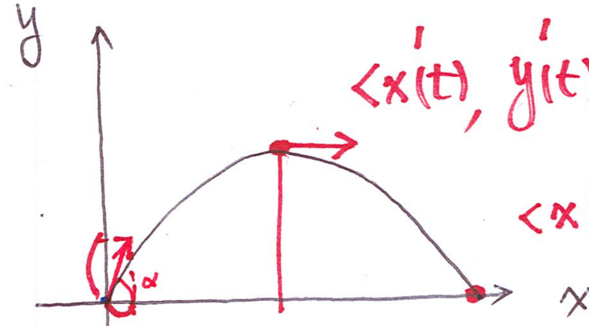
$\vec{r}(t) = \langle x(t), y(t) \rangle$

initial position $\vec{r}(0) = \underline{\langle 0, 0 \rangle}$

initial speed $|\vec{v}_0|$ at the angle α

$\vec{v}_0 = \langle |\vec{v}_0| \cos\alpha, |\vec{v}_0| \sin\alpha \rangle$



$\langle x'(t), y'(t) = 0 \rangle$

$\langle x(T), y(T) = 0 \rangle$

$\vec{r}(t) = \langle u_0 t + x_0, -\frac{1}{2} g t^2 + v_0 t + y_0 \rangle$

$= \langle |\vec{v}_0| t \cos\alpha, -\frac{1}{2} g t^2 + |\vec{v}_0| t \sin\alpha \rangle$

- <u>time of flight</u> $y(T) = 0$

$$0 = y(T) = -\frac{1}{2} g T^2 + |\vec{v}_0| T \sin\alpha = \left(-\frac{1}{2} g T + |\vec{v}_0| \sin\alpha\right) T$$

$$0 = -\frac{1}{2} g T + |\vec{v}_0| \sin\alpha \Longrightarrow \underline{T = \frac{2|\vec{v}_0| \sin\alpha}{g}}$$

- <u>range of the object</u> $x(T) = ?$

$$x(T) = |\vec{v}_0| \cos\alpha \cdot \frac{2|\vec{v}_0| \sin\alpha}{g} = \frac{|\vec{v}_0|^2 \sin 2\alpha}{g}$$

- <u>maximum height of the object</u> $y'(t) = 0$

$$0 = y'(t) = -g t + |\vec{v}_0| \sin\alpha$$

$$\Longrightarrow t = \frac{|\vec{v}_0| \sin\alpha}{g} = \frac{T}{2}$$

$$\Longrightarrow H = y\left(\frac{T}{2}\right) = -\frac{1}{2} g \left(\frac{T}{2}\right)^2 + |\vec{v}_0| \frac{T}{2} \sin\alpha$$

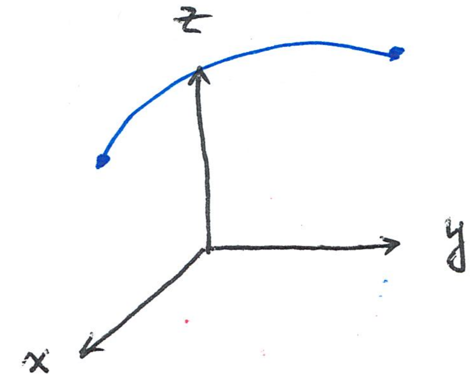
$$= -\frac{1}{2} g \left(\frac{|\vec{v}_0| \sin\alpha}{g}\right)^2 + |\vec{v}_0| \frac{|\vec{v}_0| \sin\alpha}{g} \sin\alpha$$

$$\boxed{\frac{|\vec{v}_0|^2 \sin^2\alpha}{2g} = y\left(\frac{T}{2}\right)}$$

Lesson 8

$$(a-b)^2+(a+b)^2=[a^2+b^2-\cancel{2ab}]+[a^2+b^2+\cancel{2ab}]$$
$$=2(a^2+b^2)$$

