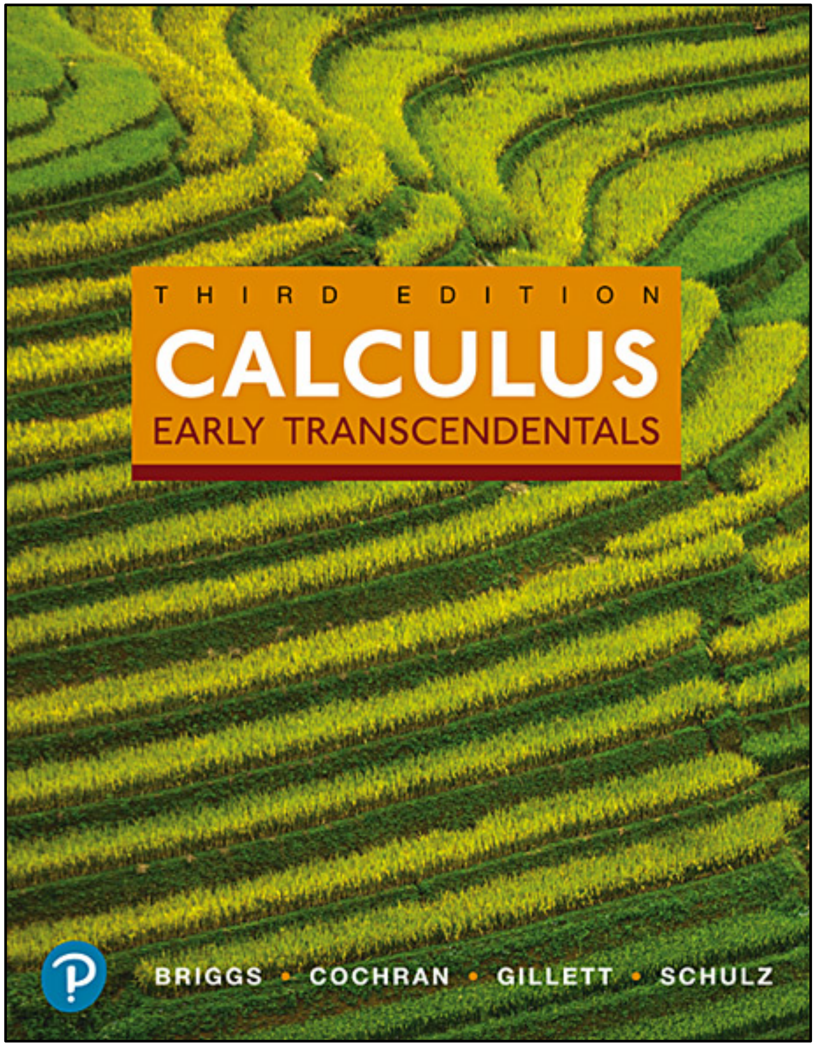
Calculus Early Transcendentals

Third Edition



Chapter 13

Vectors and the Geometry of Space

Lesson 1 (§13.1-4) Vectors in Plane and Space
Dot- and Cross-Product
Lesson 2 (§13.5) Lines and Planes in Space
Lesson 3-4 (§13.6) Cylinders and Quadric Surfaces

Three-dimensional rectangular coordinate system

Cartesian

Figure 13.25

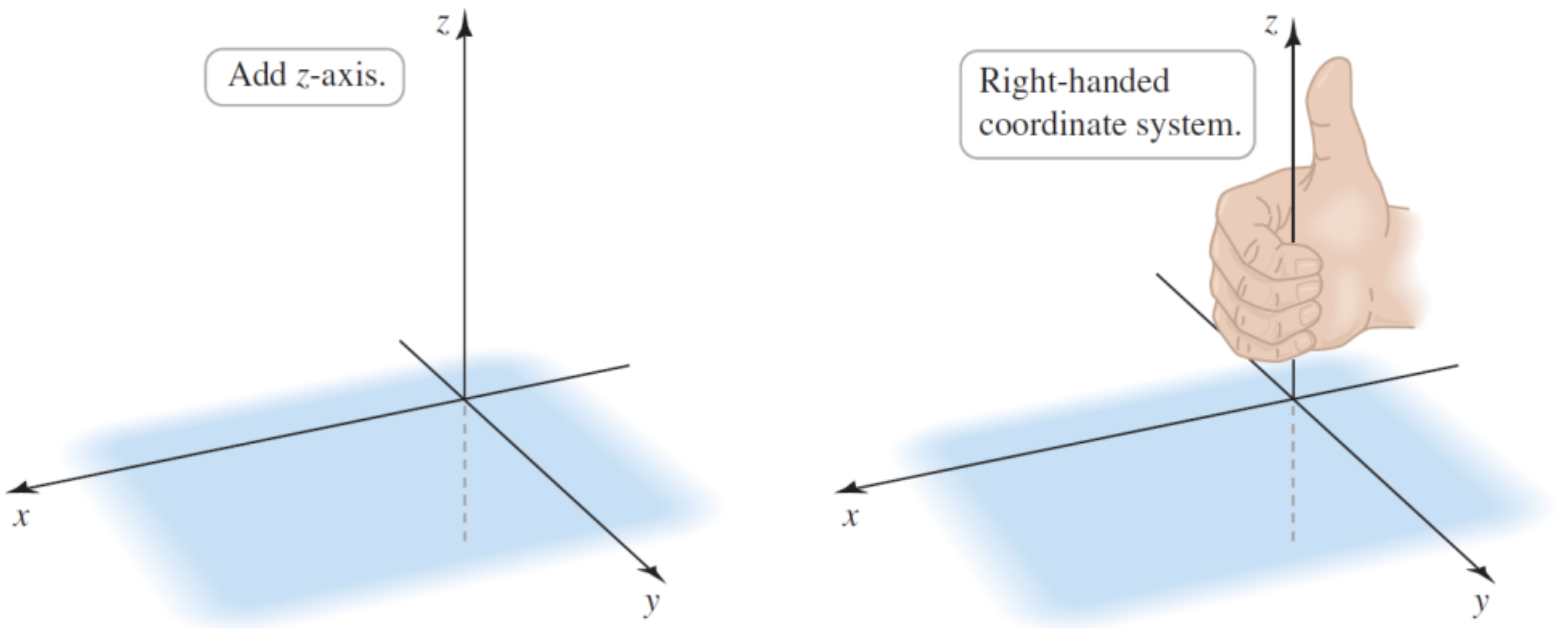
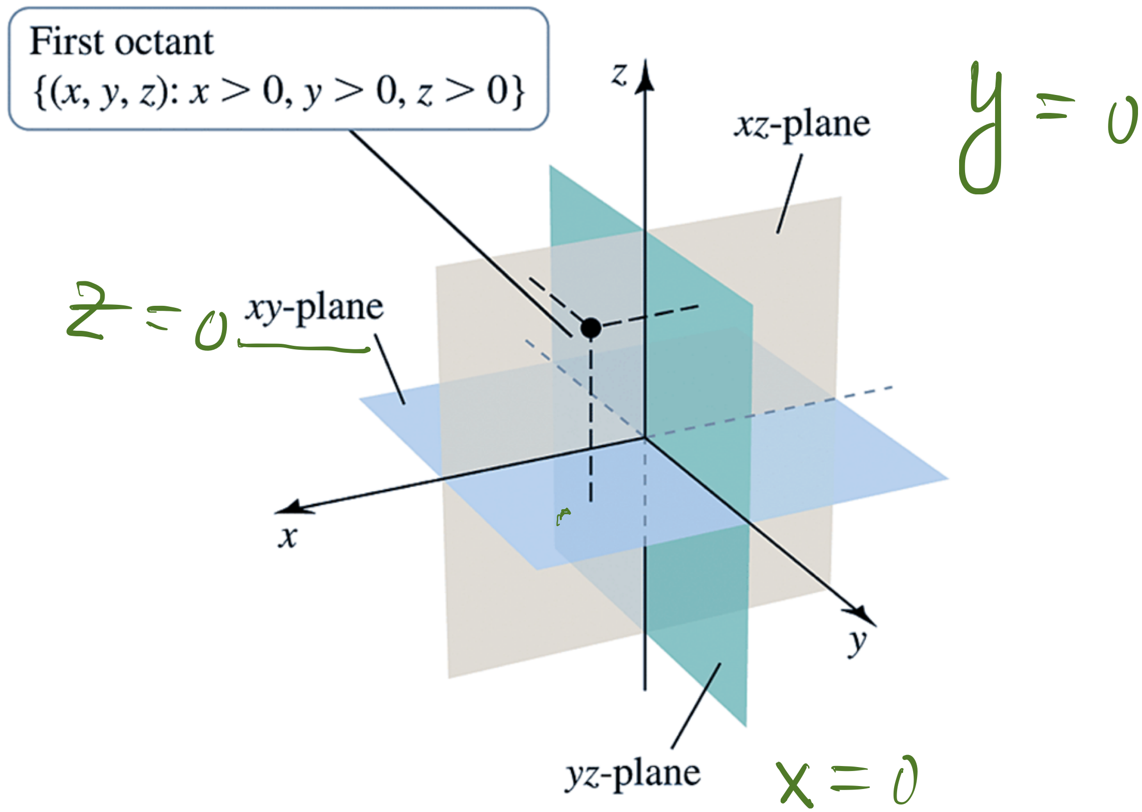


