Solutions for practice problems

Qinfeng Li

April 23, 2016

Problem 1. *If the line l has symmetric equations*

$$\frac{x-1}{2} = \frac{y}{-3} = \frac{z+2}{7},$$

find a vector equation for the line l' such that l' contains the pint (2,1,-3) and is parallel to l.

Solution. *Recall that if a line has symmetric equations*

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c},$$

then the line passes the point (x_0, y_0, z_0) and has the direction vector (a, b, c). By hypothesis, l has the direction $\boldsymbol{v} = (2, -3, 7)$, and so does l' since $l \parallel l'$. Also, since l' passes the point $\boldsymbol{r_0} = (2, 1, -3)$, therefore the vector equation for l' is

$$\boldsymbol{r} = \boldsymbol{r_0} + t\boldsymbol{v} = (2+2t)\boldsymbol{i} + (1-3t)\boldsymbol{j} + (-3+7t)\boldsymbol{k}.$$

Remark 1. *The initial position vector $\boldsymbol{r_0}$ and the direction vector $\boldsymbol{v}$ directly determine the line written in the vector equation form $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v}$, from which we can also deduce the parametric equation form.*

Problem 2. *Find parametric equations of the line containing the points $P = (1, -1, 0)$ and $Q = (-2, 3, 5)$.*

Solution. *Clearly one candiate for $\boldsymbol{r_0}$ is $P = (1, -1, 0)$ and one candidate for the direction vector $\boldsymbol{v}$ is $Q - P = (-3, 4, 5)$. Therefore the parametric equations is*

$$x = 1 - 3t, y = -1 + 4t, z = 0 + 5t.$$

Problem 3. *Find an equation of the plane that contains the point (1,-1,-1) and has normal vector $\frac{1}{2}\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$.*

Solution. *Recall that a plane is determined by its normal vector $\boldsymbol{n}$ and any point $\boldsymbol{r_0}$ in itself, and the equation for the plane is $(\boldsymbol{r} - \boldsymbol{r_0}) \cdot \boldsymbol{n} = \boldsymbol{0}$. Applying the formula above, one can obtain the plane satisfying the hypothesis is*

$$(x-1, y-(-1), z-(-1)) \cdot (\frac{1}{2}, 2, 3) = 0,$$

that is,

$$x + 4y + 6z + 9 = 0$$

.

Problem 4. *Find an equation of the plane that contains the points* $P = (1, 0, -1)$*,* $Q = (-5, 3, 2)$ *and* $R = (2, -1, 4)$*.*

Solution. *Let* $\boldsymbol{a} = Q - P = (-6, 3, 3)$ *and* $\boldsymbol{b} = R - P = (1, -1, 5)$*, then* $\boldsymbol{a} \times \boldsymbol{b} = (18, 33, 3)$ *gives a candidate for the normal vector* $\boldsymbol{n}$*. We can just choose* P *as the point* $\boldsymbol{r_0}$ *in the plane. Therefore the equation for the plane is*

$$(x - 1, y - 0, z - (-1)) \cdot (18, 33, 3) = 0,$$

that is

$$6x + 11y + z = 5.$$

Remark 2. *Actually the normal vector we have obtained is* $(18, 33, 3)$*, which is parallel to* $(6, 11, 1)$*. Only B fits this condition, so the answer must be B. We don't even need to calculate the equation for the plane.*

Problem 5. *Find parametric equations of the line tangent to the curve* $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$ *at the point* $(2, 4, 8)$*.*

Solution. *Clearly at the point* $(2, 4, 8)$*,* $t = 2$*. Then the direction vector* $\boldsymbol{v}$ *for the tangent line is the first derivative of* $\boldsymbol{r}(t)$ *at* $t = 2$*, which is* $(1, 2t, 3t^2)|_{t=2} = (1, 4, 12)$*. Therefore by Remark 1, the parametric equation is* $x = 2 + t, y = 4 + 4t, z = 8 + 12t$*.*

Problem 6. *The position function of an object is* $\boldsymbol{r}(t) = cost\boldsymbol{i} + 3sint\boldsymbol{j} - t^2\boldsymbol{k}$*. Find the velocity, acceleration, and speed of the object when* $t = \pi$*.*

Solution. *This is an easy problem. The velocity is just the first derivative of the position vector function, the acceleration is the second derivative of the position vector function, and the speed is the magnitude of the velocity vector.*

Problem 7. *A smooth parametrization of the semicircle which passes through the points* $(1, 0, 5), (0, 1, 5)$ *and* $(-1, 0, 5)$ *is*

Solution. *Observe that these three points given have the same* z*-coordiante 5, so the semicircle is contained in the plane* $z = 5$*, which is parallel to the* xy*-plane. Therefore we only need to use their* x, y *coordinates to parametrize the semicircle passing the three points. Since the semicircle passing* $(1, 0), (0, 1)$ *and* $(-1, 0)$ *in the* xy *plane is* $x = cost, y = sint, 0 \leq t \leq \pi$*, the answer has to be B, that is,* $cost\boldsymbol{i} + sint\boldsymbol{j} + 5\boldsymbol{k}, 0 \leq t \leq \pi$*.*

Remark 3. *Generally any three points who are not contained in a line can uniquely determine a triangle and thus uniquely determine the circumscribed circle. However it's a little complicated to find the equation for the circle. Although in this problem the condition is unbelievably nice, it's not likely in your final exam a general problem can appear.*

Problem 8. *The length of the curve* $\boldsymbol{r}(t) = \frac{2}{3}(1+t)^{\frac{3}{2}}\boldsymbol{i} + \frac{2}{3}(1-t)^{\frac{3}{2}}\boldsymbol{j} + t\boldsymbol{k}, -1 \leq t \leq 1$ *is*

Solution. *The formula for the length* L *is given by*

$$L = \int_{-1}^{1} |\boldsymbol{r}'(t)| dt = \int_{-1}^{1} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt = \int_{-1}^{1} \sqrt{1+t+1-t+1} dt = 2\sqrt{3}.$$

Problem 9. *Find the level curves of the function* $f(x,y) = \sqrt{1 - x^2 - 2y^2}$.

