#7 Find the point on the plane $3x+4y+z=8$ that is nearest to $(2,0,1)$

(x,y,z) on the plane : $z = 8-3x-4y$

$$d(x,y) = \sqrt{(x-2)^2 + y^2 + (8-3x-4y-1)^2}$$ critical pts

$$g(x,y) = d^2(x,y) = (x-2)^2 + y^2 + (3x+4y-7)^2$$

$$g_x = 2(x-2) + 2(3x+4y-7)\cdot 3 = 2\left[10x+12y-23\right] = 0$$

$$g_y = 2y + 2(3x+4y-7)\cdot 4 = 2\left[12x+17y-28\right] = 0$$

$$\begin{cases} 10x+12y = 23 \\ 12x+17y = 28 \end{cases} \Longrightarrow (x,y) = \left(\frac{55}{26}, \frac{2}{13}\right)$$

$$\begin{vmatrix} 10 & 12 \\ 12 & 17 \end{vmatrix} = 170 - 12^2 = 26 \neq 0$$

$$d\left(\frac{55}{26}, \frac{2}{13}\right) =$$

1. Identify the surface defined by $x^2 - y^2 - 4x + z^2 = 4$.

A. hyperboloid of one sheet

B. hyperbolic paraboloid

C. hyperboloid of two sheets

D. ellipsoid

E. cone

$$(x^2 - 4x + 4 - 4) - y^2 + z^2 = 4$$

$$(x-2)^2 - 4 - y^2 + t^2 = 4$$

$$(x-2)^2 - y^2 + z^2 = \underline{\underline{8}} > 0$$

$$\begin{cases} 2t-1=3 \Rightarrow t=2 \\ t^2 = 4 \\ t^2-2=2 \end{cases}$$

$\vec{r}(t) = \langle 2t-1, t^2, t^2-2\rangle$

$= ?$

2. If L is the tangent line to the curve $\vec{\mathbf{r}}(t) = \langle 2t-1,\ t^2,\ t^2-2\rangle$ at $(3,4,2)$, find the point where L intersects the xy-plane.

A. $(2,1,0)$

B. $(1,2,0)$

C. $(2,-2,0)$

D. $(2,2,0)$

E. $(0,0,0)$

$L: (3,4,2),\ \vec{v} = ?$

$\vec{r}'(t) = \langle 2, 2t, 2t\rangle$

$\vec{v} = \vec{r}'(2) = \langle 2, 4, 4\rangle = 2\boxed{\langle 1, 2, 2\rangle}$

$\vec{l}(t) = \langle 3,4,2\rangle + t\langle 2,4,4\rangle$

$= \langle 3+2t, 4+4t, 2+4t\rangle$

xy-plane: $z = 0 = 2+4t \Rightarrow t = -\frac{1}{2}$

$\vec{l}(-\frac{1}{2}) = \langle 3-1, 4-2, 2-2\rangle$

$= \langle 2, 2, 0\rangle$

3. Let $\vec{\mathbf{v}} = \int_0^1 \left(\frac{1}{2}\vec{\imath} + 2t^3\vec{\jmath} + (t-3t^2)\vec{\mathbf{k}}\right)\,dt$. Compute $|\vec{\mathbf{v}}|$.

A. 1

B. $\frac{3}{2}$

C. $\frac{1}{4}$

D. $\frac{1}{2}$

E. $\frac{\sqrt{3}}{2}$

$$= \left\langle \frac{1}{2}t\Big|_0^1,\ \frac{1}{2}t^4\Big|_0^1,\ \frac{1}{2}t - t^3\Big|_0^1 \right\rangle$$

$$= \left\langle \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{1}{2}\langle 1, 1, -1\rangle$$

$$|\vec{v}| = \frac{1}{2}\sqrt{1^2+1^2+(-1)^2} = \frac{1}{2}\sqrt{3}$$

4. Find the area of the triangle with vertices at $P(2,2,1)$, $Q(1,-1,2)$, and $R(0,1,-1)$.

A. $\sqrt{5}$

B. $\frac{3\sqrt{10}}{2}$

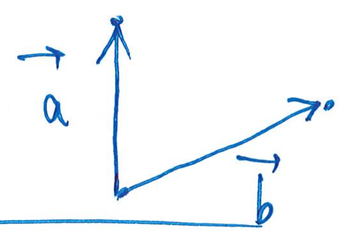
C. $\frac{\sqrt{31}}{2}$

D. $2\sqrt{5}$

E. $\frac{\sqrt{69}}{2}$

$$\vec{a} = \overrightarrow{PQ} = \langle -1, -3, 1\rangle$$

$$\vec{b} = \overrightarrow{PR} = \langle -2, -1, -2\rangle$$



$$A = \frac{1}{2}\left|\vec{a}\times\vec{b}\right| = \frac{1}{2}\sqrt{7^2+(-4)^2+(-5)^2}$$

