Lesson 6

§14.2 Calculus of vector-valued functions

- $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad t \in [a, b]$

- derivative

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{(t+\Delta t) - t = \Delta t} = \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

#11 $\frac{d}{dt} \langle 2t^3, 6\sqrt{t}, \frac{3}{t} \rangle$;

$$= \left\langle \frac{d}{dt}(2t^3), \frac{d}{dt}(6\sqrt{t}), \frac{d}{dt}\left(\frac{3}{t}\right) \right\rangle$$

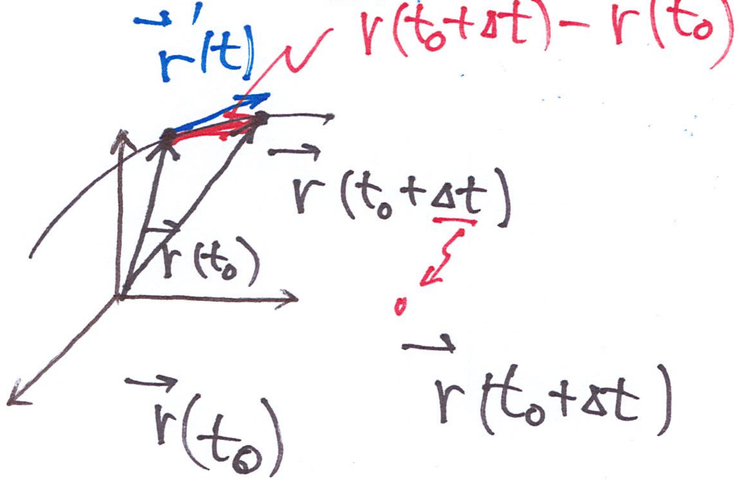
$$= \langle 6t^2, 3t^{-\frac{1}{2}}, -3t^{-2} \rangle$$

#16 $\frac{d}{dt} \langle (t+1)^{-1}, \tan^{-1} t, \ln(t+1) \rangle$

$$= \left\langle -(t+1)^{-2}, \frac{1}{1+t^2}, \frac{1}{t+1} \right\rangle$$

• tangent vector

$\vec{r}(t)$



$$\vec{r}\,'(t_0) \ ? = \lim_{\Delta t \to 0} \frac{\vec{r}(t_0+\Delta t)-\vec{r}(t_0)}{\Delta t}$$

unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Find unit tangent vector

#24 $\vec{r}(t) = \langle \cos t, \sin t, 2 \rangle$

$t \in [0, 2\pi]$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 0^2} = 1$$

$$\vec{T}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$$

#28 $\vec{r}(t) = \langle e^{2t}, 2e^{2t}, 2e^{-3t} \rangle$

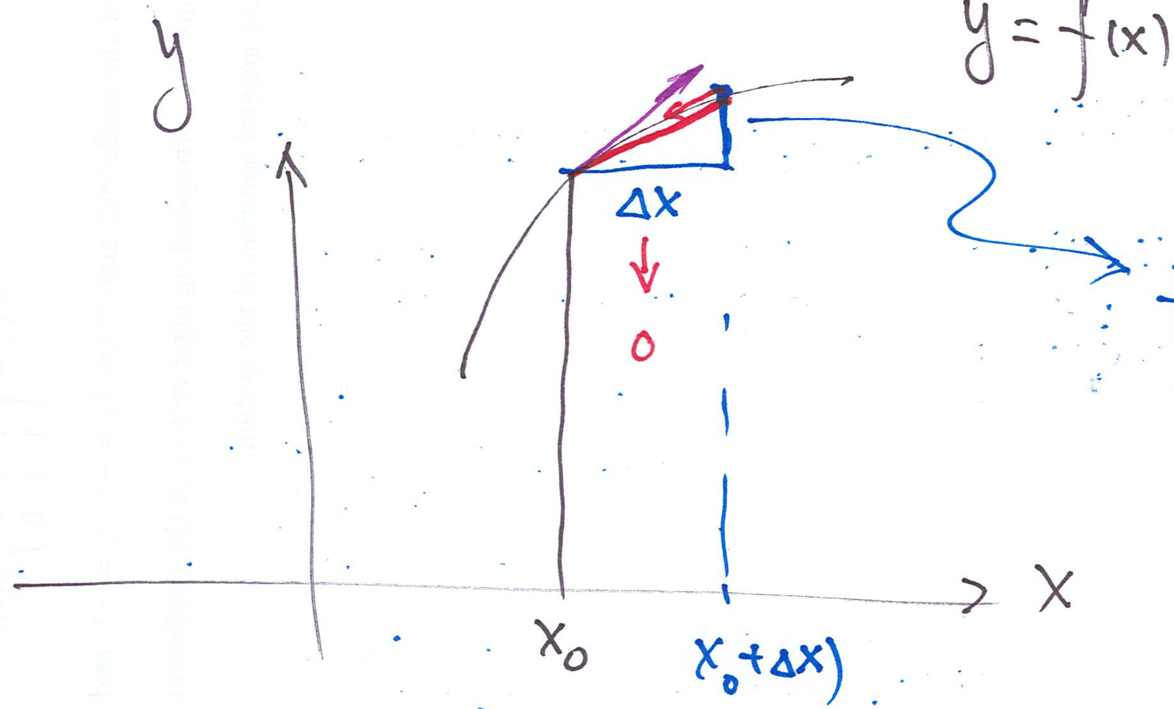
at $t=1$

$$\vec{r}\,'(t) = \langle 2e^{2t}, 4e^{2t}, -6e^{-3t} \rangle$$

$$\vec{r}\,'(1) = \langle 2e^{2}, 4e^{2}, -6e^{-3} \rangle$$

$$\vec{T}(t) = \frac{\langle 2e^2, 4e^2, -6e^{-3} \rangle}{\sqrt{4e^4+16e^4+36e^{-6}}}$$