Solution. *The family of level sets can be obtained by setting* $f(x,y) = k$, *that is,* $\sqrt{1-x^2-2y^2} = k(0 \leq k \leq 1)$. *We square both sides and after a simple algebra we can obtain* $x^2 + 2y^2 = 1 - k^2$. *Hence when* $0 \leq k < 1$, *the level sets are ellipses. When* $k = 1$, $x^2 + 2y^2 = 0$ *and hence the level set is a single point* $(0,0)$. *So none of the choices is correct.*

Problem 10. *Find the level surface of the function* $f(x,y,z) = z - x^2 - y^2$ *that passes through the point* $(1,2,-3)$ *intersects the* (x,z)*-plane* $(y = 0)$ *along the curve.*

Solution. *Set* $f(x,y,z) = z - x^2 - y^2 = k$. *Since the level surface passes* $(1,2,-3)$, $k = f(1,2,-3) = -3 - 1 - 4 = -8$. *Therefore the level surface of* f *is* $z - x^2 - y^2 = -8$. *By letting* $y = 0$, *we find the curve of the level surface in the xz-plane is* $z - x^2 = -8$, *that is,* $z = x^2 - 8$.

Problem 11. *Match the graphs of the equations with their names: (1)* $x^2 + y^2 + z^2 = 4$ *(a) paraboloid*
(2) $x^2 + z^2 = 4$ *(b) sphere*
(3) $x^2 + y^2 = z^2$ *(c) cylinder*
(4) $x^2 + y^2 = z$ *(d) double cone*
(5) $x^2 + 2y^2 + 3z^2 = 1$ *(e) ellipsoid*

Solution. *Please check the table in Section 12.6 in the textbook.*

Problem 12. *Suppose that* $w = \frac{u^2}{v}$ *where* $u = g_1(t)$ *and* $v = g_2(t)$ *are differentiable functions of* t. *If* $g_1(1) = 3, g_2(1) = 2, g_1'(1) = 5$ *and* $g_2'(1) = -4$, *find* $\frac{dw}{dt}$ *when* $t = 1$.

Solution. *By chain rule,*

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial v}\frac{dv}{dt} = \frac{2u}{v} g_1'(t) + (-\frac{u^2}{v^2}) g_2'(t).$$

When $t = 1$, $\frac{2u}{v} = 2\frac{g_1(1)}{g_2(1)} = 2 \cdot \frac{3}{2} = 3$, *and* $-\frac{u^2}{v^2} = -2\frac{g_1(1)^2}{g_2(1)^2} = -\frac{9}{4}$. *Therefore,*

$$\frac{dw}{dt} = 3 \cdot 5 + (-\frac{9}{4}) \cdot (-4) = 24.$$

Problem 13. *If* $w = e^{uv}$ *and* $u = r + s$, $v = rs$, *find* $\frac{\partial w}{\partial r}$.

Solution. *Although one can use chain rule to do this problem, a simpler way is just plug* $u = r + s, v = rs$ *into the function* w*, that is,* $w = e^{(r+s)rs} = e^{r^2s+rs^2}$*. Hence*

$$\frac{\partial w}{\partial r} = e^{r^2s+rs^2}(2rs + s^2).$$

Problem 14. *If* $f(x, y) = cos(xy)$*, find* $\frac{\partial^2 f}{\partial x \partial y}$*.*

Solution. $\frac{\partial f}{\partial y} = -xsin(xy)$*, hence*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(-xsin(xy)) = -sin(xy) - xycos(xy).$$

Problem 15. *Assuming that the equation* $xy^2 + 3z = cos(z^2)$ *defines* z *implicitly as a function of* x *and* y*, find* $\frac{\partial z}{\partial x}$*.*

Solution. *Defferentiating both sides of the equation with respect to* x*, hence the chain rule implies*

$$y^2 + 3\frac{\partial z}{\partial x} = -sin(z^2) \cdot 2z \cdot \frac{\partial z}{\partial x}.$$

Therefore,

$$\frac{\partial z}{\partial x} = \frac{-y^2}{3 + 2zsin(z^2)}.$$

Problem 16. *If* $f(x, y) = xy^2$*, then* $\nabla f(2, 3) =$

Solution. *Since* $\nabla f(x, y) = (f_x, f_y) = (y^2, 2xy)$*,* $\nabla f(2, 3) = (3^2, 2 \cdot 2 \cdot 3) = (9, 12) = 9\boldsymbol{i} + 12\boldsymbol{j}$*.*

Problem 17. *Find the directional derivative of* $f(x, y) = 5 - 4x^2 - 3y$ *at* (x, y) *towards the origin.*

