§14.3 motion in space

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = \vec{r}\,'(t)$$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

Lesson 7

Ex. 2 (comparing trajectories) Consider the trajectories described by the position functions

$$\vec{r}(t) = \langle t,\ t^2-4,\ \frac{t^3}{4}-8 \rangle$$
$$\vec{R}(t) = \langle t^2,\ t^4-4,\ \frac{t^6}{4}-8 \rangle$$

for $t \geq 0$

(a) Find the interval $[a,b]$ over which the $\vec{R}$-trajectory is the same as the $\vec{r}$ trajectory over $[0,2]$

(b) Find the velocity;

(c) Graph the speed of $\vec{r}$ over $[0,2]$ and $\vec{R}$ over $[a,b]$ ~~[c,d]~~

$[a,b] = [0,\sqrt{2}]$

(a) $\vec{R}(t) = \vec{r}(t^2)$ $\quad t^2 \in [0,2]$

$t^2 = 0 \Rightarrow t = 0.$

$t^2 = 2 \Rightarrow t = \pm\sqrt{2} \Rightarrow t = \sqrt{2}$

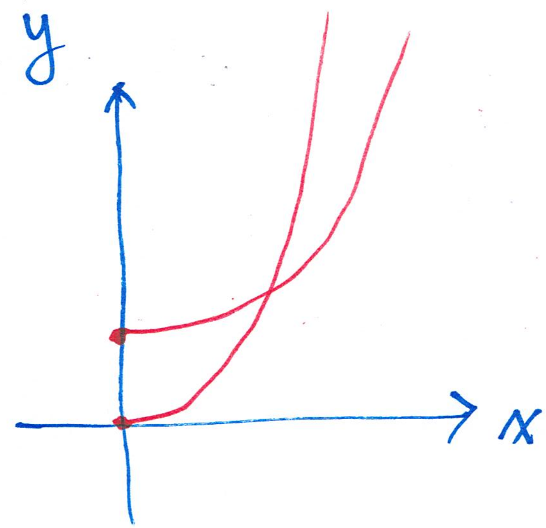
(b) $\vec{v}(t) = \vec{r}\,'(t) = \langle 1,\ 2t,\ \frac{3}{4}t^2 \rangle$

$\vec{V}(t) = \vec{R}\,'(t) = \langle 2t,\ 4t^3,\ \frac{3}{2}t^5 \rangle$

$s = |\vec{v}(t)| = \sqrt{1 + 4t^2 + \frac{9}{16}t^4} \sim \frac{3}{4}t^2$ $\quad s(0) = 1$

$S = |\vec{V}(t)| = \sqrt{4t^2 + 16t^6 + \frac{9}{4}t^{10}}$, $S(0) = 0$

$\sim \frac{3}{2}t^5$



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$$\vec{a}(t) = \langle \sin t, \cos t, 1\rangle, \quad \overset{\vec{v}(0)}{\overset{\|}{\langle u_0, v_0, w_0\rangle}} = \langle 0, 2, 0\rangle, \quad \overset{\vec{r}(0)}{\overset{\|}{\langle x_0, y_0, z_0\rangle}} = \langle 0, 0, 0\rangle$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \left\langle \int \sin t\,dt, \int \cos t\,dt, \int 1\,dt \right\rangle$$

$$= \langle -\cos t + c_1, \sin t + c_2, t + c_3\rangle$$

$$\langle 0, 2, 0\rangle = \vec{v}(0) = \langle -1 + c_1, c_2, c_3\rangle \implies c_1 = 1, \; c_2 = 2, \; 0$$

$$\implies \vec{v}(t) = \langle 1 - \cos t, \sin t + 2, t\rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \int (1 - \cos t)\,dt, \int (\sin t + 2)\,dt, \int t\,dt \right\rangle$$

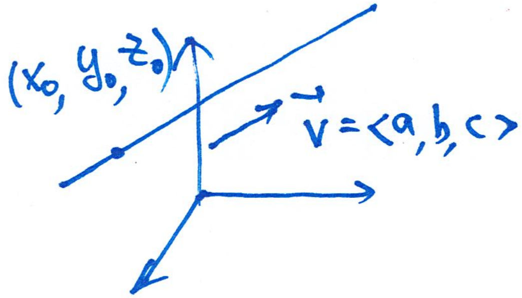
$$= \langle t - \sin t + d_1, -\cos t + 2t + d_2, \tfrac{1}{2}t^2 + d_3\rangle$$

$$\langle 0, 0, 0\rangle = \vec{r}(0) = \langle d_1, -1 + d_2, d_3\rangle \implies d_1 = 0, \; d_2 = 1, \; d_3 = 0$$

$$\implies \vec{r}(t) = \langle t - \sin t, -\cos t + 2t + 1, \tfrac{1}{2}t^2\rangle$$