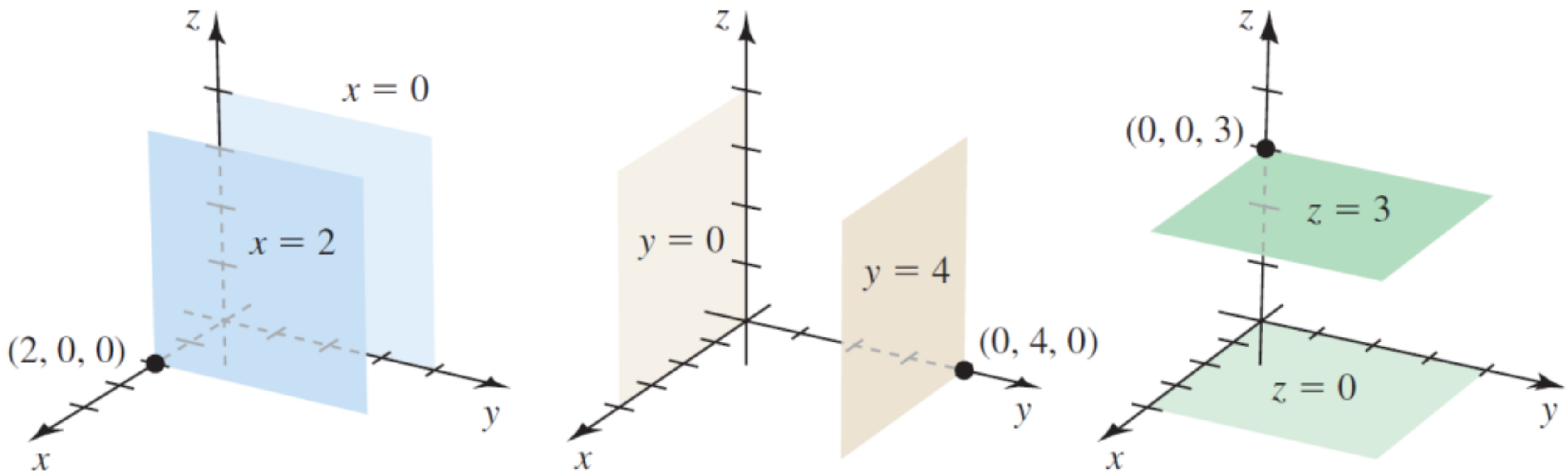
Figure 13.26



xyz-space is divided into octants.

Figure 13.30

Simple planes



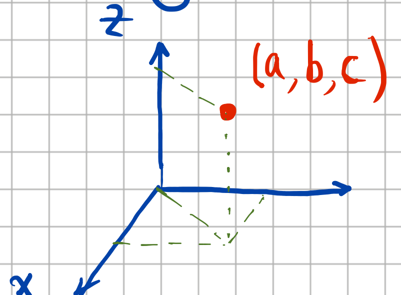
Calculus: Early Transcendentals, 3rd edition

by Briggs, Cochran, Gillett, Schulz

Chapter 13 Vectors and Geometry of Space

Lesson 1 (Review §13.1–13.4): vectors in 2D, 3D; dot and cross products

(1) 3D Rectangular Coordinate System



(3) equations of simple planes

$x = c$, $y = c$, or $z = c$;

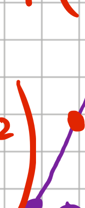
$x + y + z = 1$

(2) distance and midpts between two pts

$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

$$\text{distance} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$\text{midpt} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$



#20 (p824) Sketch the plane parallel to the xz-plane containing the pt (1, 2, 3).

$y = 2$

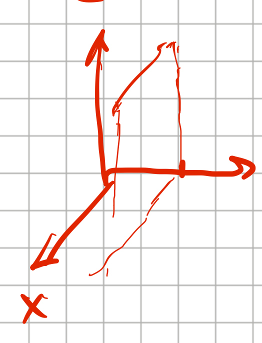
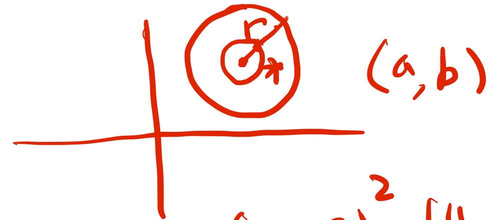
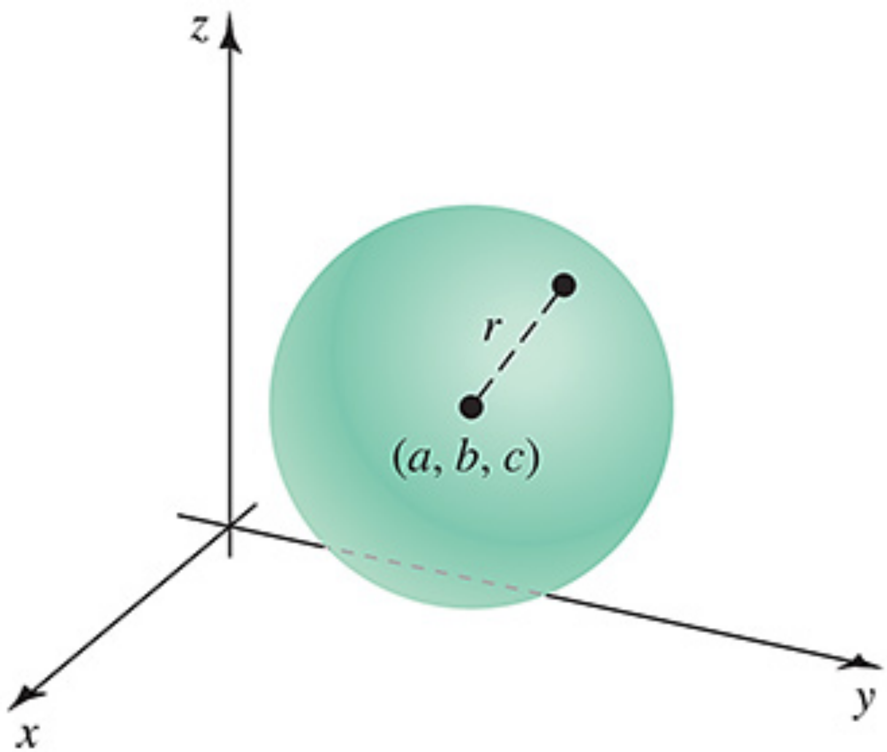


Figure 13.34



$(x-a)^2+(y-b)^2 \le r^2$



Sphere: $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

Ball: $(x-a)^2+(y-b)^2+(z-c)^2 \le r^2$

(4) equations of a sphere/ball

sphere: $(x-a)^2+(y-b)^2+(z-c)^2=r^2$; ball: $(x-a)^2+(y-b)^2+(z-c)^2\le r^2$

Examples

- #28 Find an equation of the sphere passing through $P(-4,2,3)$ and $Q(0,2,7)$ with its center at the midpt of PQ.

$$\text{mid pt} = \frac{Q+P}{2} = \left(\frac{-4+0}{2}, \frac{2+2}{2}, \frac{3+7}{2}\right)$$
$$= (-2, 2, 5) = R$$
$$r = |\overrightarrow{PR}| = \sqrt{(-4-(-2))^2+(2-2)^2+(3-5)^2}$$
$$= \sqrt{4+0+4} = \sqrt{8}$$
$$(x+2)^2+(y-2)^2+(z-5)^2 = 8$$

- Give a geometric description of the following sets of pts:

- $x^2+y^2+z^2-2y-4z-4=0$

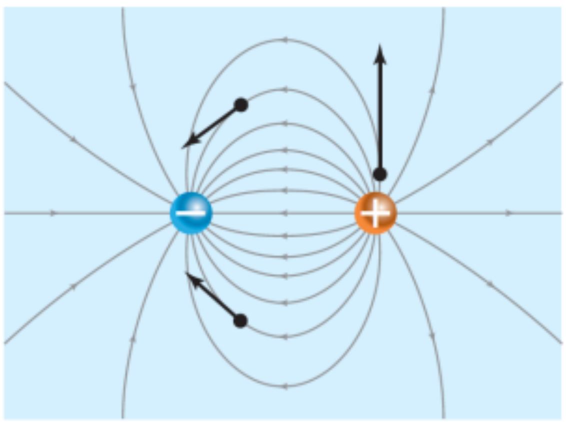
$$x^2+(y^2-2y+1)+(z^2-4z+4)=4+1+4$$
$$x^2+(y-1)^2+(z-2)^2=3^2$$
$$(0,1,2),\ r=3$$