Solution. *The direction at* (x, y) *towards the origin is parallel to that of* $-(x, y)$*, hence the unit directional vector* $\boldsymbol{u}$ *is* $-\frac{(x,y)}{\sqrt{x^2+y^2}}$*. Therefore*

$$D_{\boldsymbol{u}}f(x, y) = \nabla f(x, y) \cdot \boldsymbol{u} = (-8x, -3) \cdot -\frac{(x, y)}{\sqrt{x^2 + y^2}} = E.$$

Problem 18. *For the function* $f(x, y) = x^2y$*, find a unit vector* $\boldsymbol{u}$ *for which the directional derivative* $D_{\boldsymbol{u}}f(2, 3)$ *is zero.*

Solution. *We want to find a unit vector* $\boldsymbol{u}$ *such that* $\nabla f(2, 3) \cdot \boldsymbol{u} = 0$*. Since* $\nabla f(2, 3) = (2xy, x^2)|_{(x,y)=(2,3)} = (12, 4)$*,* $\boldsymbol{u} = \frac{(1,-3)}{\sqrt{10}} = D$*.*

Problem 19. *Find a vector pointing in the direction in which* $f(x, y, z) = 3xy - 9xz^2 + y$ *increases most rapidly at the point* $(1, 1, 0)$*.*

Solution. *Recall that the gradient vector has the direction along which the function has the maximal rate of change. Therefore, the desired vector must be parallel to* $\nabla f(1, 1, 0)$*. Since* $\nabla f(1, 1, 0) = (3y - 9z^2, 3x + 1, -18xz)|_{(x,y,z)=(1,1,0)} = (3, 4, 0)$*, the answer is A.*

Problem 20. *Find a vector that is normal to the graph of the equation* $2cos(\pi xy) = 1$ *at the point* $(\frac{1}{6}, 2)$.

Solution. *From the hypothesis, the graph is just the 0-level curve of the function* $f(x, y) = 2cos(\pi xy) - 1$*, therefore the normal of the graph must be parallel to the gradient vector. Computing* $\nabla f(x, y) = (-2sin(\pi xy)\pi y, -2sin(\pi xy)\pi x) = -2\pi sin(\pi xy)(y, x)$*, which has the same direction as that of* (y, x) *(if we don't care about the negative sign). Now that we know that the normal of the graph at* (x, y) *is parallel to* (y, x)*, hence we conclude that at* $(x, y) = (\frac{1}{6}, 2)$*, one candidate for the normal is* $(y, x) = (2, \frac{1}{6})$*. Only* C *satisfies our conclusion.*

Problem 21. *Find an equation of the tangent plane to the surface* $x^2 + 2y^2 + 3z^2 = 6$ *at the point* $(1, 1, -1)$.

Solution. *The surface is the 0-level set of the function* $f(x, y, z) = x^2 + 2y^2 + 3z^2 - 6$*, hence the normal of the tangent plane to the surface is parallel to* ∇f*. At* $(1, 1, -1)$*,* $\nabla f = (2, 4, -6)$*, hence the equation of the tangent plane is*

$$(x - 1, y - 1, z + 1) \cdot (2, 4, -6) = 0.$$

That is, $x + 2y - 3z = 6$.

Problem 22. *Find an equation of the plane tangent to the graph of* $f(x, y) = \pi + sin(\pi x^2 + 2y)$ *when* $(x, y) = (2, \pi)$.

Solution. *The graph is the 0-level surface of the function* $g(x, y, z) = z - f(x, y)$*. The normal of the tangent plane at* (x, y, z) *is parallel to* $\nabla g(x, y, z) = (-f_x, -f_y, 1) = (-2\pi x cos(\pi x^2 + 2y), -2cos(\pi x^2 + 2y), 1)$*. At* $(x, y) = (2, \pi)$*,* $z = \pi$ *and* $\nabla g = (-4\pi, -2, 1)$*, therefore the equation of the tangent plane to the graph at* $(x, y) = (2, \pi)$ *is*

$$((x, y, z) - (2, \pi, \pi)) \cdot (-4\pi, -2, 1) = 0,$$

that is,

$$4\pi x + 2y - z = 9\pi.$$

Hence the answer is A.

Remark 4. *One can also just use the formula* $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ *to find the equation for the tangent plane to the graph of* $z = f(x, y)$ *at the point* (x_0, y_0).

Remark 5. *As another kind remark, the linear approximation of the function* $f(x, y)$ *near* (x_0, y_0) *is* $f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

Problem 23. *The differential df of the function* $f(x, y, z) = xe^{y^2 - x^2}$ *is*

Solution. $f_x = e^{y^2 - z^2}, f_y = 2xye^{y^2 - z^2}$ *and* $f_z = -2xze^{y^2 - z^2}$*, so the answer is* D.

Problem 24. *Classify the critical points of the function* $f(x, y) = 2x^3 - 6xy - 3y^2$.

Solution. $f_x = 6x^2 - 6y, f_y = -6x - 6y, f_{xx} = 12x, f_{xy} = -6, f_{yy} = -6$, *and* $D = f_{xx}f_{yy} - f_{xy}^2 = -72x - 36$. *Let* $f_x = 0, f_y = 0$, *we find (from* $f_y = 0$ *we get* $y = -x$ *and then replacing* $y = -x$ *in the equation* $f_x = 0$ *and soving a quadradic equation) the critical points are* $(0, 0)$ *and* $(-1, 1)$. *At* $(0, 0)$, $D < 0$, *hence* $(0, 0)$ *is a saddle point. At* $(-1, 1)$, $f_{xx} < 0$ *and* $D > 0$, *hence* $(-1, 1)$ *is a local maximal point. The answer is* B.