§14.4 Length of Curves



curve C : $\vec{r}(t)=\langle f(t), g(t), h(t)\rangle$

$a\le t\le b$

- arc length

$$L=\int_a^b |\vec{r}\,'(t)|\,dt$$

$$|\vec{r}\,'(t)| = \sqrt{(-5\sin t^2\cdot 2t)^2+(5\cos t^2\cdot 2t)^2+(24}$$

#14 $\vec{r}(t)=\langle \cos t+\sin t, \cos t-\sin t\rangle$

$t\in[0,2\pi]$

$$L=\int_0^{2\pi}\left|\langle -\sin t+\cos t, -\sin t-\cos t\rangle\right| dt$$

$$=\int_0^{2\pi}\sqrt{(\cos t-\sin t)^2+(\cos t+\sin t)^2}\,dt$$

$$=\int_0^{2\pi}\sqrt{2}\,dt=\sqrt{2}\cdot 2\pi$$

#24 Find the speed and the length of the trajectory

$\vec{r}=\langle 5\cos t^2, 5\sin t^2, 12t^2\rangle$, $t\in[0,2]$

$$|\vec{r}\,'(t)|=\sqrt{10^2+24^2}\,t$$

$$L=\int_0^2\sqrt{10^2+24^2}\,t\,dt$$

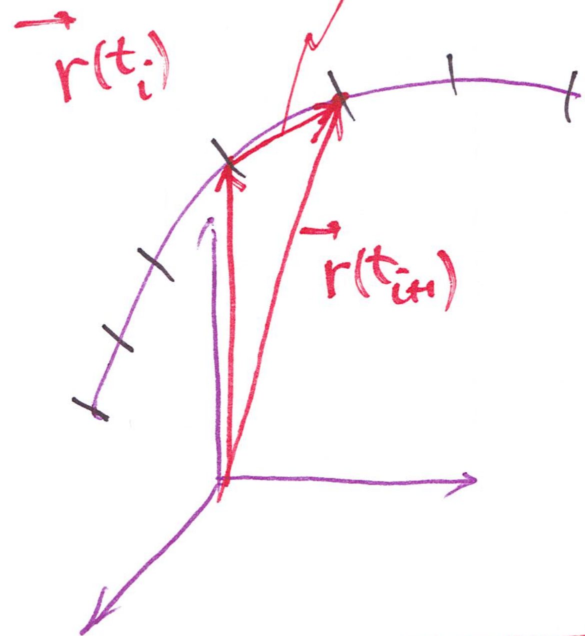
$$=\sqrt{10^2+24^2}\,\frac{1}{2}t^2\Big|_0^2=4\sqrt{169}=52$$

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$a \le t \le b$

$$\left|\vec{r}(t_{i+1}) - \vec{r}(t_i)\right| = \left|\langle x(t_{i+1}) - x(t_i),\ y(t_{i+1}) - y(t_i) \rangle\right|$$

$$= \sqrt{\left(x(t_{i+1}) - x(t_i)\right)^2 + \left(y(t_{i+1}) - y(t_i)\right)^2}$$

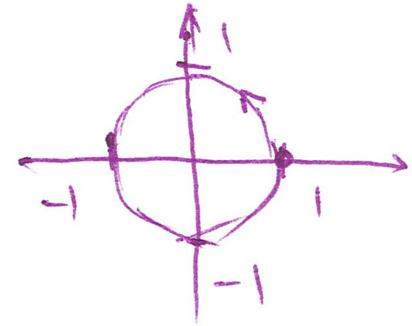


$$\lim_{\Delta t \to 0} \sum_i \sqrt{\frac{\left(x(t_{i+1}) - x(t_i)\right)^2 + \left(y(t_{i+1}) - y(t_i)\right)^2}{(\Delta t)^2}}\ \Delta t$$

$$= \lim_{\Delta t \to 0} \sum_i \sqrt{\left(\frac{x(t_{i+1}) - x(t_i)}{\Delta t}\right)^2 + \left(\frac{y(t_{i+1}) - y(t_i)}{\Delta t}\right)^2}\ \Delta t$$

