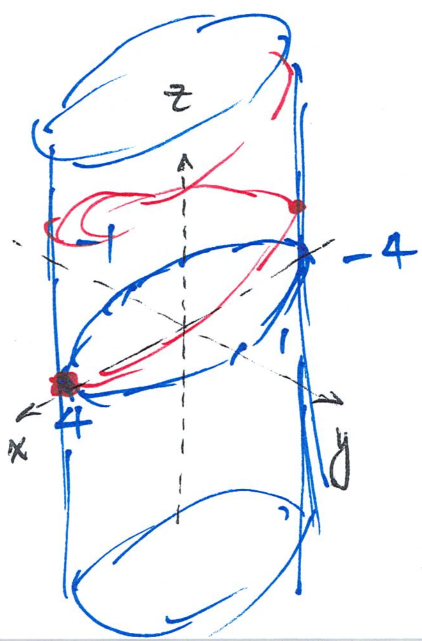
$\begin{cases} x = 4\cos t \\ y = \sin t \\ z = \dfrac{t}{2\pi}\end{cases} > \left(\dfrac{x}{4}\right)^2 + y^2 = \cos^2 t + \sin^2 t = 1$

$t=0 \quad \langle 4,0,0\rangle$

$t=\frac{\pi}{2} \quad \langle 0,1,\frac{1}{4}\rangle$

$t=\pi \quad \langle 4,0,\frac{1}{2}\rangle$





Ex. 3 (Roller coaster Curve) graph

$\vec{r}(t) = \langle \cos t,\ \sin t,\ 0.4\sin 2t\rangle,\quad t\in[0,2\pi]$

$\begin{cases} x=\cos t \\ y = \sin t \\ z = 0.4\sin 2t \in (-0.4,\ 0.4)\end{cases} > x^2+y^2 = \cos^2 t + \sin^2 t = 1$

$t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$

$\vec{r}(0) = \vec{r}(2\pi) = \langle 1,0,0\rangle$

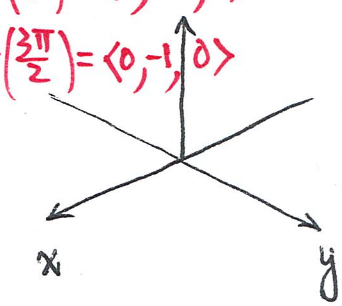
$\vec{r}(\frac{\pi}{2}) = \langle 0,1,0\rangle$

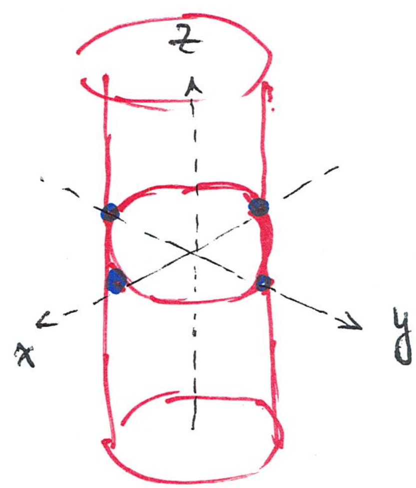
$\vec{r}(\pi) = \langle -1,0,0\rangle$

$\vec{r}(\frac{3\pi}{2}) = \langle 0,-1,0\rangle$

$\vec{r}(\frac{\pi}{4}) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0.4\rangle$

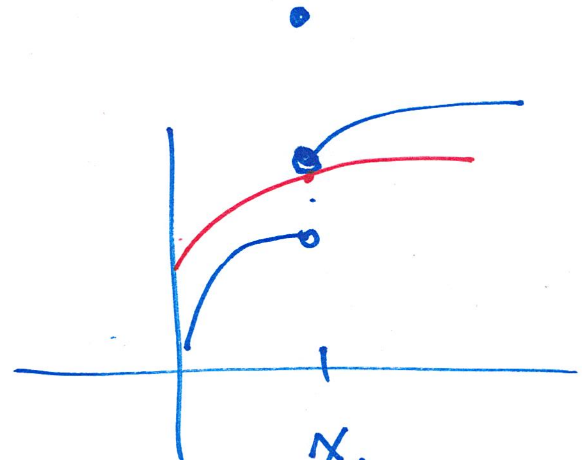
$\vec{r}(\frac{3\pi}{4}) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0.4$