- $x^2+y^2+z^2-14y\le -13$

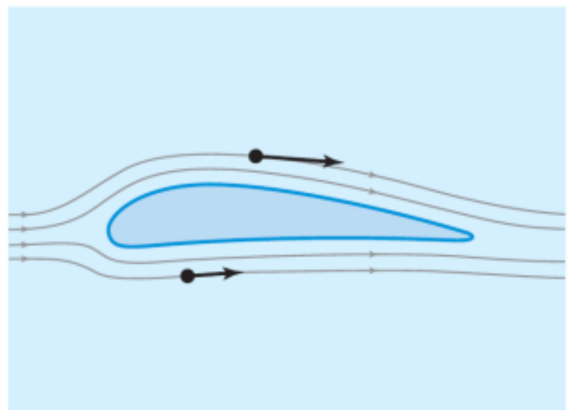
$$x^2+(y^2-2\times 7y+7^2)+z^2\le 7^2-13$$
$$x^2+(y-7)^2+z^2\le 6^2$$

ball $(0,7,0)$, $r=6$

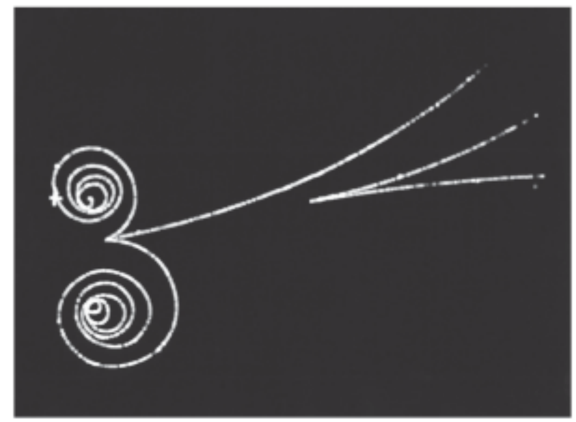
Figure 13.2



Electric field vectors due to two charges

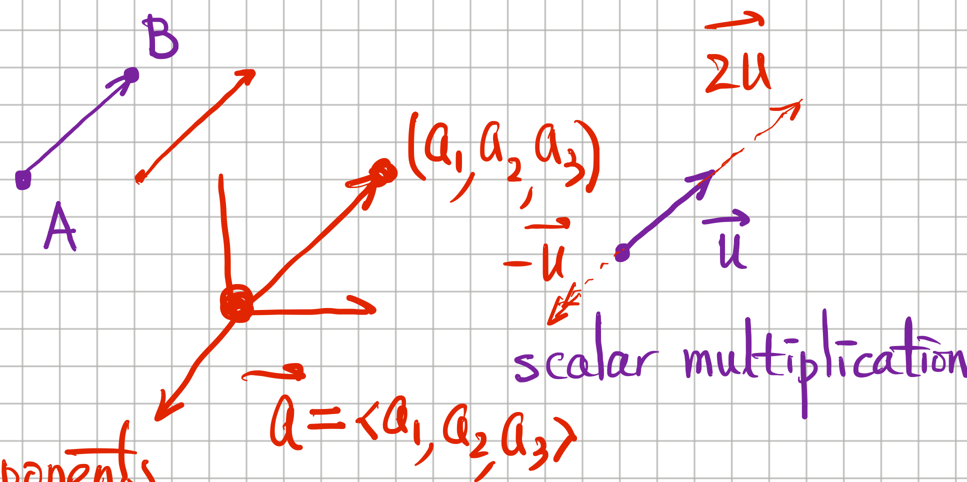


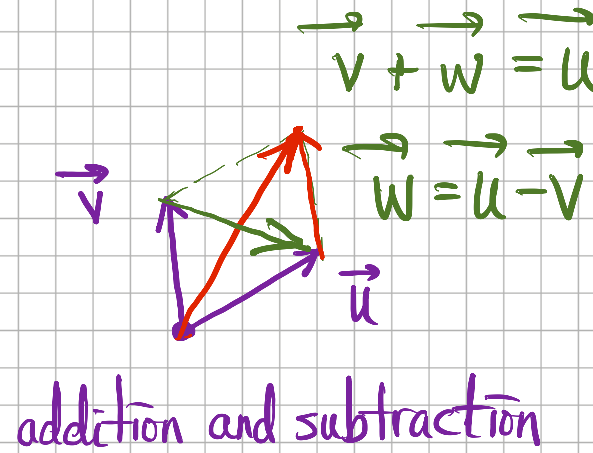
Velocity vectors of air flowing over an airplane wing



Tracks of elementary particles in a cloud chamber are aligned with the velocity vectors of the particles.

(5) Vectors (magnitude + direction)

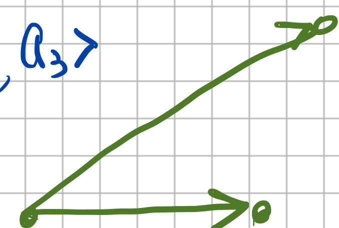




- components

$\vec{a} = \langle a_1, a_2 \rangle$; $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$=$ $=$



Example #75(a) Determine whether the pts: $P(1,6,-5)$, $Q(2,5,-3)$, and $R(4,3,1)$ are collinear (lie on a line).

$\overrightarrow{PQ} = \langle 2-1, 5-6, -3-(-5) \rangle$

$= \langle 1, -1, 2 \rangle$

$\overrightarrow{PR} = \langle 4-1, 3-6, 1-(-5) \rangle$

$= \langle 3, -3, 6 \rangle$

$= 3\langle 1, -1, 2 \rangle \; // \; \overrightarrow{PQ}$

- magnitude, manipulations, unit vector

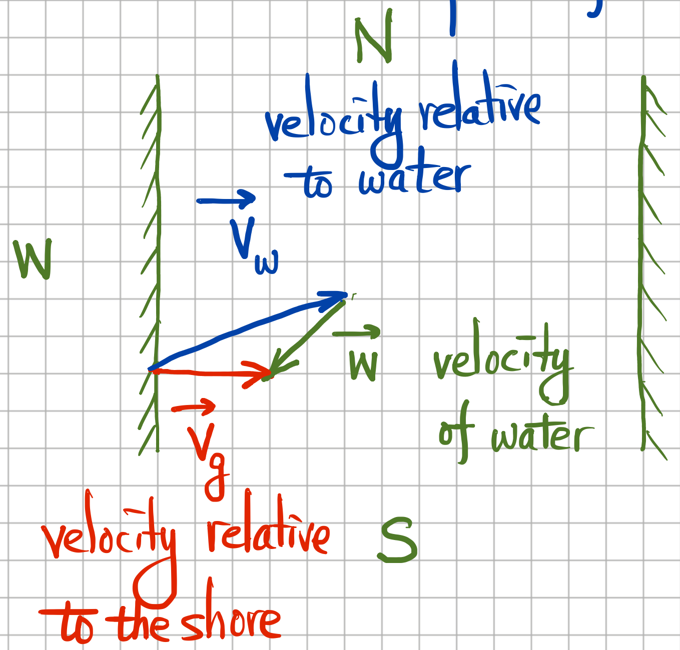
$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$$

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

Example 6 (§13.1, P811) Speed of a boat in a current

Suppose the water in a river moves south-west (45° west of south) at 4 mile/hour and a motorboat travels due east at 15 mile/hour relative to the shore. Determine the speed of the boat and its heading relative to the moving water.



Given $\vec{w}$: $|\vec{w}| = 4$, $\vec{w} = 2\sqrt{2}\langle -1, -1\rangle$

$|\vec{V}_g| = 15$

Unknown $\vec{V}_w = ?$

$$\vec{V}_w = \vec{u} - \vec{w} = \langle 15, 0\rangle - \langle -2\sqrt{2}, -2\sqrt{2}\rangle$$