$$= \frac{1}{2}\sqrt{49+16+25} = \frac{1}{2}\sqrt{90}$$

$$= \frac{3}{2}\sqrt{10}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{vmatrix} = \langle 7, -4, -5\rangle$$

$= k$ — const.

5. The level curves of $f(x,y) = \sqrt{x^2+4y^2+4} - x$ are

A. hyperbolas

B. ellipses

C. parabolas

D. sometimes lines and sometimes ellipses

E. circles

$$\sqrt{x^2+4y^2+4} = k + x$$

$$\cancel{x^2} + 4y^2 + 4 = (k+x)^2 = \cancel{x^2} + 2kx + k^2$$

$$4y^2 + 4 - k^2 = 2kx$$

6. Find the length of the curve: $\vec{r}\,'(t) = \langle 4\cos t,\ 3,\ 4\sin t \rangle$

$$\vec{\mathbf{r}}(t) = \langle 4\sin t,\ 3t,\ -4\cos t\rangle, \quad 0 \le t \le \frac{1}{2}.$$

A. $\frac{8}{3}\sinh^{-1}\left(\frac{3}{8}\right)$

B. $\frac{8}{3}\sinh^{-1}\left(\frac{3}{8}\right) + \frac{\sqrt{73}}{8}$

C. 2.5

D. 5

E. 5π

$$L = \int_0^{\frac{1}{2}} \left|\vec{r}\,'(t)\right| dt$$

$$= \int_0^{\frac{1}{2}} \sqrt{(4\cos t)^2 + 3^2 + (4\sin t)^2}\, dt$$

$$= \int_0^{\frac{1}{2}} \sqrt{4^2+3^2}\, dt = \int_0^{\frac{1}{2}} 5\, dt$$

$$= \frac{5}{2}$$

7. A particle is moving with acceleration

$$\vec{\mathbf{a}}(t) = \langle 6,\ 6t,\ 0 \rangle.$$

If at time $t = 1$, the particle has position $\vec{\mathbf{r}}(1) = \langle 2, 1, 2 \rangle$, and, at time $t = 0$ it has velocity $\vec{\mathbf{v}}(0) = \langle 0, 0, 1 \rangle$, compute $|\vec{\mathbf{r}}(2)|$, the magnitude of the position vector at $t = 2$.

A. $2\sqrt{53}$

B. $3\sqrt{21}$

C. $\sqrt{194}$

D. $\sqrt{293}$

E. $\sqrt{57}$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle \int 6, \int 6t, \int 0 \rangle$

$= \langle 6t + c_1, 3t^2 + c_2, c_3 \rangle$

$\langle 0,0,1 \rangle = \vec{v}(0) = \langle c_1, c_2, c_3 \rangle \Longrightarrow \vec{v}(t) = \langle 6t, 3t^2, 1 \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \int 6t, \int 3t^2, \int 1 \rangle = \langle 3t^2 + D_1, t^3 + D_2, t + D_3 \rangle$

$\langle 2,1,2 \rangle = \vec{r}(1) = \langle 3 + D_1, 1 + D_2, 1 + D_3 \rangle$

$\Rightarrow D_1 = -1, \ D_2 = 0, \ D_3 = 1 \quad \Longrightarrow \vec{r}(t) = \langle 3t^2 - 1, t^3, t+1 \rangle$

$\vec{r}(2) = \langle 11, 8, 3 \rangle, \quad |\vec{r}(2)| = \sqrt{11^2 + 8^2 + 3^2}$

$= \sqrt{121 + 64 + 9} = \sqrt{194}$

8. If $f(x, y) = x\sin(xy^2)$, then $f_{xy}(\pi, 1)$ is equal to

A. 4π

B. -4π

C. 2π

D. -2π

E. 0

$f_y = 2x^2y\cos(xy^2)$

$f_{xy} = 2\left[2xy\cos(xy^2) - 2x^2y \cdot y^2 \sin(xy^2)\right]$

$f_{xy}(\pi, 1) = 2\left[2\pi\cos\pi - 2\pi^2\sin\pi\right]$

$= -4\pi$

9. Let $f(x, y, z)$ be a function which is differentiable at $(1, 1, 1)$ and

$$\frac{\partial f}{\partial x}(1,1,1) = -6, \quad \frac{\partial f}{\partial y}(1,1,1) = 2, \quad \text{and} \quad \frac{\partial f}{\partial z}(1,1,1) = -1.$$