Remark 6. *As long as* $D < 0$, *it's saddle point. If* $f_{xx} > 0$ *and* $D > 0$, *then it's a local minimum point. If* $f_{xx} < 0$ *and* $D > 0$, *then it's a local maximum point.*

Problem 25. *Consider the problem of finding the minimum value of the function* $f(x, y) = 4x^2 + y^2$ *on the curve* $xy = 1$. *In using the method of Lagrange multipliers, the value of* λ *(even though it is not needed) will be*

Solution. *The constraint is* $g(x, y) = xy = 1$. *Using Lagrange multiplier method,* $f_x = \lambda g_x$ *and* $f_y = \lambda g_y$ *imply*

$$8x = \lambda y \tag{1}$$

and

$$2y = \lambda x. \tag{2}$$

Hence the multiplications of both sides are equal, that is, $16xy = \lambda^2 xy$. *Since* $xy \neq 0$, *we get* $\lambda^2 = 16$, *and thus* $\lambda = \pm 4$. *Since* $xy = 1$, x *and* y *have the same sign. Hence from* (1), λ *cannot be negative. Therefore,* $\lambda = 4$.

Problem 26. *Evaluate the iterated integral* $\int_1^3 \int_0^x \frac{1}{x} dy dx$.

Solution. *Notice* $\frac{1}{x}$ *is a constant function in terms of* y, *we evaluate the inner integral* $\int_0^x \frac{1}{x} dy = \frac{1}{x} \cdot x = 1$, *and hence* $\int_1^3 \int_0^x \frac{1}{x} dy dx = \int_1^3 1 dx = 2$. *The answer is* B.

Problem 27. *Consider the double integral,* $\iint_R f(x, y) dA$, *where* R *is the portion of the disk* $x^2 + y^2 \leq 1$, *in the upper half-plane,* $y \geq 0$. *Express the integral as an iterated integral.*

Solution. *We regard* R *as a type-I domain. The lower bound for* y *is the* x*-axis, that is* $y = 0$. *The upper bound for* y *is the circle above the* x*-axis, that is* $y = \sqrt{1 - x^2}$, *hence* $0 \leq y \leq \sqrt{1 - x^2}$. *Clearly* $-1 \leq x \leq 1$, *hence the answer is* C.

Problem 28. *Find* a *and* b *for the correct interchange of order of integration:*

$$\int_0^2 \int_{x^2}^{2x} f(x, y) dy dx = \int_0^4 \int_a^b f(x, y) dx dy.$$

Solution. *One can certainly draw the picture of the domain D. It's not convenient for me do plug the picture here, so I'll do it without picture. The argument is follows:*

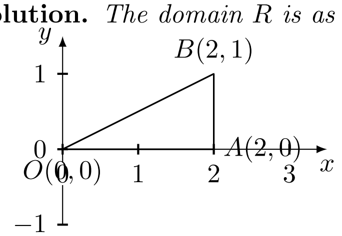
Clearly from the left hand side,

$$x^2 \leq y \leq 2x \tag{3}$$

Clearly (3) *with* $x \geq 0$ *is equivalent to* $\frac{y}{2} \leq x \leq \sqrt{y}$*, the answer is* B*.*

Problem 29. *Evaluate the double integral* $\iint_R ydA$*, where* R *is the region of the* (x, y)*-plane inside the triangle with vertices* $(0, 0), (2, 0)$ *and* $(2, 1)$*.*

Solution. *The domain* R *is as below:*



Hence

$$\iint_R ydA = \int_0^2 \int_0^{\frac{1}{2}x} ydydx = \int_0^2 \frac{1}{2}y^2|_{y=0}^{y=\frac{1}{2}x} dx = \int_0^2 \frac{1}{8}x^2 dx = \frac{1}{24}x^3|_0^2 = \frac{1}{3}.$$

Problem 30. *The volume of the solid region in the first octant bounded above by the parabolic sheet* $z = 1 - x^2$*, below by the xy plane, and on the sides by the planes* $y = 0$ *and* $y = x$ *is given by the double integral:*

Solution. *Still, it's better to draw a picture, but I've no idea how to write the code, I'll just do it without picture. Since* $0 \leq z \leq 1 - x^2$ *and* $x \geq 0$*, we obtain (by just solving* $1 - x^2 \geq 0$ *and* $x \geq 0$*)* $0 \leq x \leq 1$*. What's left is to find the bounds for* y*. From the hypothesis,* $0 \leq y \leq x$*. Paying attention to the order of integration, here we first integrate with respect to* z*, and then* y *and then* x *(why?), we get*

$$V = \int_0^1 \int_0^x \int_0^{1-x^2} 1 \cdot dzdydx == \int_0^1 \int_0^x (1 - x^2)dydx.$$

Hence the answer is A*.*

Problem 31. *The area of one leaf of the three-leaved rose bounded by the graph of* $r = 5sin3\theta$ *is*

