

Study Guide # 3

You also need Study Guides # 1 and # 2 for the Final Exam

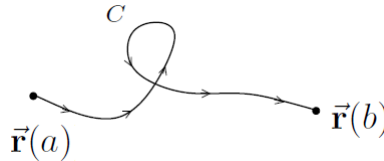
0. Line integral of a vector field $\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$ along an *oriented* curve C , parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $a \leq t \leq b$, is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \vec{T}) ds = \int_C P dx + Q dy + R dz = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

where $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$ is the unit tangent vector.

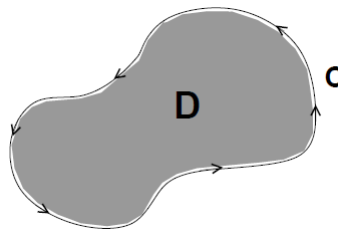
(dependent of orientation of C , other properties and applications of line integrals of $\vec{F}$)

1. FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS: $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$:



2. A vector field $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is *conservative* (i.e. $\vec{F} = \nabla f$) if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$;
how to determine a potential function f if $\vec{F} = \nabla f$.

3. GREEN'S THEOREM: $\int_C P(x, y) dx + Q(x, y) dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$ (C = boundary of D):



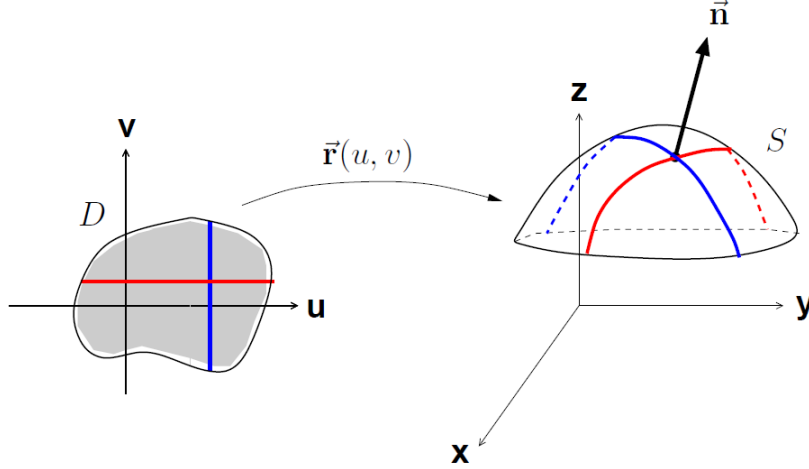
4. DEL OPERATOR: $\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$; if $\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$,
then

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Properties of curl and divergence:

- (i) If $\text{curl } \vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is a conservative vector field ($\vec{\mathbf{F}} = \nabla f$) in a simply-connected domain.
- (ii) If $\text{curl } \vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is *irrotational*; if $\text{div } \vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is *incompressible*.
- (iii) *Laplace's Equation*: $\nabla^2 f = \text{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.
- (iv) For functions with continuous partials, $\text{curl}(\nabla f) = \vec{\mathbf{0}}$, and $\text{div}(\text{curl } \vec{\mathbf{F}}) = 0$.

5. Parametric surface S : $\vec{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, where $(u, v) \in D$:



Normal vector to surface S : $\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$; tangent planes and normal lines to parametric surfaces.

6. Surface area of a surface S :

(i) $A(S) = \iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| dA$

(ii) If S is the graph of $z = h(x, y)$ above D , then $A(S) = \iint_D \sqrt{1 + (\partial h / \partial x)^2 + (\partial h / \partial y)^2} dA$;

Remark: $dS = |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| dA$ = differential of surface area; while $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) dA$

7. The surface integral of f over the surface S :

(i) $\iint_S f(x, y, z) dS = \iint_D f(\vec{\mathbf{r}}(u, v)) |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| dA$.