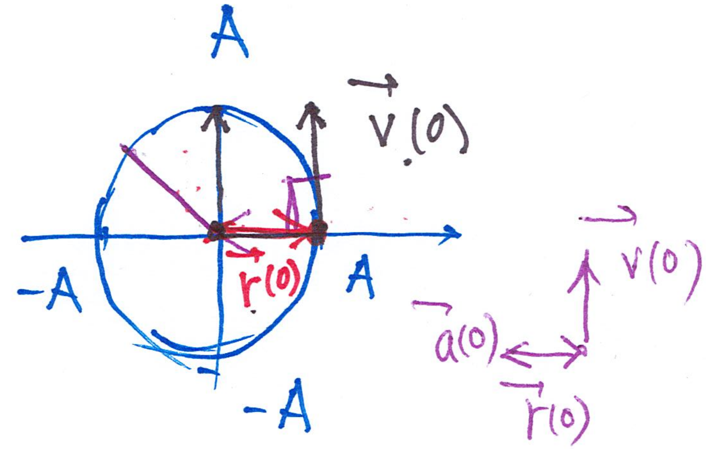
- straight-line motion

$$\vec{r}(t) = \langle x_0+at,\ y_0+bt,\ z_0+ct\rangle$$



$$\vec{v}(t)=\vec{r}\,'(t) = \langle a,\ b,\ c\rangle$$

$$\vec{a}(t)=\vec{v}\,'(t)=\langle 0,0,0\rangle$$



- circular motion

$$\vec{r}(t)=\langle A\cos t,\ A\sin t\rangle,\quad 0\le t\le 2\pi$$

$$\vec{r}(0)=\langle A, 0\rangle$$

$$\vec{a}(t)=\langle -A\cos t, -A\sin t\rangle$$

$$\vec{a}(0)=\langle -A, 0\rangle$$

$$\vec{v}(t)=\vec{r}\,'(t)=\langle -A\sin t,\ A\cos t\rangle$$

$$\vec{v}(0)=\langle 0,\ A\rangle$$

$$\begin{cases} x = A\cos t \\ y = A\sin t\end{cases}\qquad 1=\cos^2 t+\sin^2 t = \left(\frac{x}{A}\right)^2+\left(\frac{y}{A}\right)^2$$

$$\Rightarrow x^2+y^2=A^2$$

$$\sqrt{\vec{r}\cdot\vec{r}} = |\vec{r}| = c$$

$$\frac{d}{dt}\,\vec{r}\cdot\vec{r} = \frac{dc^2}{dt} = 0$$

$$\vec{r}\,'\cdot\vec{r}+\vec{r}\cdot\vec{r}\,' = 2\,\vec{r}\cdot\vec{r}\,'$$

Theorem (motion with constant $|\vec{r}|$)

$\vec{r}$ is a path on which $|\vec{r}|$ is a const $\Rightarrow \boxed{\vec{r}\perp\vec{v}} \Leftrightarrow \vec{r}\cdot\vec{v}=0$

(motion on a circular/sphere centered at the origion)

$$\vec{r}\cdot\vec{r}\,'=0$$

Ex. 3 (path on a sphere) An object moves on a trajectory described by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle 3\cos t, 5\sin t, 4\cos t \rangle \quad \text{for } t \in [0, 2\pi]$$

(a) show that it moves on a sphere and find the radius of the sphere; (b) find the velocity and the speed;

(c) Consider the curve $\vec{r}(t) = \langle 5\cos t, 5\sin t, 5\sin 2t \rangle$.

Show that it does not lie on a sphere. $\Leftrightarrow$

How could $\vec{r}$ be modified so that it describes a curve that lies on a sphere of radius 1, centered at the origion?

(a) $|\vec{r}| = \sqrt{9\cos^2 t + 25\sin^2 t + 16\cos^2 t} = \sqrt{25\cos^2 t + 25\sin^2 t} = \underset{\rho}{\underset{\parallel}{5}}$ — const.