$$= \langle 15 + 2\sqrt{2}, +2\sqrt{2}\rangle$$

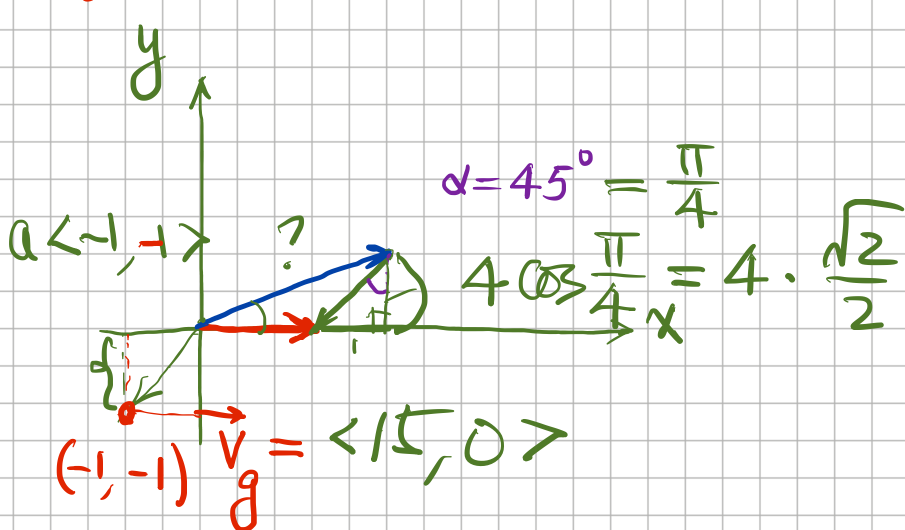
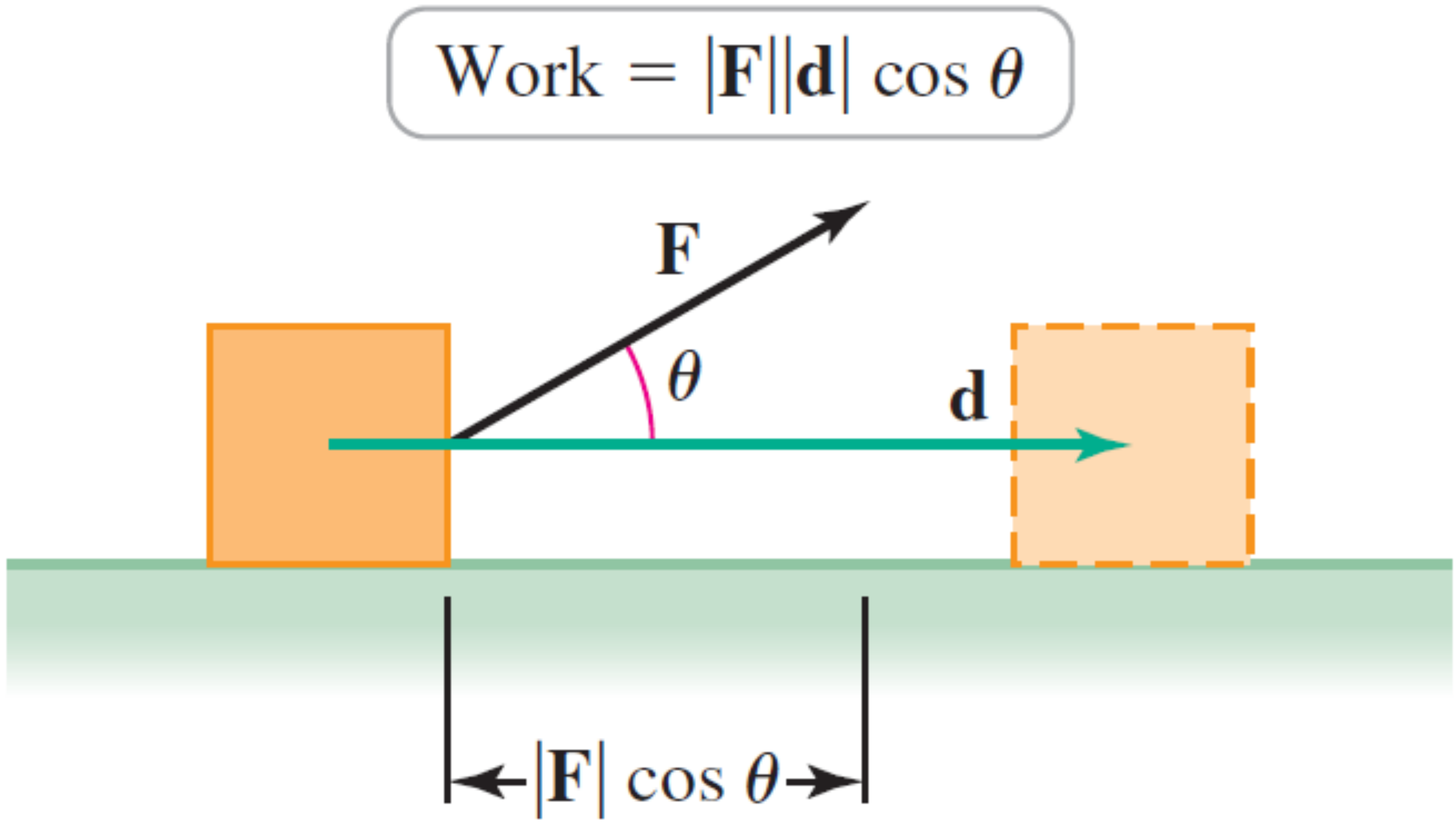
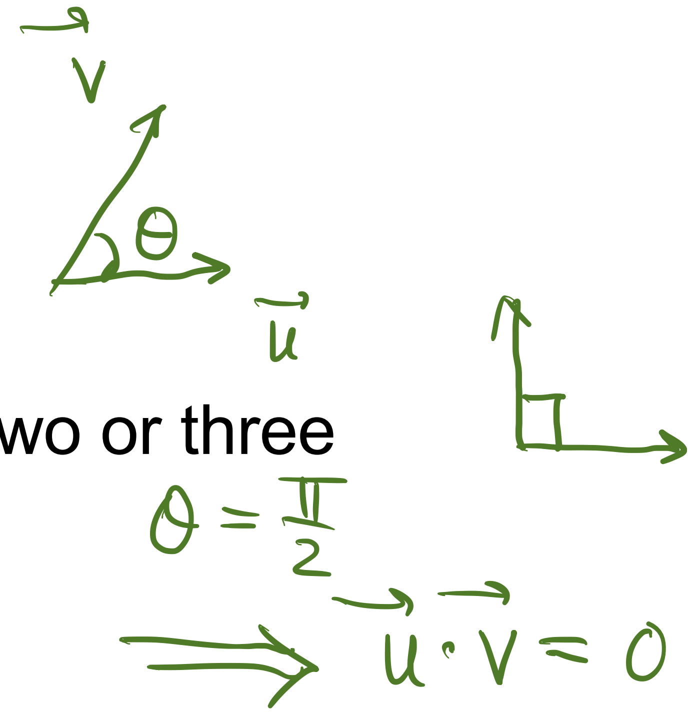


Figure 13.43 (b)



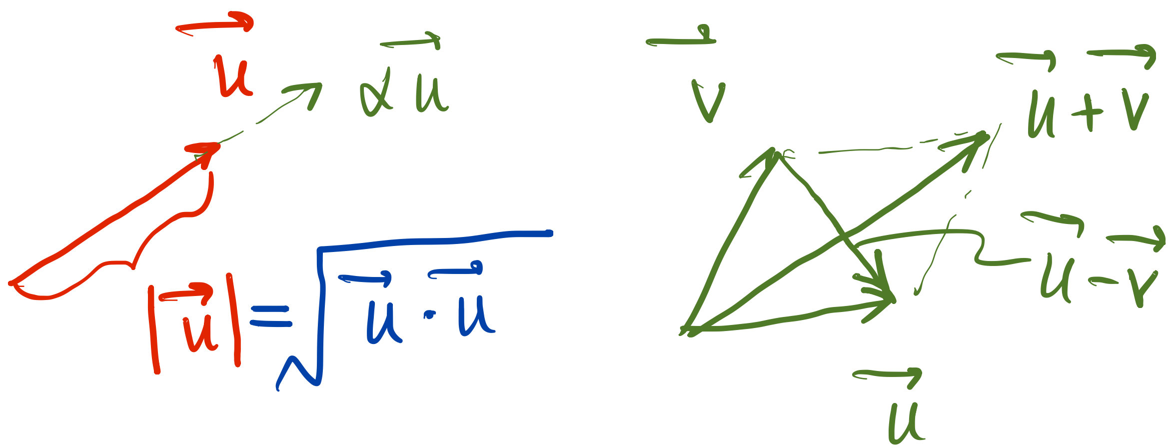
Definition Dot Product



Given two nonzero vectors **u** and **v** in two or three dimensions, their **dot product** is

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

where θ is the angle between **u** and **v** with $0 \le \theta \le \pi$ (Figure 13.44). If **u = 0** or **v = 0,** then $\mathbf{u}\cdot\mathbf{v} = 0$ and θ is undefined.



