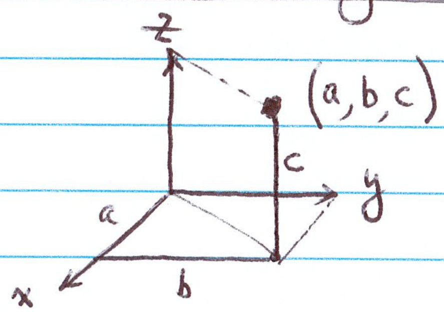
Calculus: Early Transcendentals, 3rd edition

by Briggs, Cochran, Gillett, Schulz

Chapter 13 Vectors and the Geometry of Space

Lesson 1 (Review §13.1 – 13.4)

- three-dimensional rectangular coordinate system



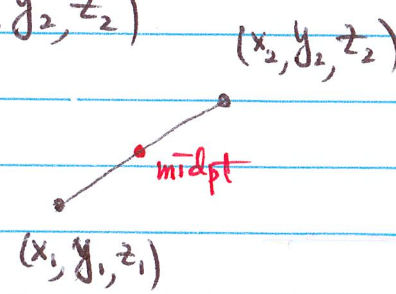
(1) Equations of a simple planes

$$z = c, \quad x = c, \quad y = c$$

(2) Distances between two pts (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z-z_1)^2}$$

$$\text{midpt} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$



(3) Equation of a sphere/ball

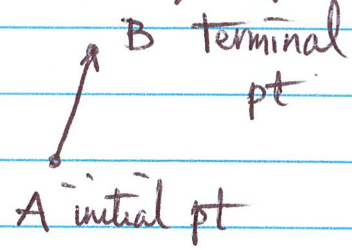
sphere $\quad (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ p824 #31, 34

ball $\quad (x-a)^2 + (y-b)^2 + (z-c)^2 \le r^2$

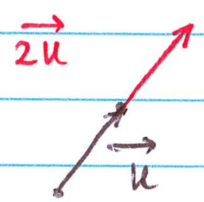
• Vectors

magnitude: mass, length, time, ...

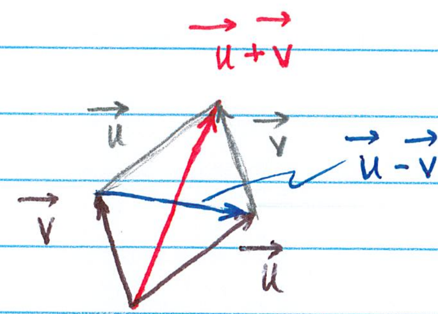
vector (magnitude & direction): force, displacement, velocity, ...



$\vec{V} = \vec{AB}$



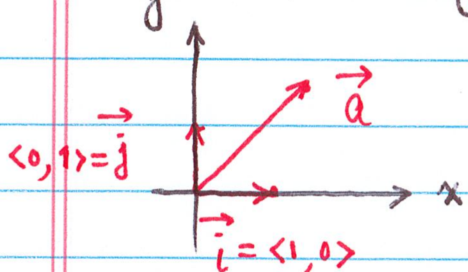
scalar multiplication

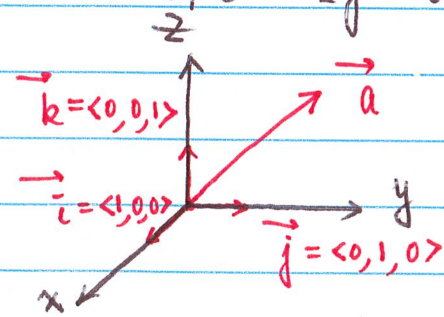


addition & subtraction

components $\quad \vec{a} = \langle a_1, a_2 \rangle$ $\qquad \vec{a} = \langle a_1, a_2, a_3 \rangle$

$= a_1 \vec{i} + a_2 \vec{j}$ $\qquad = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$





magnitude $\quad |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $\qquad c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$

$\vec{a} \pm \vec{b} = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle$

unit vector $\quad \vec{a} \,/\, |\vec{a}|$

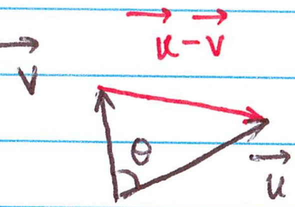
(1) the dot product $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$