Let $\vec{\gamma}(t) = \langle x(t),\ y(t),\ z(t) \rangle$ be the parametric equation of a differentiable curve in $\mathbb{R}^3$ and suppose $\vec{\gamma}(0) = \langle 1, 1, 1 \rangle$ and $\frac{d\vec{\gamma}}{dt}(0) = 3\vec{\imath} - 3\vec{\jmath} + \mathbf{k}$. We can conclude that $\frac{d}{dt} f(\vec{\gamma}(t))$ at $t = 0$ is equal to

$$f(\vec{r}(t)) = f(x(t), y(t), z(t))$$

A. -25

B. -14

C. -13

D. -11

E. -4

$$\frac{d}{dt} f(\vec{r}(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$$= \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) + \frac{\partial f}{\partial z} \cdot z'(t)$$

$$\left.\frac{d}{dt} f(\vec{r}(t))\right|_{t=0} = \frac{\partial f}{\partial x}(\vec{r}(0)) \cdot x'(0) + \frac{\partial f}{\partial y}(\vec{r}(0)) \cdot y'(0) + \frac{\partial f}{\partial z}(\vec{r}(0)) z'(0)$$

$$= f_x(1,1,1) \cdot 3 + f_y(1,1,1) \cdot (-3) + f_z(1,1,1) \cdot 1$$

$$= (-6) \cdot 3 + 2 \cdot (-3) + (-1) \cdot 1$$

$$= -18 - 6 - 1 = -25$$

10. Which of the following is an equation for the plane tangent to the surface

$$z = \tan^{-1}(x^2 + y^2) \quad \text{at the point} \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)?$$

Hint: $\frac{d}{du}(\tan^{-1} u) = \frac{1}{1+u^2}$

A. $x + y - \frac{1}{2}z = \sqrt{2} - \frac{\pi}{8}$

B. $x + y - 2z = \sqrt{2} - \frac{\pi}{2}$

C. $x + y - \sqrt{2}z = \left(1 - \frac{\pi}{4}\right)\sqrt{2}$

D. $x + y - z = \sqrt{2} - \frac{\pi}{4}$

E. $x + y - 3z = \frac{\sqrt{2}}{3} - \frac{3\pi}{4}$

9. Let $f(x, y, z)$ be a function which is differentiable at $(1,1,1)$ and

$$\frac{\partial f}{\partial x}(1,1,1) = -6, \quad \frac{\partial f}{\partial y}(1,1,1) = 2, \quad \text{and} \quad \frac{\partial f}{\partial z}(1,1,1) = -1.$$

Let $\vec{\gamma}(t) = \langle x(t),\ y(t),\ z(t)\rangle$ be the parametric equation of a differentiable curve in $\mathbb{R}^3$ and suppose $\vec{\gamma}(0) = \langle 1,1,1\rangle$ and $\frac{d\vec{\gamma}}{dt}(0) = 3\vec{\imath} - 3\vec{\jmath} + \vec{\mathbf{k}}$. We can conclude that $\frac{d}{dt} f(\vec{\gamma}(t))$ at $t = 0$ is equal to

A. -25

B. -14

C. -13

D. -11

E. -4

$$\frac{d}{dt} f(\vec{r}(t))\Big|_{t=0} = \frac{\partial f}{\partial x}\cdot x'(t) + \frac{\partial f}{\partial y}\cdot y'(t) + \frac{\partial f}{\partial z} z'(t)$$

$$= \nabla f(\vec{r}(t))\cdot \vec{r}\,'(t)\Big|_{t=0} = \nabla f(\vec{r}(0))\cdot \vec{r}\,'(0)$$

$$= \langle -6, 2, -1\rangle \cdot \langle 3, -3, 1\rangle$$

$$= -18 - 6 - 1 = -25$$

10. Which of the following is an equation for the plane tangent to the surface

$$z = \tan^{-1}(x^2 + y^2) \quad \text{at the point} \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)?$$