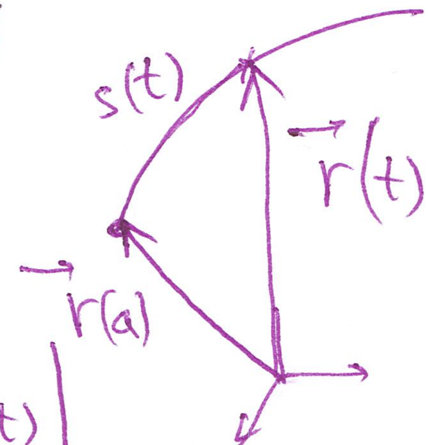
$$= \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\ dt = \boxed{\int_a^b \left|\vec{r}\,'(t)\right| dt = L}$$

- arc length as a parameter



Example C: unit circle $x^2+y^2=1$

parametrizations
$$\begin{cases} \vec{r}_1(t) = \langle \cos t, \sin t \rangle, & t \in [0, 2\pi] \\ \vec{r}_2(t) = \langle \cos 2t, \sin 2t \rangle, & t \in [0, \pi] \end{cases}$$



the arc length function

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du \Leftrightarrow \frac{ds(t)}{dt} = |\vec{r}\,'(t)|$$

ex. 3 (arc length parametrization) $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 4 \rangle$

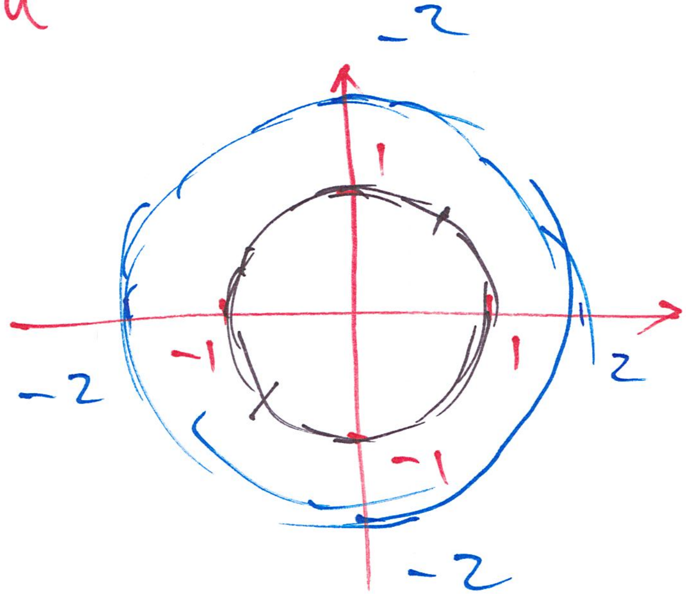
$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4t \rangle$, $t \geq 0$ (helix) $|\vec{r}\,'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 16} = \sqrt{4+16} = 2\sqrt{5}$

(a) Find $s(t) = \int_0^t 2\sqrt{5}\,du = 2\sqrt{5}\,t \Longrightarrow t = \frac{s}{2\sqrt{5}}$

(b) use arc length as the parameter to describe the helix. $\vec{R}(s) = \vec{r}(t) = \vec{r}\left(\frac{s}{2\sqrt{5}}\right) = \left\langle 2\cos\frac{s}{2\sqrt{5}}, 2\sin\frac{s}{2\sqrt{5}}, \cancel{4}\,\frac{2s}{\sqrt{5}} \right\rangle$

$$x^2 + y^2 = a^2$$

$$K = \frac{1}{a}$$



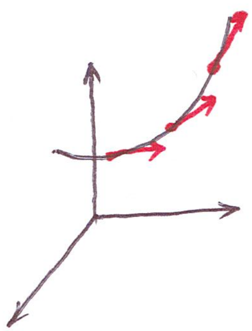
§14.5 Curvature and Normal Vectors

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}}$$

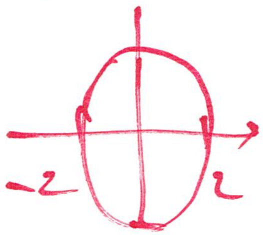
- curvature

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$\vec{r}'(t)$ unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$