Definition $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$

angel $\theta = \cos^{-1} \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} \in [0, \pi]$

orthogonality $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$

Theorem $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$



Proof the law of cosine

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 \underbrace{-2|\vec{u}|\,|\vec{v}|\cos\theta}_{-2\vec{u}\cdot\vec{v}}$$

$$\| $$

$$(u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2$$

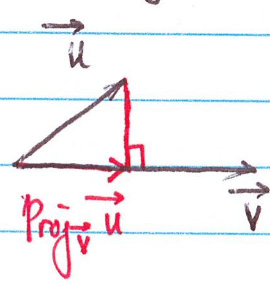
$$= (u_1^2+u_2^2+u_3^2) + (v_1^2+v_2^2+v_3^2) - 2(u_1v_1+u_2v_2+u_3v_3)$$

$\Rightarrow \vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

Orthogonal Projection $\text{proj}_{\vec{v}}\vec{u} = ? = \underbrace{|\vec{u}|\cos\theta}_{\text{length}} \underbrace{\dfrac{\vec{v}}{|\vec{v}|}}_{\text{direction}} = \left(\vec{u} \cdot \dfrac{\vec{v}}{|\vec{v}|}\right) \dfrac{\vec{v}}{|\vec{v}|}$

$\|$

$\alpha\vec{v}$ $\alpha = ?$



$\vec{u} - \text{proj}_{\vec{v}}\vec{u} \perp \vec{v} \Rightarrow 0 = (\vec{u} - \alpha\vec{v}) \cdot \vec{v}$

$= \vec{u}\cdot\vec{v} - \alpha\,\vec{v}\cdot\vec{v}$

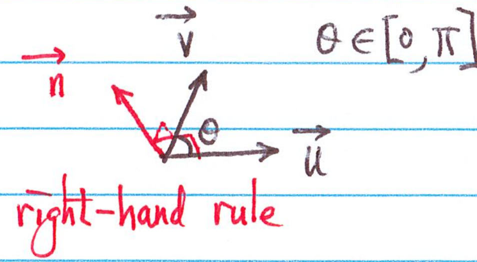
$$\Rightarrow \alpha = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2} = \frac{|\vec{u}|\cos\theta}{|\vec{v}|}$$

(2) the cross product

cross product $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$

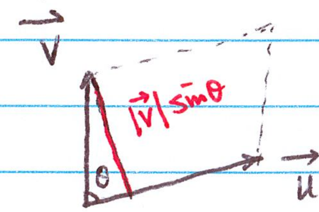
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

$$= \left(|\vec{u}|\, |\vec{v}| \sin\theta \right) \vec{n}$$



properties (a) $\vec{u} \times \vec{v} \perp \vec{u}$, $\vec{u} \times \vec{v} \perp \vec{v}$

(b) $|\vec{u} \times \vec{v}| = |\vec{u}|\, |\vec{v}| \sin\theta =$ area of parallelogram



(c) $\vec{u} \times \vec{v} = 0 \iff \vec{u} \parallel \vec{v}$

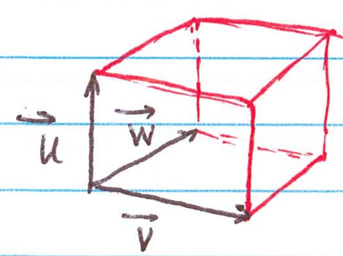
(d) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{u} \cdot \vec{v}) \vec{w}$$

(e) $\left| \vec{u} \cdot (\vec{v} \times \vec{w}) \right| = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} =$ volume of parallelopiped

(#62)