(ii) If S is the graph of $z = h(x, y)$ above D , then

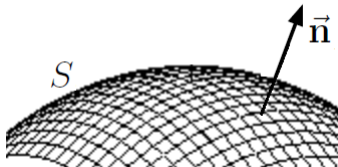
$$\iint_S f(x, y, z) dS = \iint_D f(x, y, h(x, y)) \sqrt{1 + (\partial h / \partial x)^2 + (\partial h / \partial y)^2} dA.$$

8. The surface integral of $\vec{\mathbf{F}}$ over the surface S (recall, $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) dA$):

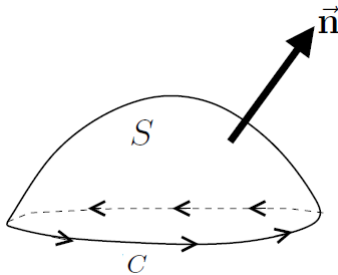
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) dA = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}) dS = \text{flux of } \vec{\mathbf{F}} \text{ across the surface } S.$$

If S is the graph of $z = h(x, y)$ above D , with $\vec{\mathbf{n}}$ oriented upward, and $\vec{\mathbf{F}} = \langle P, Q, R \rangle$, then

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \left(-P \frac{\partial h}{\partial x} - Q \frac{\partial h}{\partial y} + R \right) dA$$

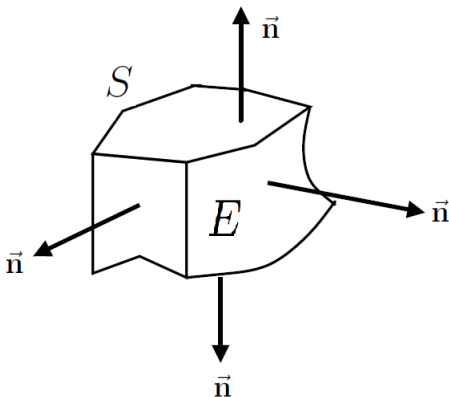


9. STOKES' THEOREM: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$ (recall, $\text{curl } \vec{F} = \nabla \times \vec{F}$).



$\int_C \vec{F} \cdot d\vec{r} = \text{circulation of } \vec{F} \text{ around } C.$

10. THE DIVERGENCE THEOREM/GAUSS' THEOREM: $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \, dV$
 (recall, $\text{div } \vec{F} = \nabla \cdot \vec{F}$).



11. Summary of Line Integrals and Surface Integrals:

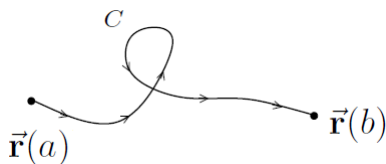
LINE INTEGRALS	SURFACE INTEGRALS
$C : \vec{\mathbf{r}}(t), \text{ where } a \leq t \leq b$	$S : \vec{\mathbf{r}}(u, v), \text{ where } (u, v) \in D$
$ds = \vec{\mathbf{r}}'(t) dt = \text{differential of arc length}$	$dS = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v dA = \text{differential of surface area}$
$\int_C ds = \text{length of } C$	$\iint_S dS = \text{surface area of } S$
$\int_C f(x, y, z) ds = \int_a^b f(\vec{\mathbf{r}}(t)) \vec{\mathbf{r}}'(t) dt$ (independent of orientation of C)	$\iint_S f(x, y, z) dS = \iint_D f(\vec{\mathbf{r}}(u, v)) \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v dA$ (independent of normal vector $\vec{\mathbf{n}}$)
$d\vec{\mathbf{r}} = \vec{\mathbf{r}}'(t) dt$	$d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) dA$
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) dt$ (depends on orientation of C)	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u, v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) dA$ (depends on normal vector $\vec{\mathbf{n}}$)
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}) ds$ The <i>circulation</i> of $\vec{\mathbf{F}}$ around C	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}) dS$ The <i>flux</i> of $\vec{\mathbf{F}}$ across S in direction $\vec{\mathbf{n}}$

12. Integration Theorems:

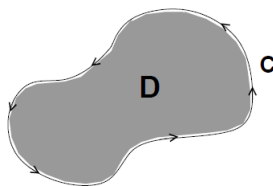
FUNDAMENTAL THEOREM OF CALCULUS: $\int_a^b F'(x) dx = F(b) - F(a)$



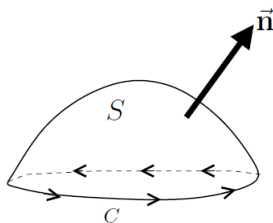
FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS: $\int_a^b \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$



GREEN'S THEOREM: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P(x, y) dx + Q(x, y) dy$



STOKES' THEOREM: $\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$



DIVERGENCE THEOREM: $\iiint_E \text{div } \vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$

