Study Guide # 2

1. Relative/local extrema; critical points (points where $\nabla f = \vec{\mathbf{0}}$ or ∇f does not exist).

2. <u>2^{nd} Derivatives Test</u>: Suppose the 2^{nd} partials of $f(x,y)$ are continuous in a disk with center (a,b) and $\nabla f(a,b) = \tilde{\mathbf{0}}$. Let $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(a,b)}$.

(a) If $D > 0$ and $f_{xx}(a,b) > 0 \implies f(a,b)$ is a local minimum value.

(b) If $D > 0$ and $f_{xx}(a,b) < 0 \implies f(a,b)$ is a local maximum value.

(c) If $D < 0 \implies f(a,b)$ is a *not* a local min or local max value. So (a,b) is a **saddle point** of f.

If $D = 0$ (or if $\nabla f(a,b)$ does not exist or f has more than 2 variables) the test gives no information and you need to do something else to determine if a relative extremum exists.

3. Absolute extrema; Max-Min Problems.

4. Constrained extreme values via **Lagrange Multipiers:** Max/min -ize $f(\mathbf{v})$ subject to constraint $g(\mathbf{v}) = C$, solve the system $\nabla f = \lambda \nabla g$ and $g(\mathbf{v}) = C$.

5. Double integrals; Double Riemann sums: $\displaystyle\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\ \Delta A$;

6. Type I region $R : \begin{cases} g_1(x) \le y \le g_2(x) \\ a \le x \le b \end{cases}$; Type II region $R : \begin{cases} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{cases}$;

iterated integrals over Type I and II regions: $\displaystyle\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ and

$\displaystyle\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

7. Polar: $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$, $\tan\theta = \dfrac{y}{x}$ (make sure θ in correct quadrant).

Change of Variables Formula in Polar Coordinates: if $R : \begin{cases} h_1(\theta) \le r \le h_2(\theta) \\ \alpha \le \theta \le \beta \end{cases}$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta.$$

↑

8. Applications of double integrals:

(a) Area of region R is $A(R) = \iint_R dA$

(b) Volume of solid under graph of $z = f(x,y)$, where $f(x,y) \geq 0$, is $V = \iint_R f(x,y)\,dA$

(c) Mass of R is $m = \iint_R \rho(x,y)\,dA$, where $\rho(x,y) =$ density (per unit area).

(d) Moment about the x-axis $M_x = \iint_R y\,\rho(x,y)\,dA$; moment about the y-axis $M_y = \iint_R x\,\rho(x,y)\,dA$.

(e) Center of mass $(\overline{x}, \overline{y})$, where $\overline{x} = \dfrac{M_y}{m} = \dfrac{\iint_R x\,\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$, $\overline{y} = \dfrac{M_x}{m} = \dfrac{\iint_R y\,\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$

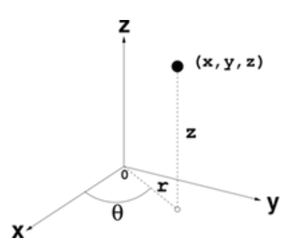
9. Elementary solids $D \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids D:

$$\iiint_D f(x,y,z)\,dV = \iint_R \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\,dA \text{ for } D = \{(x,y) \in R,\ u(x,y) \leq z \leq v(x,y)\};$$

volume of solid D is $V(D) = \iiint_D dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes M_{xy}, M_{xz}, M_{yz}, center of mass of a solid $(\overline{x}, \overline{y}, \overline{z})$.

10. Cylindrical Coordinates (r, θ, z):

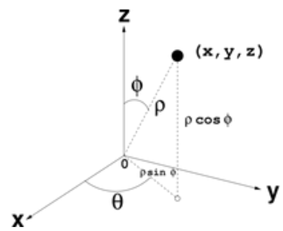
From CC to RC : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$



Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$ (make sure θ is in correct quadrant).

11. Spherical Coordinates (ρ, θ, ϕ), where $0 \le \phi \le \pi$:

From SC to RC : $\begin{cases} x = (\rho \sin\phi)\cos\theta \\ y = (\rho \sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$



Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan\theta = \dfrac{y}{x}$ and $\cos\phi = \dfrac{z}{\rho}$.

12. Triple integrals in Cylindrical Coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\boldsymbol{dV = r\,dz\,dr\,d\theta}$

$$\iiint\limits_D f(x,y,z)\ dV = \iiint\limits_D f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$
$$\uparrow$$

13. Triple integrals in Spherical Coordinates: $\begin{cases} x = (\rho \sin\phi)\cos\theta \\ y = (\rho \sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$, $\boldsymbol{dV = \rho^2 \sin\phi\ \ d\rho\, d\phi\, d\theta}$

$$\iiint_D f(x,y,z)\ dV = \iiint_D f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$
$$\uparrow$$

14. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\mathbf{F}(x,y) = \langle P(x,y),\ Q(x,y)\rangle$ and
$\mathbf{F}(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle$;
$\mathbf{F}$ is a conservative vector field if $\mathbf{F} = \nabla f$, for some real-valued function f (potential).

15. Line integral of a function $f(x, y)$ along C, parameterized by $x = x(t), \quad y = y(t)$ and $a \leq t \leq b$, is

$$\int_C f(x, y)\; ds = \int_a^b f(x(t),\, y(t))\, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

<u>**Remarks**</u>:

(a) $\int_C f(x, y)\, ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$

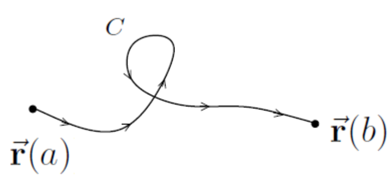
(c) $\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$

16. Line integral of a vector field $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\tilde{\mathbf{k}}$ along an *oriented* curve C, parameterized by $\mathbf{r}(t) = \langle x(t),\, y(t), z(t)\rangle$ and $a \leq t \leq b$, is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\mathbf{F} \cdot \mathbf{T})\; ds = \int_C P\, dx + Q\, dy + R\, dz = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\; dt$$

where $\mathbf{T}(t) = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector.
(dependent of orientation of C, other properties and applications of line integrals of $\mathbf{F}$)

17. <u>FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS</u>: $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$:



18. A vector field $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is *conservative* (i.e. $\mathbf{F} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$;
a vector field $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\tilde{\mathbf{k}}$ is *conservative* if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y};$$

how to determine a potential function f if $\mathbf{F} = \nabla f$.