$y = f(x)$ $\qquad$ $y' = f'(x)$



$f'(x_0)$?

$$\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

- derivative rules $\quad \vec{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle, \ \vec{v}(t) = \langle v_1(t), v_2(t), v_3(t) \rangle$

(1) $\frac{d}{dt}\vec{c} = \frac{d}{dt}\langle c_1, c_2, c_3 \rangle = \langle 0, 0, 0 \rangle = \vec{0}$

(2) $\left[\vec{u}(t) + \vec{v}(t)\right]' = \frac{d}{dt}\langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle = \langle u_1'+v_1', u_2'+v_2', u_3'+v_3' \rangle$

$= \langle u_1', u_2', u_3' \rangle + \langle v_1', v_2', v_3' \rangle$

$= \vec{u}'(t) + \vec{v}'(t)$

(3) $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

(4) $\frac{d}{dt}\vec{u}(f(t)) = \vec{u}'(f(t))\, f'(t)$

(5) $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

(6) $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

#34 $\frac{d}{dt}\left[(4t^8-6t^3)\vec{v}(t)\right] = (32t^7-18t^2)\vec{v}(t) + (4t^8-6t^3)\langle e^t, -2e^{-t}, -2e^{2t}\rangle$

$\vec{v}(t) = \langle e^t, 2e^{-t}, -e^{2t}\rangle$

#38 $\vec{u}(t) = \langle 2t^3, t^2-1, -8\rangle$

$\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) =$

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 2t^3 & t^2-1 & -8 \\ e^t & 2e^{-t} & -e^{2t} \end{vmatrix} = \langle -(t^2-1)e^{2t}+16e^{-2},\ 2t^3e^{2t}+8e^t,\ 4t^3e^{-t}-(t^2-1)e^{t}$$

$\vec{u}\cdot\vec{v} = t^2 - 2t^2 + t^2 = 0$

#48 $\vec{u}(t) = \langle 1, t, t^2\rangle$

$\vec{v}(t) = \langle t^2, -2t, 1\rangle$

$\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = 0$

#56 $\vec{r}(t) = \langle e^{4t}, 2e^{-t}+1, 2e^{-t}\rangle$

$\vec{r}\,'(t) = \langle 4e^{4t}, -2e^{-t}, -2e^{-t}\rangle$

$\vec{r}\,''(t) = \left(\vec{r}\,'(t)\right)' = \langle 16e^{4t}, 2e^{-t}, 2e^{-t}\rangle$

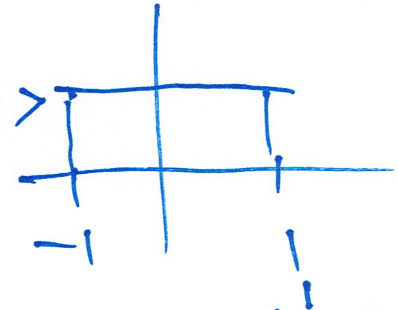
$\vec{r}\,'''(t) = \left(\vec{r}\,''(t)\right)' = \langle 64e^{4t}, -2e^{-t}, -2e^{-t}\rangle$

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

- integral

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt \right\rangle$$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$



#71 $$\int_{-1}^{1} \langle 1, t, 3t^2 \rangle\,dt = \left\langle \int_{-1}^{1} 1\,dt,\ \int_{-1}^{1} t\,dt,\ \int_{-1}^{1} 3t^2\,dt \right\rangle = \left\langle 2, 0, 2\int_0^1 3t^2\,dt \right\rangle$$

$$= \left\langle 2, 0, 2t^3 \Big|_0^1 \right\rangle = \langle 2, 0, 2 \rangle$$

#72 $$\int_0^1 \langle te^t, t\sin t, 1 \rangle\,dt = \left\langle \int_0^1 te^t\,dt,\ \int_0^1 t\sin t\,dt,\ \int_0^1 1\,dt \right\rangle$$