(b) $\vec{v} = \vec{r}\,' = \langle -3\sin t, 5\cos t, -4\sin t \rangle$

$|\vec{v}| = \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t} = 5$

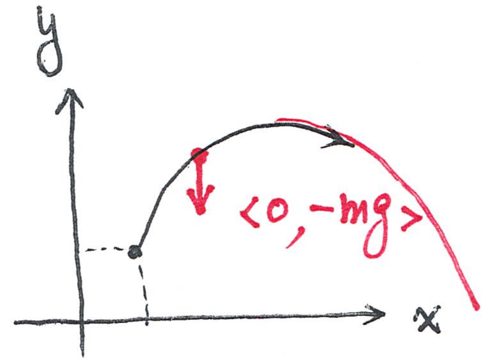
(c) $|\vec{r}| = \sqrt{25\cos^2 t + 25\sin^2 t + 25\sin^2 2t} = 5\sqrt{1 + \sin^2 2t} \neq$ constant

$\Rightarrow$ not on a sphere

$$\vec{R}(t) = \frac{\vec{r}(t)}{|\vec{r}(t)|} \Rightarrow |\vec{R}(t)| = \underline{\underline{1}}$$

$$\left|\frac{\vec{r}(t)}{|\vec{r}(t)|}\right| = \frac{1}{|\vec{r}(t)|}\,|\vec{r}(t)|$$

- Two-dimensional motion in a gravitational field



Newton's 2nd Law

$$m\vec{a} = \vec{F} = \langle 0, -mg\rangle \Rightarrow \boxed{\vec{a} = \langle 0, -g\rangle}$$

$$g = 9.8\ m/s^2 = 32\ ft/s^2$$

$\vec{a}(t)$

$$\begin{cases} \vec{v}'(t) = \vec{a}(t) \Rightarrow \vec{v}(t) = \int \vec{a}(t)\,dt = \langle c_1, -gt + c_2\rangle \\ \vec{v}(0) = \langle u_0, v_0\rangle = \vec{v}(0) = \langle c_1, c_2\rangle \Rightarrow \boxed{\vec{v}(t) = \langle u_0, -gt + v_0\rangle} \end{cases}$$

$$\begin{cases} \vec{r}'(t) = \vec{v}(t) & \vec{r}(t) = \int \vec{v}(t)\,dt = \langle \int u_0\,dt, \int(-gt + v_0)\,dt\rangle \\ \vec{r}(0) = \langle x_0, y_0\rangle & = \langle u_0 t + d_1, -\frac{1}{2}gt^2 + v_0 t + d_2\rangle \Rightarrow \boxed{\vec{r}(t) = \langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0\rangle} \end{cases}$$

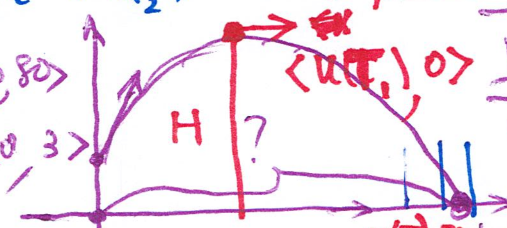
$$\vec{r}(0) = \langle d_1, d_2\rangle$$

Example 4 (~~fly~~ flight of a baseball)

A baseball is hit from 3 ft above home plate with $\vec{v}(0) = \langle 80, 80\rangle$ in ft/s. Neglect all forces other than gravity.

$\vec{v}(0) = \langle 80, 80\rangle$

$\vec{r}(0) = \langle 0, 3\rangle$



$\vec{r}(t) = \langle 80t, -16t^2 + 80t + 3\rangle$

$\vec{v}(t) = \langle 80, -32t + 80\rangle$

$$\begin{cases} x = 80t \\ y = -16t^2 + 80t + 3 \end{cases} \quad t = \frac{x}{80} \Rightarrow = -\frac{16}{80^2}x^2 + x + 3$$

(a) Find $\vec{r}(t)$, $\vec{v}(t)$

(b) Show that the trajectory of the ball is a segment of a parabola.

(c) How far does the ball travel horizontally? Plot its trajectory.

$y(T) = 0 = -16T^2 + 80T + 3 \Rightarrow T = ?$

$\Rightarrow x(T) = ?$

(d) What is the max. height of the ball? $0 = v(T_1) = -32T_1 + 80 \Rightarrow T_1 = \frac{80}{32} \Rightarrow H = y(T_1)$

(e) Does the ball clear a 20 ft fence that is 380 ft from home plate?

$x(T_2) = 380 \Rightarrow T_2 = ?$ $\quad y(T_2) > 20$