§13.5 Lines and Planes in Space

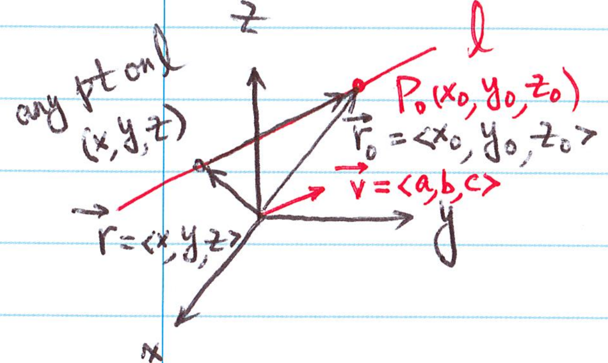
• Equation of a Line ℓ

Given (1) a point $P_0(x_0, y_0, z_0)$ on ℓ
(2) a vector $\vec{v} = \langle a, b, c \rangle \parallel \ell$

or two points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$

$\Downarrow$

a pt $P_0(x_0, y_0, z_0)$ on ℓ
a vector $\overrightarrow{P_0P_1} \parallel \ell$



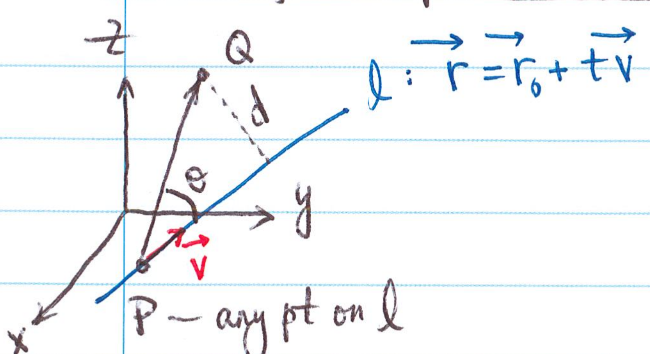
$$\vec{r} = \vec{r}_0 + t\vec{v} \quad \text{(vector equation)}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \iff \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

parametric equation.

examples #12, 20, 28, 34, 38

• Distance from a point to a line



$$d = |\overrightarrow{PQ}| \sin\theta$$

$$|\vec{v} \times \overrightarrow{PQ}| = |\vec{v}|\,|\overrightarrow{PQ}| \sin\theta = |\vec{v}|\, d$$

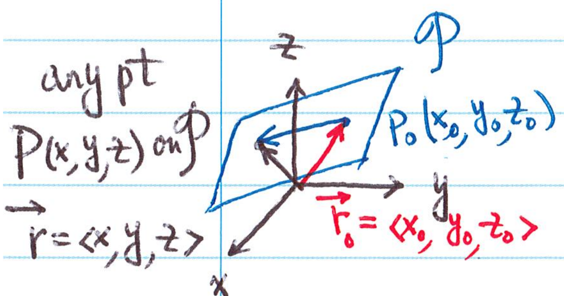
$$\Rightarrow d = \frac{|\vec{v} \times \overrightarrow{PQ}|}{|\vec{v}|}$$

- <u>Equation of a plane in R^3</u> $\mathcal{P}$

<u>Given</u> $\begin{cases} \text{(1) a point } P_0(x_0,y_0,z_0) \text{ on } \mathcal{P} \\ \text{(2) a vector } \vec{n} = \langle a,b,c\rangle \perp \mathcal{P} \end{cases}$ or $\left[\begin{array}{l} \text{three pts} \\ P_0,\ P_1,\ P_2 \end{array}\right.$

$$\Downarrow$$

$$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}$$



$$\overrightarrow{P_0P} = \vec{r} - \vec{r}_0 \text{ on } \mathcal{P}$$

$$\Rightarrow 0 = \left(\vec{r} - \vec{r}_0\right) \cdot \vec{n}$$

$$= a(x - x_0) + b(y - y_0) + c(z - z_0)$$

or $\quad ax + by + cz = d \equiv ax_0 + by_0 + cz_0$

<u>examples</u> #46, 66, 74, 79, 87

§13.6 Cylinders and Quadratic Surfaces

by Katty Yackman

Def. A **trace** is the set of points at which a surface intersects a plane that is parallel to one of the coordinate planes.

xy-trace: intersection with $z=0$ (the xy-plane)
yz-trace: intersection with $x=0$ (the yz-plane)
xz-trace: intersection with $y=0$ (the xz-plane)