$(uv)' = u'v + uv'$

$\int uv'\,dt = uv - \int u'v\,dt$

$u = t,\ v' = e^t$
$u' = 1,\ v = e^t$

$u = t,\ v' = \sin t$
$u' = 1,\ v = -\cos t$

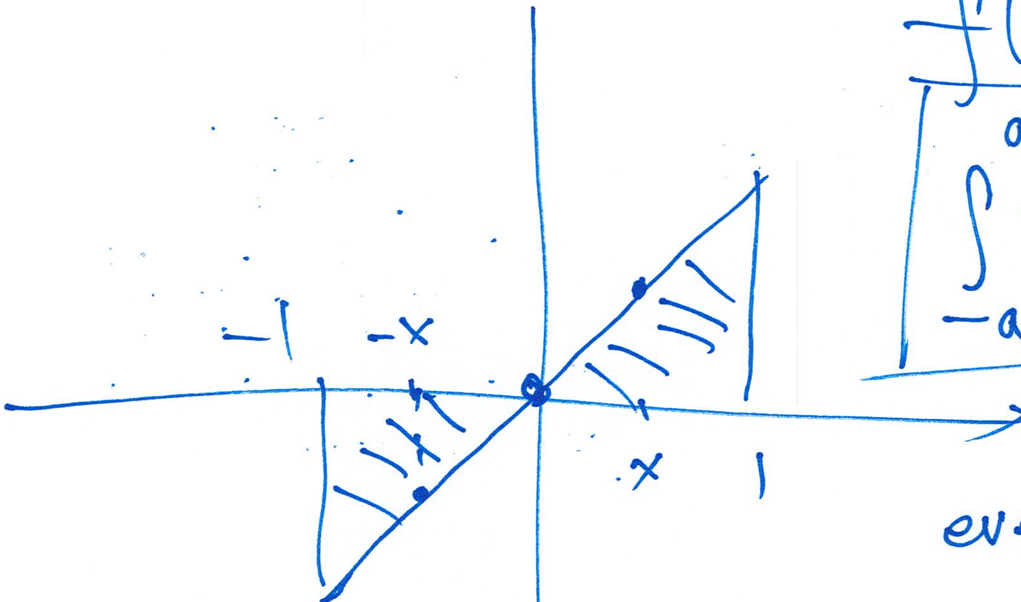
$$\left\langle te^t\Big|_0^1 - \int_0^1 e^t\,dt,\ -t\cos t\Big|_0^1 + \int_0^1 \cos t\,dt,\ 1 \right\rangle$$

$$= \left\langle e - e^t\Big|_0^1,\ -\cos 1 + \sin t\Big|_0^1,\ 1 \right\rangle = \langle 1, \sin 1 - \cos 1, 1 \rangle$$

odd function

$f(-x) = -f(x)$

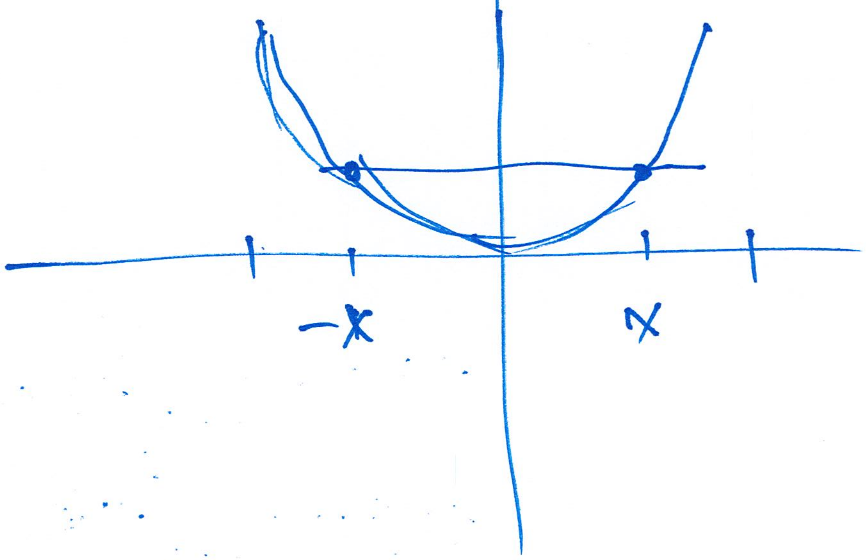
$$\boxed{\int_{-a}^{a} f(x)\,dx = 0}$$



even function

$f(-x) = f(x)$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



§14.3 Motion in Space

- position of a moving particle

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

- velocity $\quad \vec{v}(t) = \vec{r}\,'(t)$

  speed $\quad |\vec{v}(t)| = |\vec{r}\,'(t)|$

- acceleration $\quad \vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

$$\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}$$

$$1 = \cos^2 t + \sin^2 t = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 =$$

Ex. 1 Consider two-dimensional motion given by the position vector

$$\vec{r}(t) = \langle x(t), y(t) \rangle = \langle 3\cos t, 3\sin t \rangle \quad t \in [0, 2\pi]$$

(a) sketch the trajectory of the object; (b) find velocity, speed, acceleration of the object.

(c) sketch $\vec{r}(t)$, $\vec{v}(t)$, $\vec{a}(t)$ at $t = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$.

$$\vec{r}(0) = \langle 3, 0 \rangle$$

$$\vec{v}(0) = \langle -3\sin t, 3\cos t \rangle \Big|_{t=0} = \langle 0, 3 \rangle$$

$$\vec{a}(0) = \langle -3\cos t, -3\sin t \rangle \Big|_{t=0} = \langle -3, 0 \rangle$$