$$K = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right| = \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|}$$



$$\boxed{K = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{r}'(t)|}}$$

#12 $\vec{r}(t) = \langle 2\cos t, -2\sin t \rangle$

Compute $\vec{T}$ and K

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -2\sin t, -2\cos t \rangle}{\sqrt{4\sin^2 t + 4\cos^2 t}}$$

$$= \langle -\sin t, -\cos t \rangle$$

$$\frac{d\vec{T}}{dt} = \langle -\cos t, \sin t \rangle \qquad \left|\frac{d\vec{T}}{dt}\right| = 1$$

$$K = \frac{1}{2}$$

#14 $\vec{r} = \langle \cos t^2, \sin t^2 \rangle$

Compute $\vec{T}$ and $K = 1$ $\qquad |\vec{r}'(t)| = 2t$

$$\vec{r}'(t) = \langle -2t\sin t^2, 2t\cos t^2 \rangle$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \langle -\sin t^2, \cos t^2 \rangle$$

$$\vec{T}'(t) = \langle -2t\cos t^2, -2t\sin t^2 \rangle$$

$$= -2t\langle \cos t^2, \sin t^2 \rangle$$

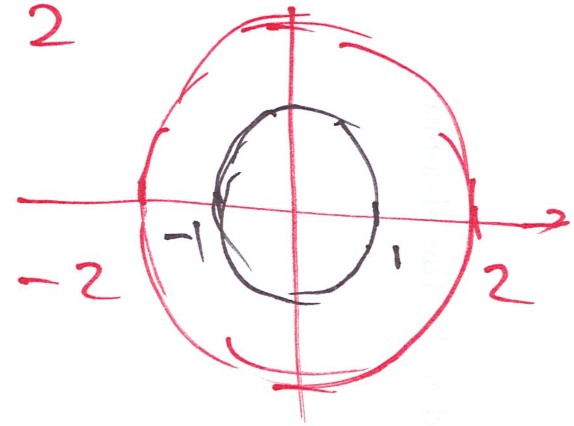
$$|\vec{T}'(t)| = 2t \Rightarrow K = \frac{1}{|\vec{r}'(t)|}|\vec{T}'(t)| = 1$$

$\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$

$x = 2\cos t$, $y = 2\sin t$

$x^2 + y^2 = 2^2$

$\kappa = \frac{1}{2}$



$\vec{r}(t) = \langle \cos t^2, \sin t^2 \rangle$

$x = \cos t^2$, $y = \sin t^2$

$$x^2 + y^2 = \left(\cos t^2\right)^2 + \left(\sin t^2\right)^2 = 1$$

$\kappa = 1$

$\vec{r}(t) = \langle a\cos t, a\sin t \rangle$

$x^2 + y^2 = \underline{\underline{a}}^2$

$\kappa = \frac{1}{a}$

#22 $\vec{r}(t) = \langle 4t, 3\sin t, 3\cos t \rangle$

$\vec{v} = \vec{r}\,'(t)$

$\vec{a} = \vec{r}\,''(t)$

Compute $\kappa = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$

$\vec{v}(t) = \vec{r}\,'(t) = \langle 4, 3\cos t, -3\sin t \rangle$, $|\vec{v}| = \sqrt{4^2 + 3^2(\cos^2 t + \sin^2 t)} = \sqrt{16 + 9} = 5$

$\vec{a}(t) = \vec{v}\,'(t) = \langle 0, -3\sin t, -3\cos t \rangle$

$$\vec{v} \times \vec{a} = \begin{vmatrix} i & j & k \\ 4 & 3\cos t & -3\sin t \\ 0 & -3\sin t & -3\cos t \end{vmatrix} = \langle -9, 12\cos t, -12\sin t \rangle$$

$$|\vec{v} \times \vec{a}| = \sqrt{9^2 + 12^2(\cos^2 t + \sin^2 t)} = \sqrt{9^2 + 12^2} = 15$$

$$\kappa = \frac{15}{5^3} = \frac{3}{25}$$