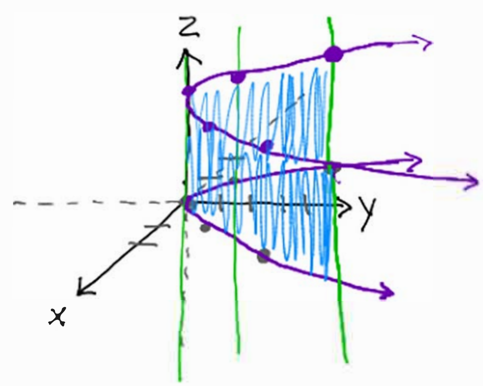
Note: Intersection with coordinate axes

x-axis: Set $y=0$ and $z=0$
y-axis: Set $x=0$ and $z=0$
z-axis: Set $x=0$ and $y=0$

Def. A **cylinder** is a surface that consists of all lines that are parallel to a given line and pass through a given curve.

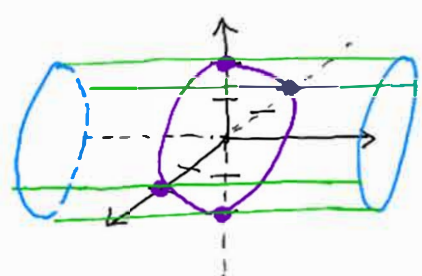
Ex. $y = x^2$

Because there is no restriction on z, this surface consists of all lines parallel to the z-axis that pass through the curve $y=x^2$ (in the xy-plane).



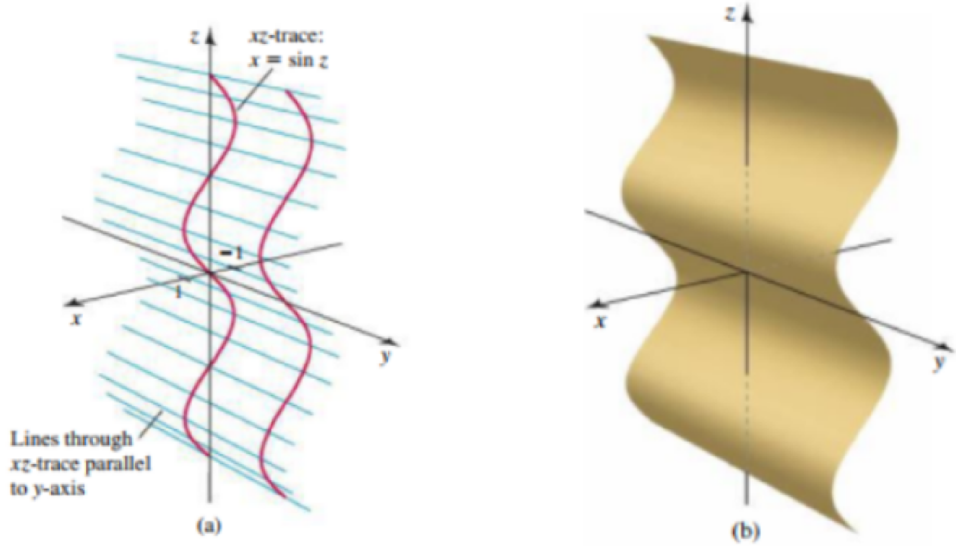
Ex. $x^2 + z^2 = 4$

No restriction on y, so parallel to y-axis through the circle $x^2+z^2=4$ (in the xz-plane).



Ex. $x - \sin z = 0$

See Fig. 13.82 (below) from Pg. 857 of the text.



Def. Quadratic surfaces have the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where not all of A, B, C, D, E, and F are 0.

We will focus on a smaller class where
$D = E = F = 0$.

Note: $ax + by + cz = d$ is a plane

all variables linear $\rightarrow$ Not a quadratic surface.

$x^2 + y^2 + z^2 = r^2$ is a sphere

all squared $\longrightarrow$ Quadratic Surface

Also, some cylinders are quadratic surfaces (like $y = x^2$ and $x^2 + z^2 = 4$, but $x = \sin z$ is not).

Def. Ellipsoids have the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.
All traces are ellipses. A sphere when $a=b=c$.