Solution. *When* θ *goes from 0 to* $\frac{\pi}{6}$*,* r *goes from* 0 *to its maximum* 5*. When When* θ *goes from* $\frac{\pi}{6}$ *to* $\frac{\pi}{3}$*,* r *goes from its maximum* 5 *to 0. Hence* $0 \leq r \leq 5sin(3\theta), 0 \leq \theta \leq \frac{\pi}{3}$ *describe one leave of the rose. (It's very clear from the* *picture.) Hence the area is given by*

$$\begin{aligned}\int_0^{\frac{\pi}{3}}\int_0^{5sin(3\theta)} rdrd\theta &= \int_0^{\frac{\pi}{3}}\frac{25sin^2(3\theta)}{2}d\theta\\ &= \frac{25}{2}\int_0^{\frac{\pi}{3}}\frac{1-cos(6\theta)}{2}d\theta\\ &= \frac{25}{4}\left(\theta-\frac{sin(6\theta)}{6}\right)\Big|_0^{\frac{\pi}{3}}\\ &= \frac{25\pi}{12}\end{aligned}$$

Problem 32. *Find the area of the portion of the plane $x+3y+2z=6$ that lies in the first octant.*

Solution. *By letting $z=0$ in the plane $x+3y+2z=6$, we find the intersection of the plane and the xy-plane is the line $x+3y=6$. Hence the xy-domain D is enclosed by $x+3y=6$ and the x,y axes, which is a triangle with two edges 6 and 2. The area of the triangle $A(D)=\frac{1}{2}\cdot 2\cdot 6=6$, hence the area asked in the problem is*

$$\iint_D\sqrt{1+z_x^2+z_y^2}dA=\iint_D\sqrt{1+(-\frac{1}{2})^2+(-\frac{3}{2})^2}dA=\frac{\sqrt{14}}{2}A(D)=3\sqrt{14}.$$

Problem 33. *A solid region in the first octant is bounded by the surfaces $z=y^2, y=x, y=0, z=0$ and $x=4$. The volume of the region is*

Solution. *Since $0\le z\le y^2, 0\le y\le x$ and $0\le x\le 4$ (by letting $y=0$ in $y=x$ we know the lower bound for x is 0),*

$$V=\int_0^4\int_0^x\int_0^{y^2}dzdydx=\int_0^4\int_0^x y^2dydx=\int_0^4\frac{y^3}{3}\Big|_0^x dx=\int_0^4\frac{x^3}{3}dx=\frac{64}{3}.$$

Problem 34. *An object occupies the region Ω bounded above by the sphere $x^2+y^2+z^2=32$ and below by the upper nappe of the cone $z^2=x^2+y^2$. The mass density at any point of the object is equal to its distance from the xy plane. Set up a triple integral in rectangular coordinates for the total mass m of the object.*

Solution. *Clearly $\sqrt{x^2+y^2}\le z\le\sqrt{32-x^2-y^2}$. By replacing z^2 with x^2+y^2 in the equation of the ball, we obtain $2(x^2+y^2)=32$. Hence the projection D of Σ to the xy-plane is the disk centered at origin with radius 4. Using rectangular coordinates, $D: -\sqrt{16-x^2}\le y\le\sqrt{16-x^2}$, $-4\le x\le 4$. The integrand is the distance from the xy-plane, which is z. Therefore, the answer is*

$$\iint_D\int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} zdzdA=\int_{-4}^4\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} zdzdydx,$$

which is B.

Problem 35. *Do Problem 34 in spherical coordinates.*

Solution. $z^2 = x^2 + y^2$ *implies* $(\rho cos\phi)^2 = (\rho sin\phi)^2$, *hence* $\phi = \frac{\pi}{4}$. *Since the region is above the cone,* $0 \le \phi \le \frac{\pi}{4}$. *Clearly* $0 \le \rho \le \sqrt{32}$ *and* $0 \le \theta \le 2\pi$. *Therefore the mass is*

$$\iiint_\Omega zdV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{32}} \rho cos\phi \cdot \rho^2 sin\phi d\rho d\phi d\theta.$$

The answer is A.

Problem 36. *Convert* $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2(x^2+y^2)^3 dydx$ *into polar coordinates form.*

Solution. *Noticing* $x, y \ge 0$, *the domain* D *is part of the disk centered at origin with radius 1 in the first quadrant, hence* $0 \le r \le 1$ *and* $0 \le \theta \le \frac{\pi}{2}$. *Also,* $y^2(x^2 + y^2)^3 = (rsin\theta)^2(r^2)^3$ *and* $dxdy = rdrd\theta$, *therefore*

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2(x^2 + y^2)^3 dydx = \int_0^{\frac{\pi}{2}} \int_0^1 (rsin\theta)^2(r^2)^3 \cdot rdrd\theta,$$

which is E.

Problem 37. *Convert the triple integrals*

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 dzdydx$$

from rectangular to cylindrical coordinates.

Solution. *Use* $\sqrt{x^2 + y^2} = r$ *we know* $r \le z \le 2$. *The projection of the solid region to the xy plane is the disk centered at origin with radius 2, hence* $0 \le r \le 2, 0 \le \theta \le 2\pi$. *Replacing* $dzdydx$ *with* $rdzdrd\theta$, *one can easily see the answer is* $\int_0^{2\pi} \int_0^2 \int_r^2 rdzdrd\theta$.