(6) the dot product

$\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2 = |\vec{u}|^2$$

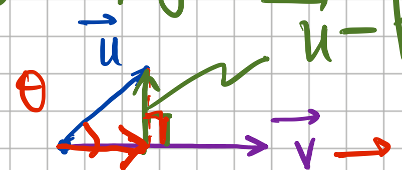
$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$$

- orthogonal projection

$$\vec{u} - \mathrm{Proj}_{\vec{v}} \vec{u} \perp \vec{v}$$

$$\mathrm{Proj}_{\vec{v}} \vec{u} = \alpha \frac{\vec{v}}{|\vec{v}|} = \left(\vec{u} \cdot \frac{\vec{v}}{|\vec{v}|} \right) \frac{\vec{v}}{|\vec{v}|}$$



$$|\vec{u}| \cos\theta \frac{\vec{v}}{|\vec{v}|} = \mathrm{Proj}_{\vec{v}} \vec{u} = \alpha \frac{\vec{v}}{|\vec{v}|}$$

(7) the cross product

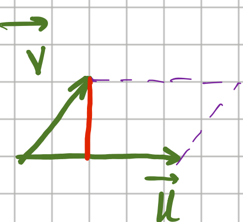
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

- properties

(a) $\vec{u} \times \vec{v} \perp \vec{u}$, $\vec{u} \times \vec{v} \perp \vec{v}$

(b) $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$

area of parallelogram



(c) $\vec{u} \times \vec{v} = 0 \iff \vec{u} \parallel \vec{v}$

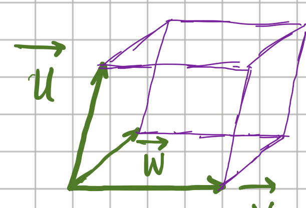
(d) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

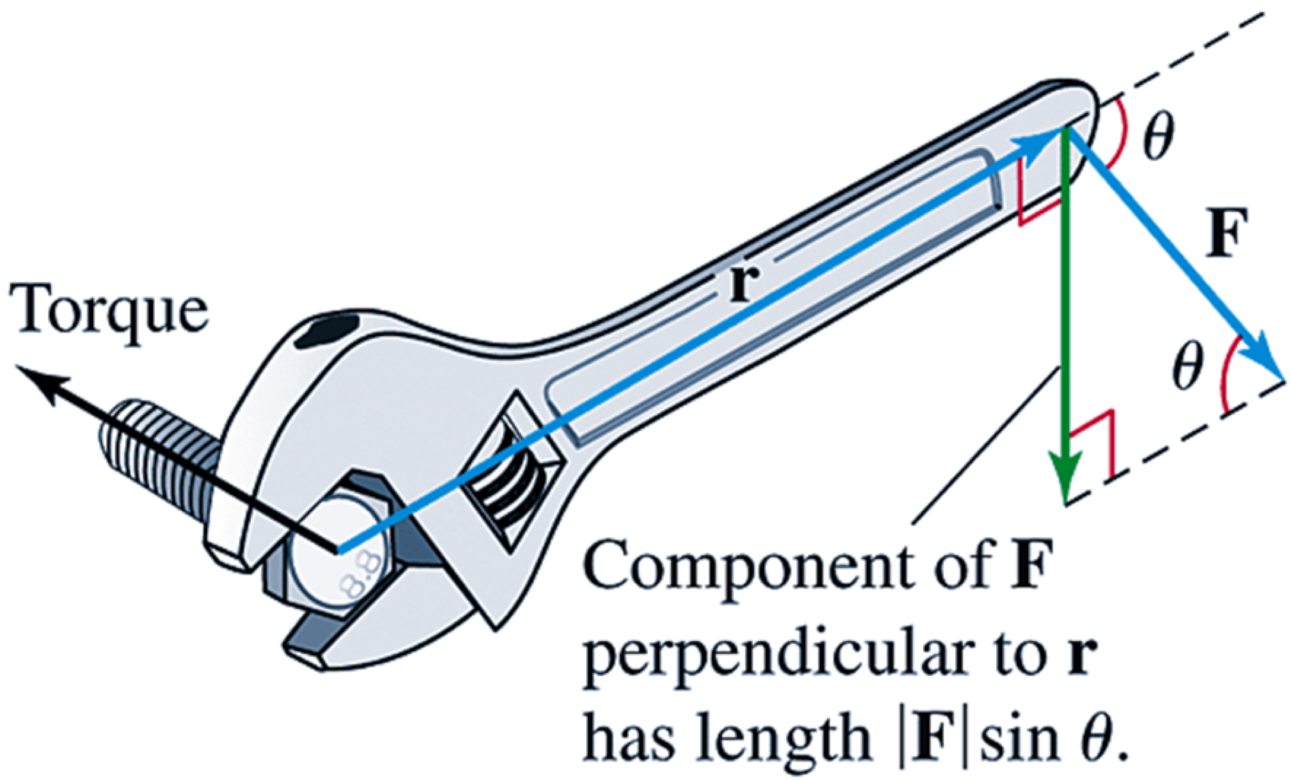
(e) $|\vec{u} \cdot (\vec{v} \times \vec{w})|$

$$= \left| \begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{matrix} \right|$$



= volume of parallelopiped

Figure 13.55


Torque
r
θ
F
θ
Component of **F** perpendicular to **r** has length |**F**| sin θ.

Definition Cross Product

Given two nonzero vectors **u** and **v** in $\circ^{3}$, the **cross product** $\mathbf{u} \times \mathbf{v}$

is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where $0 \le \theta \le \pi$ is the angle between **u** and **v**. The direction of $\mathbf{u} \times \mathbf{v}$

is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from **u** to **v**, the direction of $\mathbf{u} \times \mathbf{v}$

is the direction of your thumb, orthogonal to both **u** and **v** (Figure 13.56).

When $\mathbf{u} \times \mathbf{v} = \mathbf{0},$ the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.

Pearson

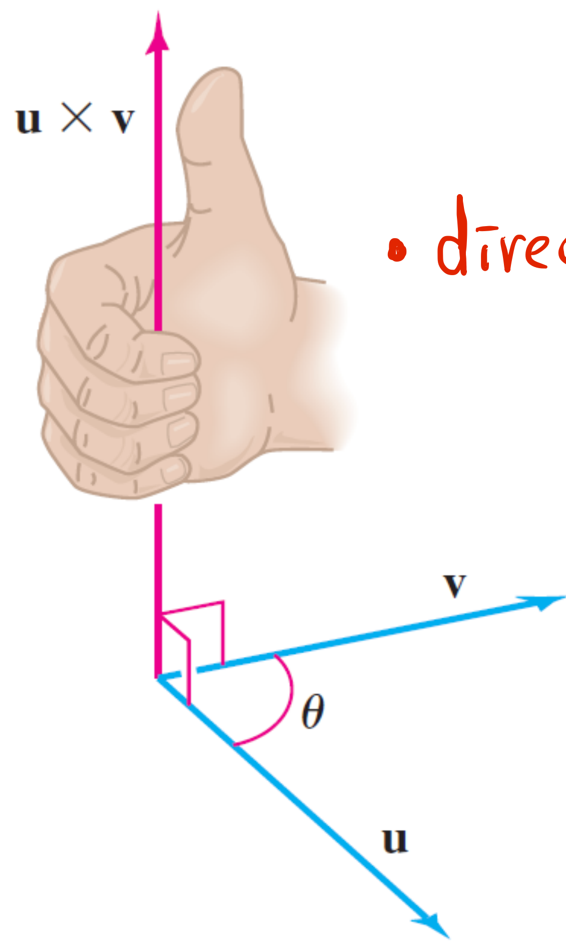
Figure 13.56

$\vec{u} \times \vec{V}$ cross product

- magnetude

$$|\vec{u} \times \vec{V}| = |\vec{u}||\vec{v}| \sin\theta$$

- direction



Theorem 13.6 Evaluating the Cross Product

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$

$$= \left(u_2 v_3 - u_3 v_2\right)\vec{i} - \left(u_1 v_3 - u_3 v_1\right)\vec{j} + \left(u_1 v_2 - u_2 v_1\right)\vec{k}$$

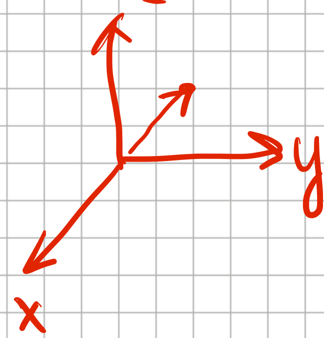
Examples

$\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$

(a) #54 (P835) Find two vectors that are orthogonal to $\langle 0,1,1 \rangle$ and to each other.

$\vec{u} = \langle 1, 0, 0 \rangle$ $\quad \vec{v} = \vec{u} \times \langle 0,1,1 \rangle$

$$\begin{cases} \vec{u} \cdot \langle 0,1,1 \rangle = 0 \\ \vec{v} \cdot \langle 0,1,1 \rangle = 0 \\ \vec{u} \cdot \vec{v} = 0 \end{cases}$$



(b) #60 (P844) Vector Equation

Find all vectors $\vec{u}$ that satisfy $\langle 1,1,1 \rangle \times \vec{u} = \langle 0,0,1 \rangle$

$$\langle 1,1,1 \rangle \times \vec{u} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ u_1 & u_2 & u_3 \end{vmatrix} = \langle u_3 - u_2, u_1 - u_3, u_2 - u_1 \rangle$$

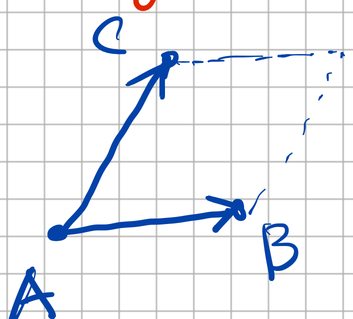
$$\begin{cases} u_3 - u_2 = 0 \\ u_1 - u_3 = 0 \\ u_2 - u_1 = 1 \end{cases} \Rightarrow \begin{matrix} u_3 = u_2 = u_1 \\ u_1 = u_3 \end{matrix} \qquad \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$u_2 - u_1 = 0$

$0 \neq 1 \Rightarrow$ no $\vec{u}$

(c) #33 (P843) Find the area of the triangle T with vertices:

A(0,0,0), B(3,0,1), and C(1,1,0)



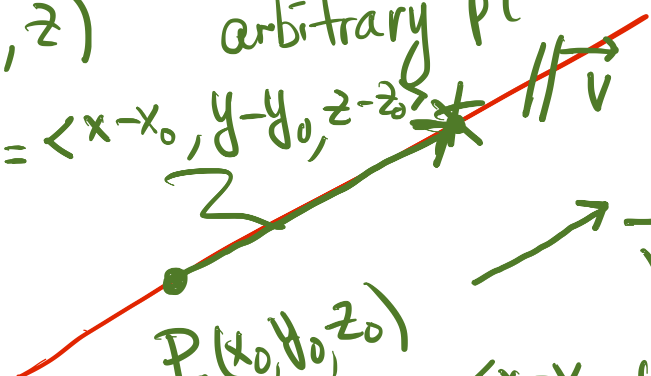
$$\frac{1}{2} \left| \vec{AB} \times \vec{AC} \right| = \frac{1}{2} \left\| \begin{vmatrix} i & j & k \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \right\| = \frac{1}{2} \left| \langle -1, 1, 3 \rangle \right| = \frac{1}{2}\sqrt{1+1+3^2}$$

$$= \frac{1}{2}\sqrt{11}$$

Section 13.5 Lines and Planes in Space

$P(x, y, z)$ arbitrary pt

ℓ



$\overrightarrow{P_0P} = \langle x-x_0, y-y_0, z-z_0 \rangle \parallel \vec{v}$

$P_0(x_0, y_0, z_0)$

$\vec{v} = \langle a, b, c \rangle$

Given pt $P_0(x_0, y_0, z_0)$ on ℓ

vector $\vec{v} = \langle a, b, c \rangle \parallel \ell$

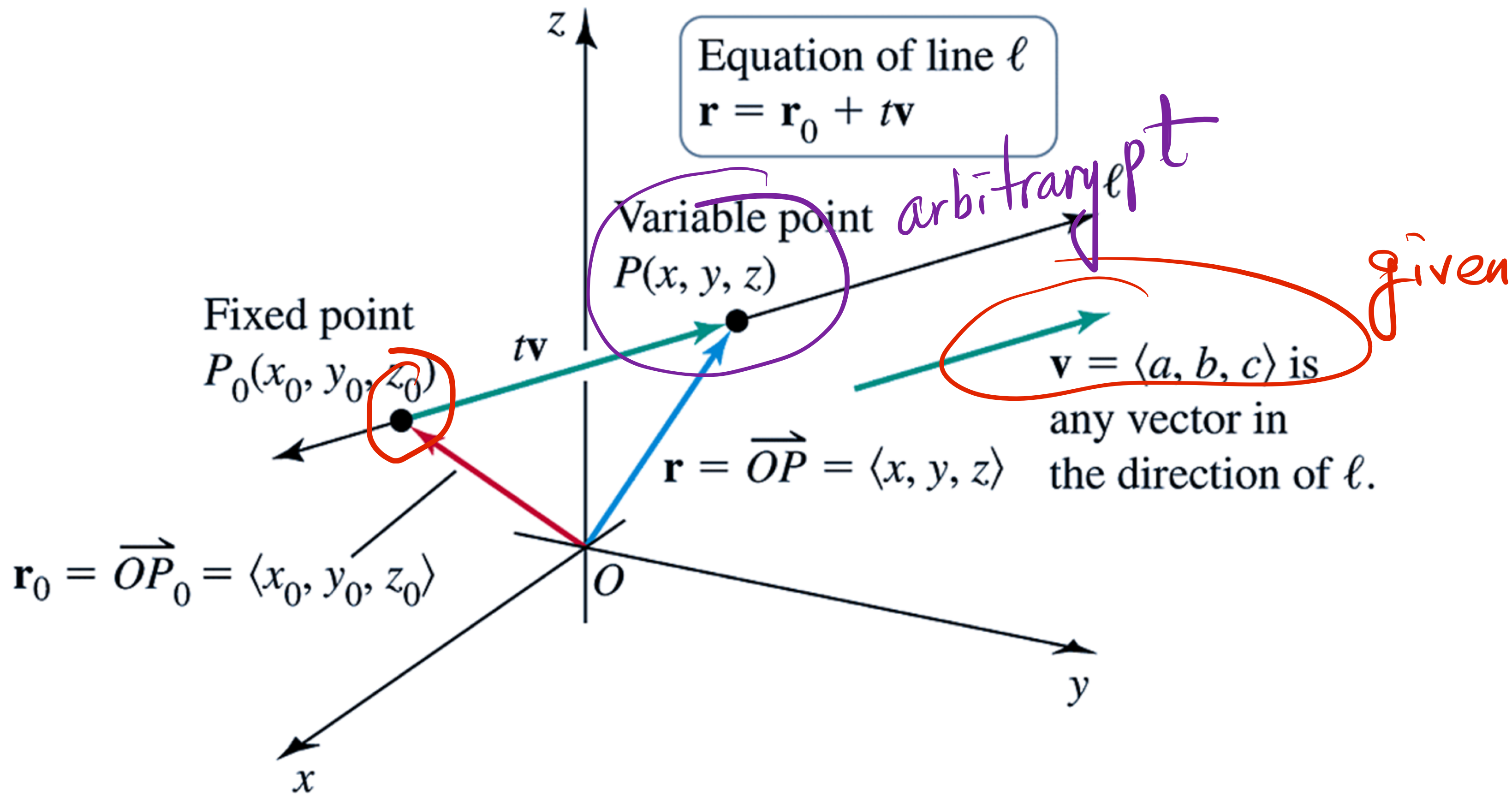
$\overrightarrow{P_0P} \parallel \vec{v} \Rightarrow \boxed{\overrightarrow{P_0P} = t\vec{v}}$

$\langle x-x_0, y-y_0, z-z_0 \rangle = t \langle a, b, c \rangle$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

Equations of Lines

Figure 13.66



Equation of a Line

A **vector equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c \rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

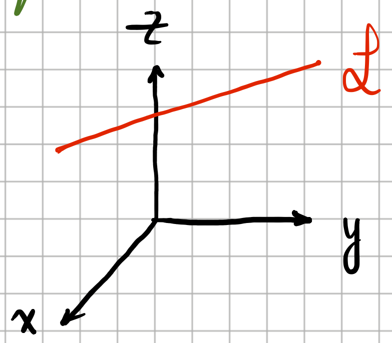
$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle, \text{ for } -\infty < t < \infty.$$

Equivalently, the corresponding **parametric equations of the line** are

$$x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct, \text{ for } -\infty < t < \infty.$$

Lesson 2 Lines and Planes in Space (§13.5)

- equation of a line $\mathcal{L}$



Given

Examples Find equations of lines

- #12 the line through $(0,0,1)$ in the direction $\vec{v} = \langle 1, -2, 0 \rangle$

$$\begin{cases} x = t \\ y = -2t \\ z = 1 \end{cases}$$

- #20 the line through $(-3,4,2)$ that is perpendicular to both $\vec{u} = \langle 1, 1, -5 \rangle$ and $\vec{v} = \langle 0, 4, 0 \rangle$

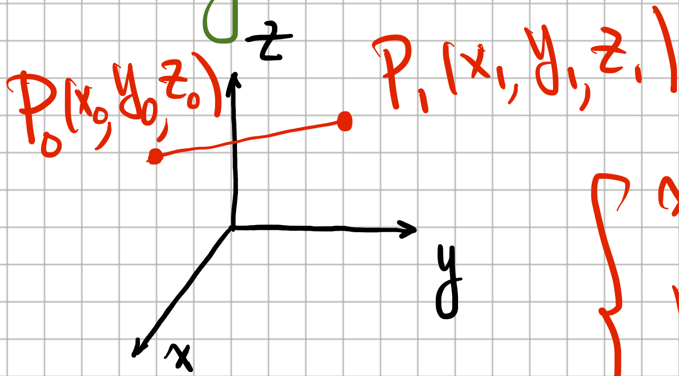
pt

vector: $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & -5 \\ 0 & 4 & 0 \end{vmatrix}$

$= \langle 20, 0, 4 \rangle = 4 \langle 5, 0, 1 \rangle$

$$\begin{cases} x = -3 + 5t \\ y = 4 \\ z = 2 + t \end{cases}$$

- line segments



$$\begin{cases} x = x_0 + (x_1 - x_0)t \\ y = y_0 + (y_1 - y_0)t \\ z = z_0 + (z_1 - z_0)t \end{cases}$$

$\boxed{0 \le t \le 1}$

pt $P_0(x_0, y_0, z_0)$

vector $\vec{v} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

- #28 Fine parametric equations for the line segment joining (1,0,1) and (0,-2,1)

$$\begin{cases} x = 1 - t \\ y = 0 - 2t \\ z = 1 \end{cases}$$

- parallel, intersecting, or skew lines

- #34 Determine whether the following lines:

$$(1)\begin{cases} x = 4 + 5t \\ y = -2t \\ z = 1 + 3t \end{cases} \quad \text{and} \quad (2)\begin{cases} x = 10s \\ y = 6 + 4s \\ z = 4 + 6s \end{cases}$$

are parallel, interset at a single pt, or skew.

the same line?

pt of intersection

Colliding particles?