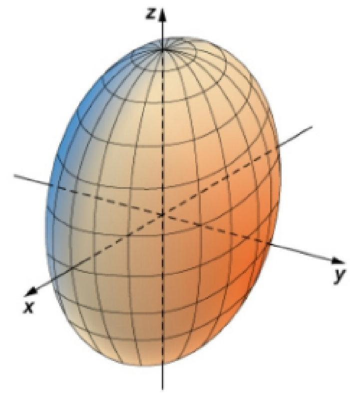
Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Traces
In plane $z=p$: an ellipse
In plane $y=q$: an ellipse
In plane $x=r$: an ellipse

Ellipsoid

If $a=b=c$, then this surface is a sphere.



Elliptic paraboloids have the form $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.
Traces in two directions are parabolas. In this case, when $y=0$: $\frac{z}{c}=\frac{x^2}{a^2}$ and when $x=0$: $\frac{z}{c}=\frac{y^2}{b^2}$.
Traces in one direction are ellipses. In this case, when $z=c$: $1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.
When $z=0$, the "ellipse" is the point $(0,0,0)$.

Enter as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$

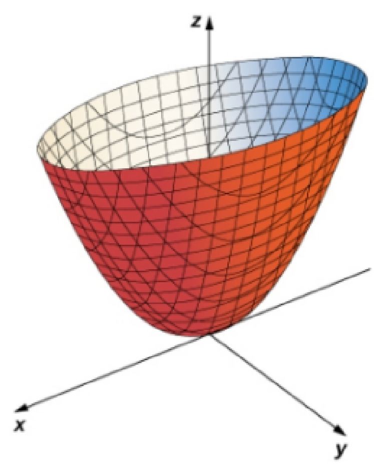
Elliptic Paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Traces
In plane $z=p$: an ellipse — Elliptic
In plane $y=q$: a parabola
In plane $x=r$: a parabola

Paraboloid

The axis of the surface corresponds to the linear variable.



Hyperbolic paraboloids have the form $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b}$

Traces in two directions are parabolas. In this case, when $y=0: \frac{z}{c} = \frac{x^2}{a^2}$ and when $x=0: \frac{z}{c} = -\frac{y^2}{b^2}$.

Traces in one direction are hyperbolas. In this case, when $z=c: 1 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.

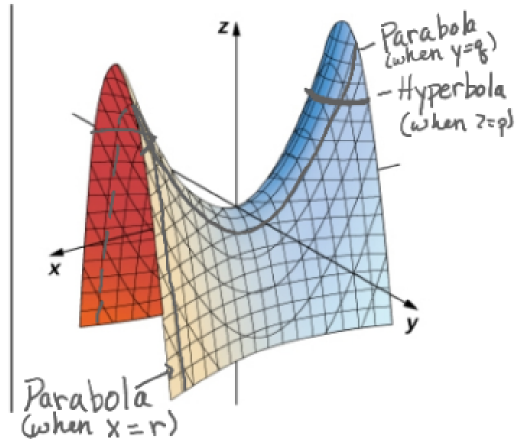
Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces

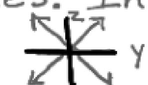
In plane $z = p$: a hyperbola — Hyperbolic
In plane $y = q$: a parabola
In plane $x = r$: a parabola — Paraboloid

The axis of the surface corresponds to the linear variable.



(Elliptic) cones have the form $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Traces in two directions are two lines. In this case, when $x=0: \frac{z^2}{c^2} = \frac{y^2}{b^2} \Rightarrow \frac{|z|}{|c|} = \frac{|y|}{|b|} \Rightarrow$



$y=0: \frac{z^2}{c^2} = \frac{x^2}{a^2} \Rightarrow \frac{|z|}{|c|} = \frac{|x|}{|a|} \Rightarrow$

Enter the squared versions, not absolute values

Traces in one direction are ellipses. In this case, when $z=c: 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Again, when $z=0$, the "ellipse" is the point $(0,0,0)$. This point is the center of the cone.

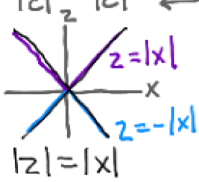
Enter as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$.