Problem 38. *If* D *is the solid region above the xy-plane that is between* $z = \sqrt{4 - x^2 - y^2}$ *and* $z = \sqrt{1 - x^2 - y^2}$, *then* $\iiint_D \sqrt{x^2 + y^2 + z^2} dV$ *is*

Solution. *Clearly* D *is the shell between the balls centered at origin with radius 1 and 2 above xy plane, hence* $1 \le \rho \le 2, 0 \le \phi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi$. *Use* $\sqrt{x^2 + y^2 + z^2} = \rho$ *and* $dV = \rho^2 sin\phi d\rho d\phi d\theta$, *we obtain*

$$\iiint_D \sqrt{x^2 + y^2 + z^2} dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 \rho \cdot \rho^2 sin\phi d\rho d\phi d\theta = \left(\frac{\rho^4}{4} \Big|_1^2 \right) 2\pi \cdot \int_0^{\frac{\pi}{2}} sin\phi d\phi = \frac{15}{2}\pi.$$

Problem 39. *Determine which of the vector fields below are conservative, i.e.,* $\boldsymbol{F} = \nabla f$ *for some function* f.

Solution. *Just check one by one whether* $\nabla \times \boldsymbol{F} = \boldsymbol{0}$.

Problem 40. *Let F be any vector field whose components have continuous partial derivatives up to second order, let f be any real valued function with continuous partial derivatives up to second order, and let $\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$. Find the incoorect statement.*
A. $curl(\nabla f) = \mathbf{0}$ *B.* $div(curl\boldsymbol{F}) = \mathbf{0}$ *C.* $\nabla(div\boldsymbol{F}) = 0$
D. $\boldsymbol{F} = \nabla \times \boldsymbol{F}$ *E.* $div\boldsymbol{F} = \nabla \cdot \boldsymbol{F}$

Solution. *C is generally false.*
D and E are definitions. A and B are important facts. Especially A implies if a vector field is conservative, then by Stoke's theorem and fundamental theorem of line integrl, then the second type of line integral over the boundary of a surface is zero. B implies the flux of the vector field curl$\boldsymbol{F}$ over a closed solid region is always 0 as a result of the divergence theorem.

Problem 41. *A wire lies on the xy-plane along the curve $y = x^2, 0 \le x \le 2$. The mass density (per unit length) at any point (x, y) of the wire is equal to x. The mass of the wire is*

Solution. *The mass is*

$$\int \rho ds = \int_0^2 x\sqrt{1+y_x^2}dx = \int_0^2 x\sqrt{1+4x^2}dx = \frac{2}{3}\cdot\frac{1}{8}(1+4x^2)^{\frac{3}{2}}\Big|_0^2 = \frac{17\sqrt{17}-1}{12}.$$

Problem 42. *Evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $F(x, y) = y\boldsymbol{i} + x^2\boldsymbol{j}$ and C is composed of the line segments from $(0, 0)$ to $(1, 0)$ and from $(1, 0)$ to $(1, 2)$.*

Solution. *Let C_1 be the horizontal segment and C_2 be the vertical segments. C_1 can be parametrized by $(x, 0), 0 \le x \le 1$, and C_2 can be parametrized by $(1, y), 0 \le y \le 2$. Hence*

$$\int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^1 \boldsymbol{F}(x, 0)\cdot(1, 0)dx = \int_0^1 (0, x^2)\cdot(1, 0)dx = 0,$$

and

$$\int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^2 \boldsymbol{F}(1, y)\cdot(0, 1)dy = \int_0^2 (y, 1)\cdot(0, 1)dy = 2.$$

Hence

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r} = 2.$$

Problem 43. *Evaluate the line integral*

$$\int_C xdx + ydy + xydz$$

where C is parametrized by $\boldsymbol{r}(t) = cost\boldsymbol{i} + sint\boldsymbol{j} + cost\boldsymbol{k}$ for $-\frac{\pi}{2} \le t \le 0$.

Solution. *Since* $d\boldsymbol{r} = \boldsymbol{r}'(t)dt = (-sint, cost, -sint)dt$,

$$\begin{aligned}\int_C xdx + ydy + xydz &= \int_{-\frac{\pi}{2}}^{0} (cost, sint, costsint) \cdot (-sint, cost, -sint)dt \\ &= \int_{-\frac{\pi}{2}}^{0} -costsin^2 tdt \\ &= -\frac{sin^3 t}{3}\Big|_{-\frac{\pi}{2}}^{0} \\ &= 0 - \left(-\frac{1}{3}sin(\frac{-\pi}{2})\right) = -\frac{1}{3}.\end{aligned}$$

Notice that the last step we have used $sin(-\frac{\pi}{2}) = -1$. *(It's easy to make mistakes here.)*

Problem 44. *Are the following statements true or false?*
1. The line integral $\int_C (x^3 + 2xy)dx + (x^2 - y^2)dy$ *is independent of path in the xy-plane.*
2. $\int_C (x^3 + 2xy)dx + (x^2 - y^2)dy = 0$ *for every closed oriented curve* C *in the xy-plane*
3. There is a function $f(x, y)$ *defined in the xy-plane, such that* $(x, y) = (x^3 + 2xy)\boldsymbol{i} + (x^2 - y^2)\boldsymbol{j}$.