$\vec{v}_1 = \langle 5, -2, 3 \rangle$

$\vec{v}_2 = \langle 10, 4, 6 \rangle = 2\langle 5, 2, 6 \rangle$

$\vec{v}_1 \parallel \vec{v}_2$? $\iff$ $\vec{v}_1 \overset{?}{=} \alpha \vec{v}_2$

$\vec{v}_1 \perp \vec{v}_2$? $\iff$ $\vec{v}_1 \cdot \vec{v}_2 \overset{?}{=} 0$

Equations of Planes

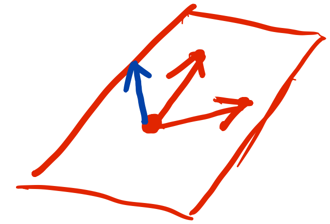


Figure 13-72

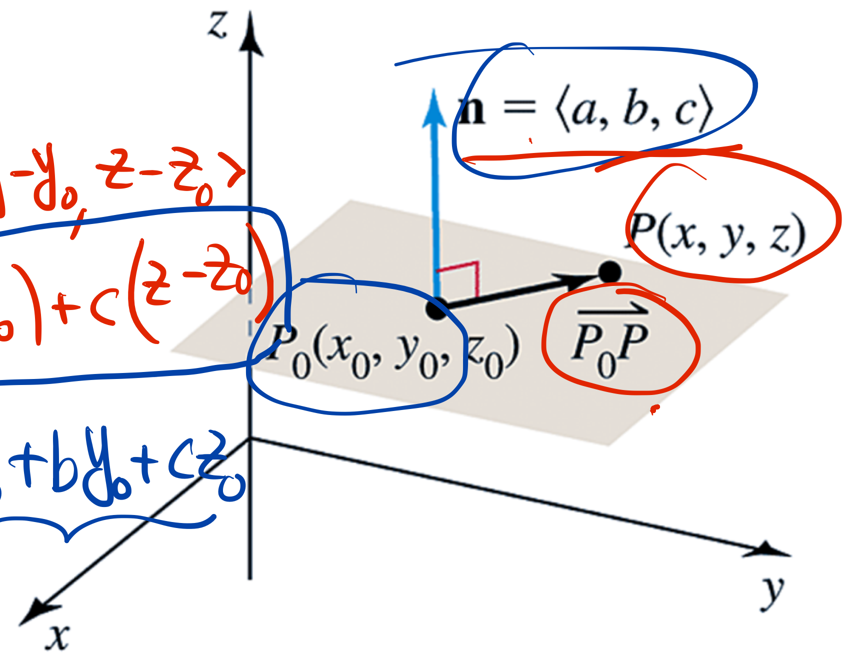
$\vec{n} \perp \overrightarrow{P_0P}$

$\Leftrightarrow 0 = \vec{n} \cdot \overrightarrow{P_0P}$

$= \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$

$$0 = a(x - x_0) + b(y - y_0) + c(z - z_0)$$

$$ax + by + cz = ax_0 + by_0 + cz_0 = d$$



The orientation of a plane is specified by a normal vector $\mathbf{n}$. All vectors $\overrightarrow{P_0P}$ in the plane are orthogonal to $\mathbf{n}$.

General Equation of a Plane in *R* Cubed

The plane passing through the point $P_0(x_0, y_0, z_0)$

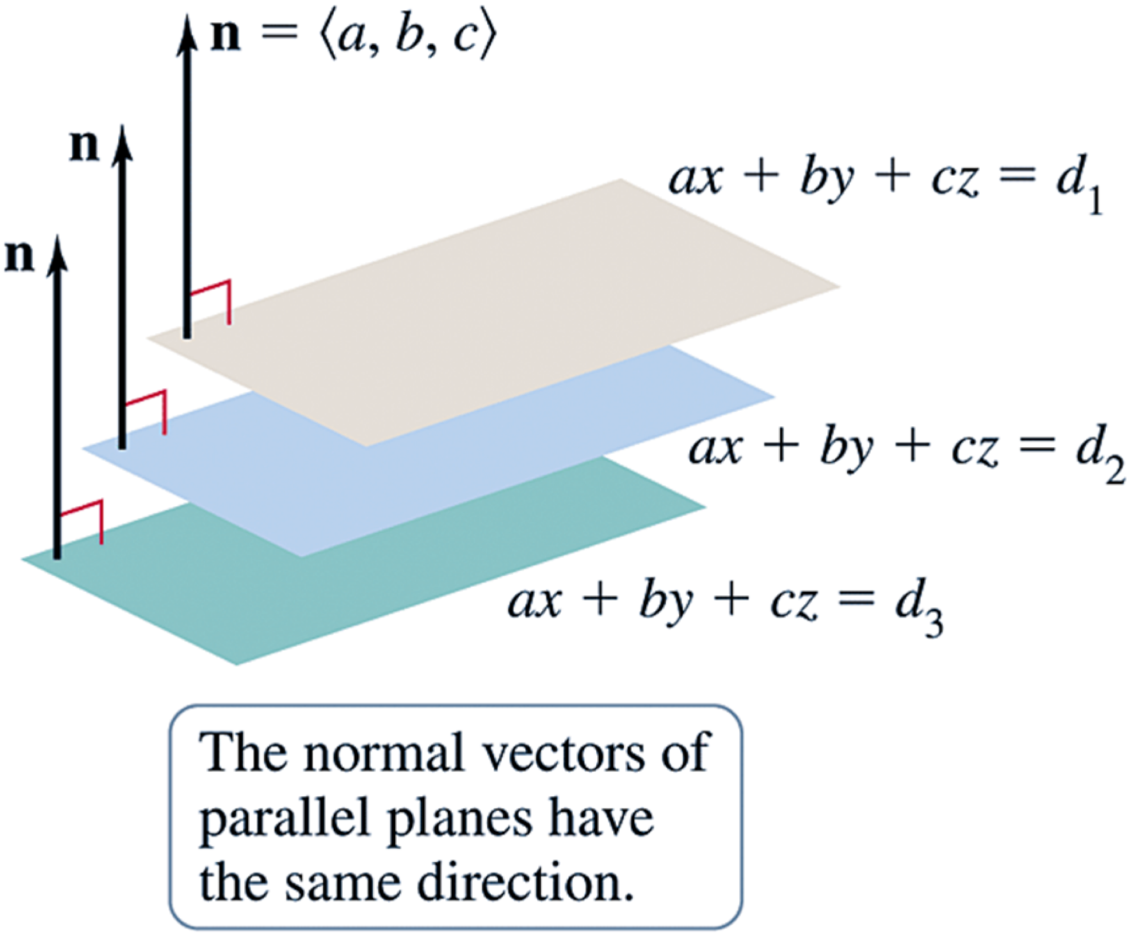
with a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$

is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \text{ or } ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

Figure 13.73



n = ⟨a, b, c⟩
n
n
ax + by + cz = d1
ax + by + cz = d2
ax + by + cz = d3
The normal vectors of parallel planes have the same direction.

Figure 13.74

Example 5 Find equation of a plane

(a) passing through $P_0(2,-3,4)$ with a normal vector $\vec{n}=\langle -1,2,3\rangle$

(b) passing through $P_0(2,-3,4)$ that is perpendicular to the line $\begin{cases} x=3+2t \\ y=-4t \\ z=1-6t \end{cases}$

$$0=-1(x-2)+2(y+3)+3(z-4)$$
$$=-1+2y+3z+(2+6-12)$$

$$\vec{v}=\langle 2,-4,-6\rangle$$
$$=2\langle 1,-2,-3\rangle$$
$$=-2\langle -1,2,3\rangle$$