Elliptic Cone

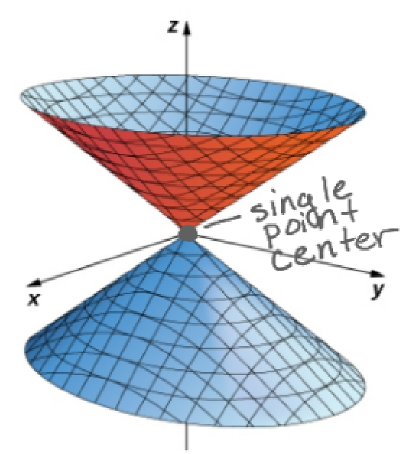
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces

In plane $z = p$: an ellipse — Elliptic
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
$\frac{|z|}{|c|} = \frac{|x|}{|c|}$ ← In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin



The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.



Hyperboloids of one sheet have the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Traces in two directions are hyperbolas. In this case, when $y=0$: $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$

when $x=0$: $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Traces in one direction are ellipses. In this case, when $z=0$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The center of this ellipse is the center of the surface.

Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

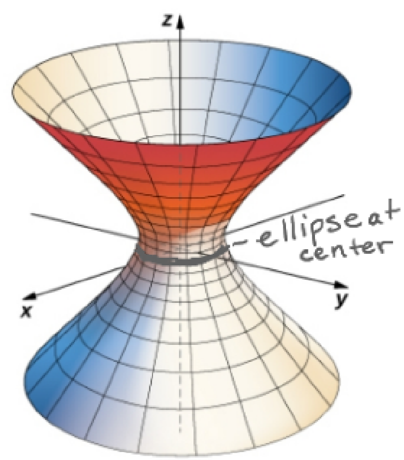
Traces

In plane $z = p$: an ellipse

In plane $y = q$: a hyperbola ⟩ Hyperboloid

In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.



Note: Center could be shifted.

For instance, $(x-2)^2 + (y+1)^2 - (z+2)^2 = 1$.

To determine if cone, hyperboloid of one sheet or hyperboloid of two sheets, check

$$\left.\begin{array}{ll} x=2: & \text{hyperbola} \\ y=-1: & \text{hyperbola} \\ z=-2: & \text{ellipse (circle)} \end{array}\right\} \text{hyperboloid of one sheet.}$$

Hyperboloids of two sheets have the form

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Traces in two directions are hyperbolas.

In this case, $x=0$: $-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$y=0$: $-\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$.

Traces in one direction are ellipses.

$\frac{z^2}{c^2}-1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ $\Rightarrow$ If $\frac{z^2}{c^2}-1<0$, $-c<z<c$, the the equation is not valid, so there are no traces.

$\Rightarrow$ When $z=\pm c$, $0=\frac{x^2}{a^2}+\frac{y^2}{b^2}$, so the "vertices" of the sheets are $(0,0,c)$ and $(0,0,-c)$, and the distance between the sheets is $2c$.

Hyperboloid of Two Sheets

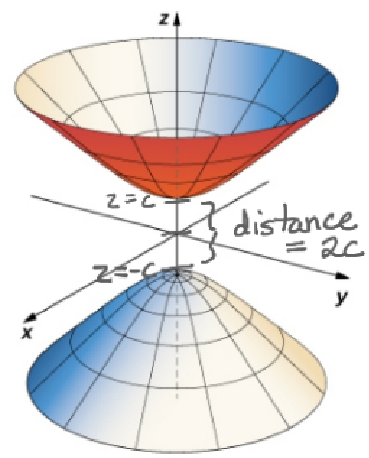
$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Traces

In plane $z=p$: an ellipse or the empty set (no trace)

In plane $y=q$: a hyperbola
In plane $x=r$: a hyperbola $\Big\}$ hyperboloid

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.



Ex. $(x-2)^2-(y-7)^2-(z+3)^2=1$

$x=2$: $-(y-7)^2-(z+3)^2=1$

$(y-7)^2+(z+3)^2=-1$ Not possible (No trace)

$y=7$: Hyperbola

$z=-3$: Hyperbola

Hyperboloid of two sheets