Solution. *This problem tests the following four equivalent conditions for the conservative vector field. A vector filed* $\boldsymbol{F}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}$ *is conservative if and only if one of the four conditions hold:*
1 $\int_C Pdx + Qdy$ *is independent of path.*
2 $\int_C Pdx + Qdy = 0$ *for every closed curve* C *in the xy-plane.*
3 There is a function $f(x, y)$ *defined in the xy-plane, such that* $\nabla f(x, y) = \boldsymbol{F}(x, y)$.
4 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

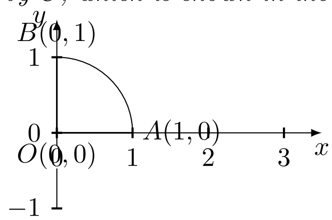
Problem 45. *Evaluate* $\int_C y^2 dx + 6xydy$ *where* C *is the boundary curve of the region bounded by* $y = \sqrt{x}, y = 0$ *and* $x = 4$, *in the counterclockwise direction.*

Solution. *Let* D *be the region enclosed by* C, *hence* D *is a type-I domain* $0 \le y \le \sqrt{x}, 0 \le x \le 4$. *Clearly,* $P = y^2$, $Q = 6xy$, *hence* $\frac{\partial Q}{\partial x} = 6y$, $\frac{\partial P}{\partial y} = 2y$. *Considering the orientation of the curve and applying Green's theorem, we have*

$$\begin{aligned}\int_C y^2 dx + 6xydy &= \int_0^4 \int_0^{\sqrt{x}} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dydx \\ &= \int_0^4 \int_0^{\sqrt{x}} 4ydydx \\ &= \int_0^4 2y^2\Big|_{y=0}^{y=\sqrt{x}} dx \\ &= \int_0^4 2xdx = 16.\end{aligned}$$

Problem 46. *If C goes along the x-axis from $(0,0)$ to $(1,0)$, then along $y = \sqrt{1-x^2}$ to $(0,1)$, and then back to $(0,0)$ along the y-axis, then $\int_C xydy =$*

Solution. *Here $P(x,y) = 0$ and $Q(x,y) = xy$. Let D be the domain enclosed by C, which is shown in the following picture:*



Applying the Green's theorem, and since C is positively oriented(counterclockwise),

$$\int_C xydy = \iint_D \frac{\partial}{\partial x}(xy)dA = \int_0^1 \int_0^{\sqrt{1-x^2}} ydydx,$$

which is B.

Problem 47. *Evaluate $\int_C \boldsymbol{F}d\boldsymbol{r}$, if $\boldsymbol{F}(x,y) = (xy^2 - 1)\boldsymbol{i} + (x^2y - x)\boldsymbol{j}$ and C is the circle of radius 1 centered at $(1,2)$ and oriented counterclockwise.*

Solution. *Let D be the region enclosed by C, that is, the unit disk centered at $(1,2)$. Hence the area $A(D)$ of D is π. Considering the orientation (positive) and Applying Green's Theorem, we obtain*

$$\int_C \boldsymbol{F}d\boldsymbol{r} = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA = \iint_D ((2xy-1) - (2xy))\, dA = -A(D) = -\pi.$$

Remark 7. *Sometimes it's convenient to use the formula of the area of some common figures, for example triangles($A = \frac{1}{2}ab$), disks($A = \pi r^2$), ellipses($A = \pi ab$), etc.*

Problem 48. *Green's theorem yields the following formula for the area of a simple region R in terms of a line integral over the boundary C of R, oriented counterclockwise. Area of $R = \int_R dA =$*

Solution. *By Green's theorem, we just need to check whether the P, Q's in the choices $A - E$ satisfying $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, whence*

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_R dA = R.$$

In A, $P = -y$ and $Q = 0$, hence $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - (-1) = 1$, hence the answer is A, that is,

$$\iint_R dA = -\int_C ydx.$$

Remark 8. *$\int_R dA$ can also be written as $\int_C xdy$, $\frac{1}{2}\int_C(-ydx + xdy)$, and so forth. Generally, $\iint_R dA = \alpha(-\int_C ydx) + \beta\int_C xdy$ for all α, β such that $\alpha + \beta = 1$.*

Problem 49. *Evaluate the surface integral $\iint_\Sigma xdS$ where Σ is part of the plane $2x + y + z = 4$ in the first octant.*

Solution. *Clearly $z = 4 - 2x - y$ and thus $z_x = -2, z_y = -1$. The projection of Σ to the xy-plane is enclosed by $x = 0, y = 0$ and $2x + y = 4$, that is $0 \le y \le -2x + 4, 0 \le x \le 2$. Therefore,*

$$\iint_\Sigma xdS = \int_0^2 \int_0^{-2x+4} x\sqrt{1 + z_x^2 + z_y^2}dydx = \sqrt{6}\int_0^2 x(4-2x)dx = \sqrt{6}\ (2x^2 - \frac{2}{3}x^3)\Big|_0^2 = \frac{8}{3}\sqrt{6}.$$

Problem 50. *If Σ is part of the paraboloid $z = x^2 + y^2$ with $z \le 4$, $\boldsymbol{n}$ is the unit normal vector on Σ directed upward, and $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then evaluate $\iint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}dS$.*

Solution. *Recall the formula that $\boldsymbol{n}dS = d\boldsymbol{S} = (-z_x, -z_y, 1)dA$. Since $z = x^2+y^2$, $z_x = 2x$ and $z_y = 2y$. Hence $\boldsymbol{F}\cdot\boldsymbol{n}dS = (x, y, x^2+y^2)\cdot(-2x, -2y, 1)dA = -(x^2 + y^2)dA$. The projection of Σ to the xy-plane is the disk $D : x^2 + y^2 \le 4$. Therefore,*

$$\iint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}dS = \iint_D -(x^2 + y^2)dA = \int_0^{2\pi}\int_0^2 -r^2 \cdot rdrd\theta = -8\pi.$$

Remark 9. *Generally if the surface is parametrized by*

$$\boldsymbol{r} = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in D,$$

then $dS = |\boldsymbol{r}_u \times \boldsymbol{r}_v|dudv$ and $d\boldsymbol{S} = (\boldsymbol{r}_u \times \boldsymbol{r}_v)dudv$.