$y = f(x)$ $\qquad x \in [a, b] \iff a \le x \le b$

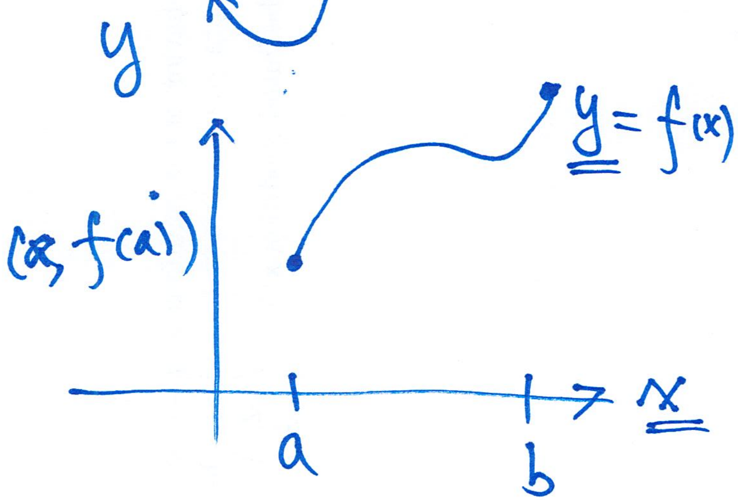
scalar-valued function



$f(x)$ is cont. at $x = x_0$

$$\iff \lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right) = f(x_0)$$

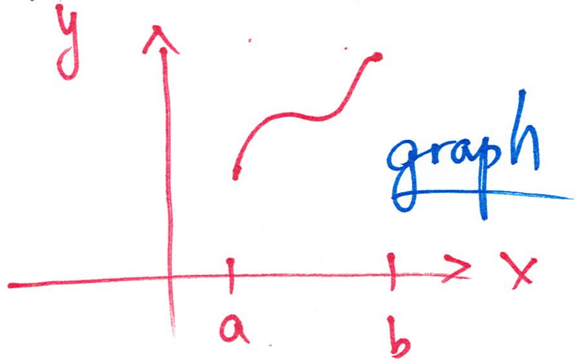
$$\{ (x, f(x)) \mid x \in [a, b] \}$$



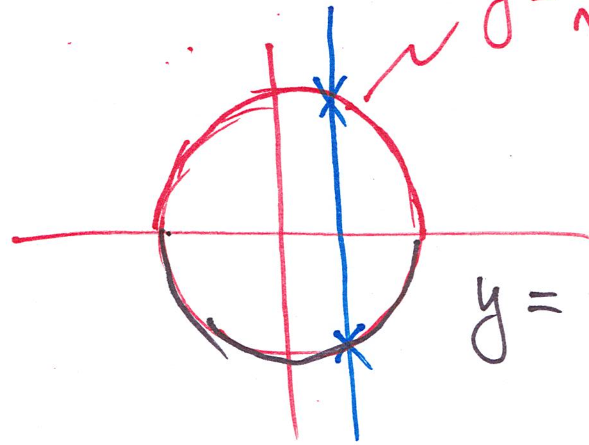
- Representations of curves in plane and space

in plane

$y=f(x)$, $x\in[a,b]$

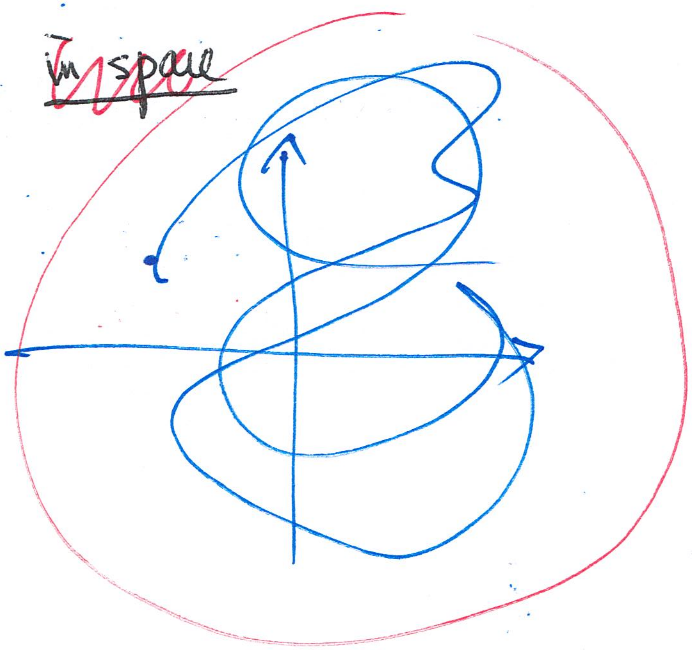


$y=\sqrt{1-x^2}$



$y=-\sqrt{1-x^2}$

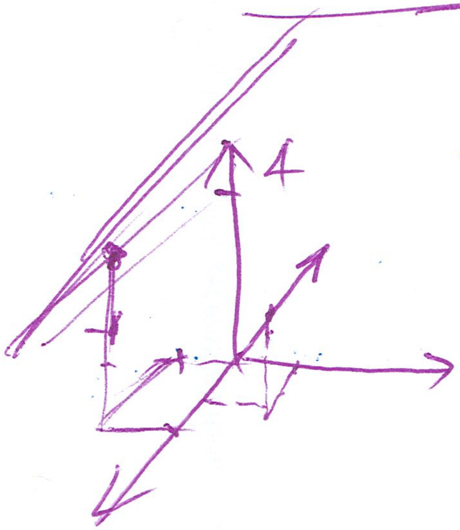
in space



$f(x,y)=x^2+y^2=1$

level curve

$\vec{r}(t) = \langle 2+t, -1+t, 4+2t \rangle$ $\langle 1, 1, 2 \rangle$



$x = 2+t$

$y = -1+t$

$z = 4+2t$