Hint: $\frac{d}{du}(\tan^{-1} u) = \frac{1}{1+u^2}$

$z = f(x, y)$

~~$\vec{n} = \nabla f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$~~

A. $x + y - \frac{1}{2}z = \sqrt{2} - \frac{\pi}{8}$

B. $x + y - 2z = \sqrt{2} - \frac{\pi}{2}$

C. $x + y - \sqrt{2}z = \left(1 - \frac{\pi}{4}\right)\sqrt{2}$

D. $x + y - z = \sqrt{2} - \frac{\pi}{4}$

E. $x + y - 3z = \frac{\sqrt{2}}{3} - \frac{3\pi}{4}$

$$\frac{\partial f}{\partial x} = \frac{2x}{1+(x^2+y^2)^2}\Bigg|_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} = \frac{2\frac{1}{\sqrt{2}}}{1+\left(\frac{1}{2}+\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{1+(x^2+y^2)^2} = \frac{1}{\sqrt{2}}$$

$$z - z_0 = \nabla f(x_0, y_0)\cdot \langle x - x_0, y - y_0\rangle$$

$$z - \frac{\pi}{4} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle \cdot \left\langle x - \frac{1}{\sqrt{2}}, y - \frac{1}{\sqrt{2}}\right\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\left(x - \frac{1}{\sqrt{2}}\right) + \left(y - \frac{1}{\sqrt{2}}\right)\right]$$

$$x + y - \sqrt{2}z = \frac{2}{\sqrt{2}} - \frac{\pi}{4}\sqrt{2}$$

$$= \sqrt{2}\left(1 - \frac{\pi}{4}\right)$$

11. The absolute minimum value of $f(x, y) = 2 + x^2y^2$ in the region $\frac{x^2}{2} + y^2 \leq 1$ equals 2. The absolute maximum value of f in this region is

A. 4.5

B. 4

C. 3.5

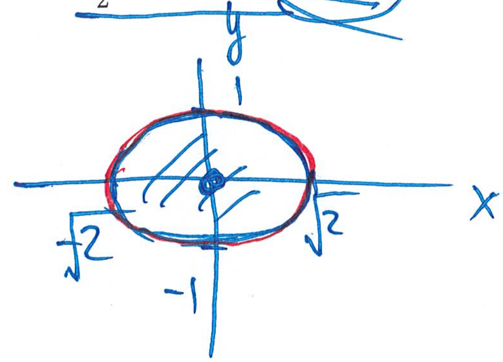
D. 3

E. 2.5

$f_x = 2xy^2 = 0$

$f_y = 2x^2y = 0 \Rightarrow (0,0)$

$f(0,0) = 2$



$\frac{x^2}{2} + y^2 = 1 \Rightarrow y^2 = 1 - \frac{x^2}{2}$ $\quad g' =$

$-\sqrt{2} \leq x \leq \sqrt{2}$

$g(x) = f\Big|_{\partial\Omega} = 2 + x^2\left(1 - \frac{x^2}{2}\right)$

$= 2 + x^2 - \frac{1}{2}x^4 = -\frac{1}{2}\left(x^4 - 2x^2 + 1 - 1\right) + 2$

$= -\frac{1}{2}\left[(x^2-1)^2\right] + 2 + \frac{1}{2}$

$\frac{5}{2}$

12. Let $z(x, y)$ be the function implicitly defined as the solution to

$$x + y + z + \sin(xyz) = 3 + \frac{\pi}{2}$$

$z(x,y)$

that satisfies $z(1, 1) = \frac{\pi}{2}$. Find $z_x(1, 1)$.

A. 1

B. -1

C. 2

D. 0

E. $\frac{3}{2}$

$0 = \frac{\partial}{\partial x}\left[x + y + z + \sin(xyz)\right]$

$= 1 + \frac{\partial z}{\partial x} + \left[yz + xy\frac{\partial z}{\partial x}\right]\cos(xyz)$

$(1, 1, \frac{\pi}{2})$

$\left[yz + xy\frac{\partial z}{\partial x}\cos(xyz)\right]$

$= 1 + yz\cos(xyz) + \left[1 + xy\cos(xyz)\right]\frac{\partial z}{\partial x}$

$\frac{\partial z}{\partial x}\Big|_{(1,1)} = -\frac{1 + yz\cos(xyz)}{1 + xy\cos(xyz)}\Big|_{(1,1)} = \frac{-(1 + \frac{\pi}{2})\cos\frac{\pi}{2}}{1 + \cos\frac{\pi}{2}} = -1$