In particular, if the surface equation is given as $z = g(x, y)$, then $dS = \sqrt{1 + g_x^2 + g_y^2}dA$, and $d\boldsymbol{S} = (-g_x, -g_y, 1)dA$. This is better to be remembered.

For the second type of surface integral, one has to check whether the direction of $\boldsymbol{r}_u \times \boldsymbol{r}_v$ matches the direction of the orentation of the surface. If not, one has to add a negative sign to the final answer.

Problem 51. *If $F = cosz\boldsymbol{i} + sinz\boldsymbol{j} + xy\boldsymbol{k}$, Σ is the complete boundary of the rectangular solid region Ω bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = \frac{\pi}{2}$, and $\boldsymbol{n}$ is the outward unit normal on Σ, then find $\iint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}dS$.*

Solution. *Since $div\boldsymbol{F} = 0$ and Σ is the complete boundary of Ω, applying the divergence theorem we obtain*

$$\iint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}dS = \iiint_\Omega div\boldsymbol{F}dV = 0.$$

Problem 52. *Find $\iint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}dS$ where Σ is the unit sphere $x^2 + y^2 + z^2 = 1$ and $\boldsymbol{n}$ is the outward unit normal on Σ.*

Solution. *The outward unit normal $\boldsymbol{n} = (x, y, z)$, and $dS = sin\phi d\phi d\theta$, hence*

$$\iint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}dS = \int_0^{2\pi}\int_0^{\pi} (x, y, z)\cdot(x, y, z)sin\phi d\phi d\theta = \int_0^{2\pi}\int_0^{\pi} sin\phi d\phi d\theta = 4\pi.$$

Remark. *Generally speaking, if the surface is part of the sphere* $x^2+y^2+z^2=a^2$*, then* $\boldsymbol{n}=\frac{(x,y,z)}{a}$ *and* $dS=a^2 sin\phi d\phi d\theta$.

Problem 53. *Evaluate* $\iint_S curl\boldsymbol{F}\cdot d\boldsymbol{S}$*, where* $\boldsymbol{F}=x^2e^{yz}\boldsymbol{i}+y^2e^{xz}\boldsymbol{j}+z^2e^{xy}\boldsymbol{k}$ *and* S *is the hemisphere* $x^2+y^2+z^2=4$*,* $z\geq 0$*, oriented upward.*

Solution. *Let* C *be the boundary of the hemisphere, thus* $C: x^2+y^2=4, z=0$*. We can parametrize* C *by* $x=2cost, y=2sint, z=0, 0\leq t\leq 2\pi$*. Hence* $\boldsymbol{r'}(t)=(-2sint, 2cost, 0)$*. Hence by Stoke s Theorem,*

$$\begin{aligned}\iint_S curl\boldsymbol{F}\cdot d\boldsymbol{S} &= \int_C \boldsymbol{F}\cdot d\boldsymbol{r}\\ &= \int_0^{2\pi}(4ecos^2t, 4esin^2t, 0)\cdot(-2sint, 2cost, 0)dt\\ &= \int_0^{2\pi} 8ecostsint(sint-cost)dt\\ &= 8e\left.\frac{sin^3t+cos^3t}{3}\right|_0^{2\pi}=0.\end{aligned}$$

Problem 54. *Evaluate* $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$*, where* $\boldsymbol{F}=x^2z\boldsymbol{i}+xy^2\boldsymbol{j}+z^2\boldsymbol{k}$ *and* C *is the curve of intersection of the plane* $x+y+z=1$ *and the cylinder* $x^2+y^2=9$ *oriented counterclockwise as viewed from above.*

Solution. *Let* S *be the part of the plane* $z=1-x-y$ *inside the cylinder* $x^2+y^2=9$*. Clearly the curve* C *is the boundary of the surface* S*, hence we can safely apply Stoke's theorem. One can easily calculate* $curl\boldsymbol{F}=(0,x^2,y^2)$*. Moreover, from Remark 9,* $d\boldsymbol{S}=(-\frac{\partial z}{\partial x},-\frac{\partial z}{\partial y},1)dxdy=(1,1,1)dA$*, and the projection of* S *to the xy-plane is the disk enclosed by the circle* $x^2+y^2=9$*. Therefore,*

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \iint_S curl\boldsymbol{F}\cdot d\boldsymbol{S}\\ &= \iint_D (0,x^2,y^2)\cdot(1,1,1)dA\\ &= \iint_D (x^2+y^2)dA\\ &= \int_0^{2\pi}\int_0^3 r^2\cdot rdrd\theta\\ &= 2\pi\left.\frac{r^4}{4}\right|_0^3=\frac{81\pi}{2}\end{aligned}$$