(b) $\vec{n}=\,?\,\langle -1,2,3\rangle$

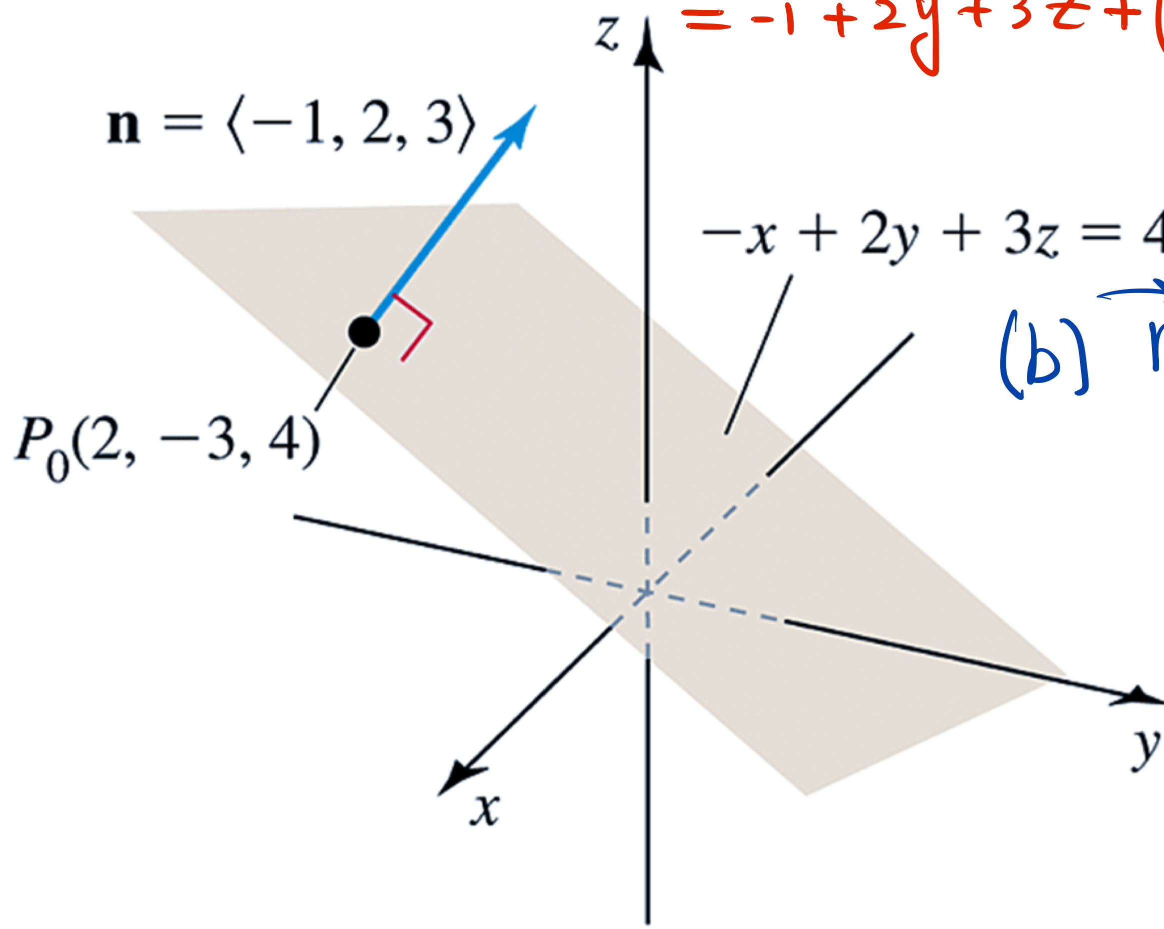


Figure 13.75

Example 6 Find an equation of the plane that passes through points: $P(2,-1,3)$, $Q(1,4,0)$, $R(0,-1,5)$

$$? = \vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -1 & 5 & -3 \\ -2 & 0 & 2 \end{vmatrix}$$

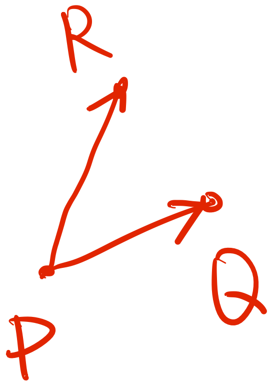
$$= \langle 10, 8, 10 \rangle$$

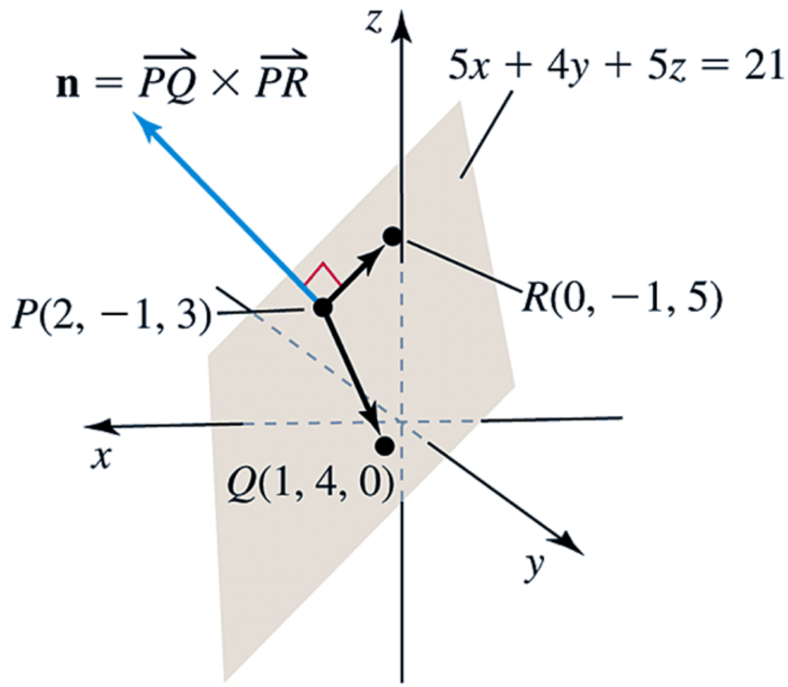
$$= 2\langle 5, 4, 5 \rangle$$

$$0 = \langle 5, 4, 5 \rangle \cdot \langle x-2, y+1, z-3 \rangle$$

$$= 5x + 4y + 5z - (10 - 4 + 15)$$

$$0 = 5x + 4y + 5z - 21$$





$\vec{PQ}$ and $\vec{PR}$ lie in the same plane.
$\vec{PQ} \times \vec{PR}$ is orthogonal to the plane.

Figure 13.76

Example 7 plane Q: $2x - 3y - z = 6$

(a) Find a vector normal to Q $\vec{n} = \langle 2, -3, -1 \rangle$

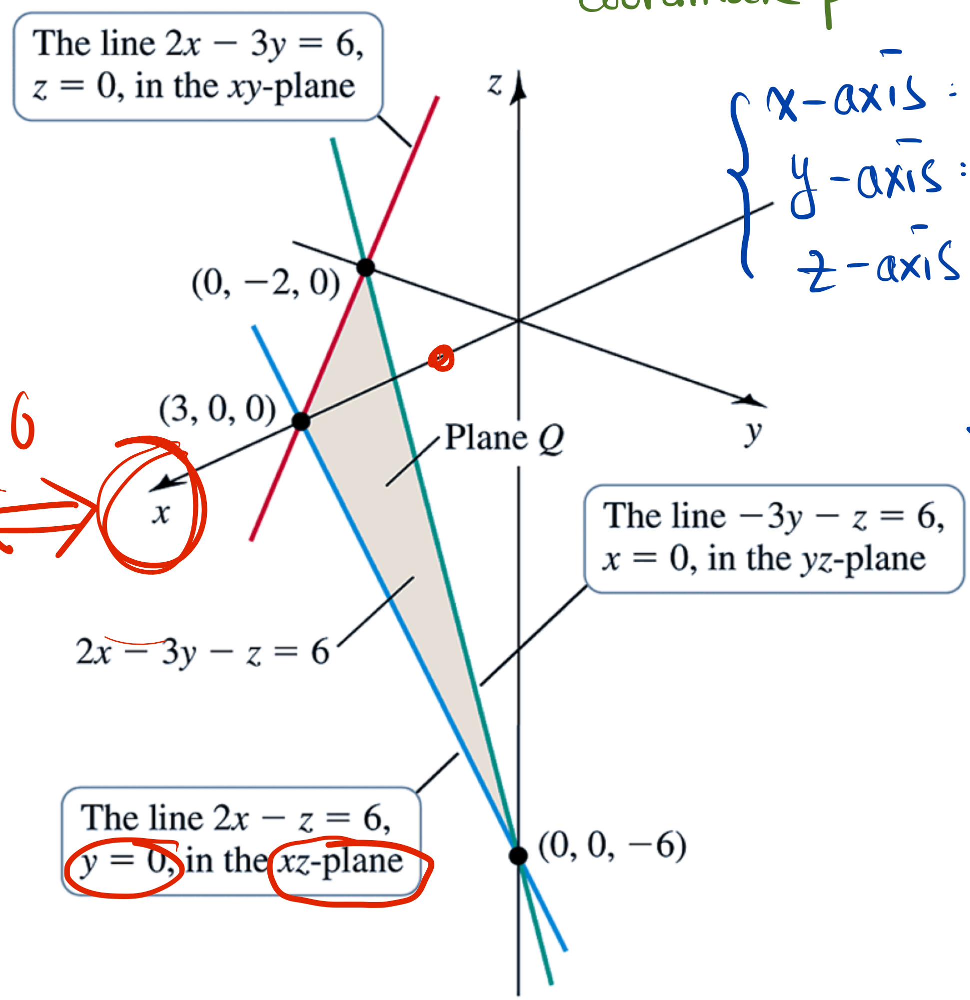
(b) Find pts at which Q intersects the coordinate axes

(c) Describe the sets of pts at which Q intersects the coordinate planes.

$$\begin{cases} x\text{-axis}: & y = z = 0 \\ y\text{-axis}: & x = z = 0 \\ z\text{-axis}: & x = y = 0 \end{cases}$$

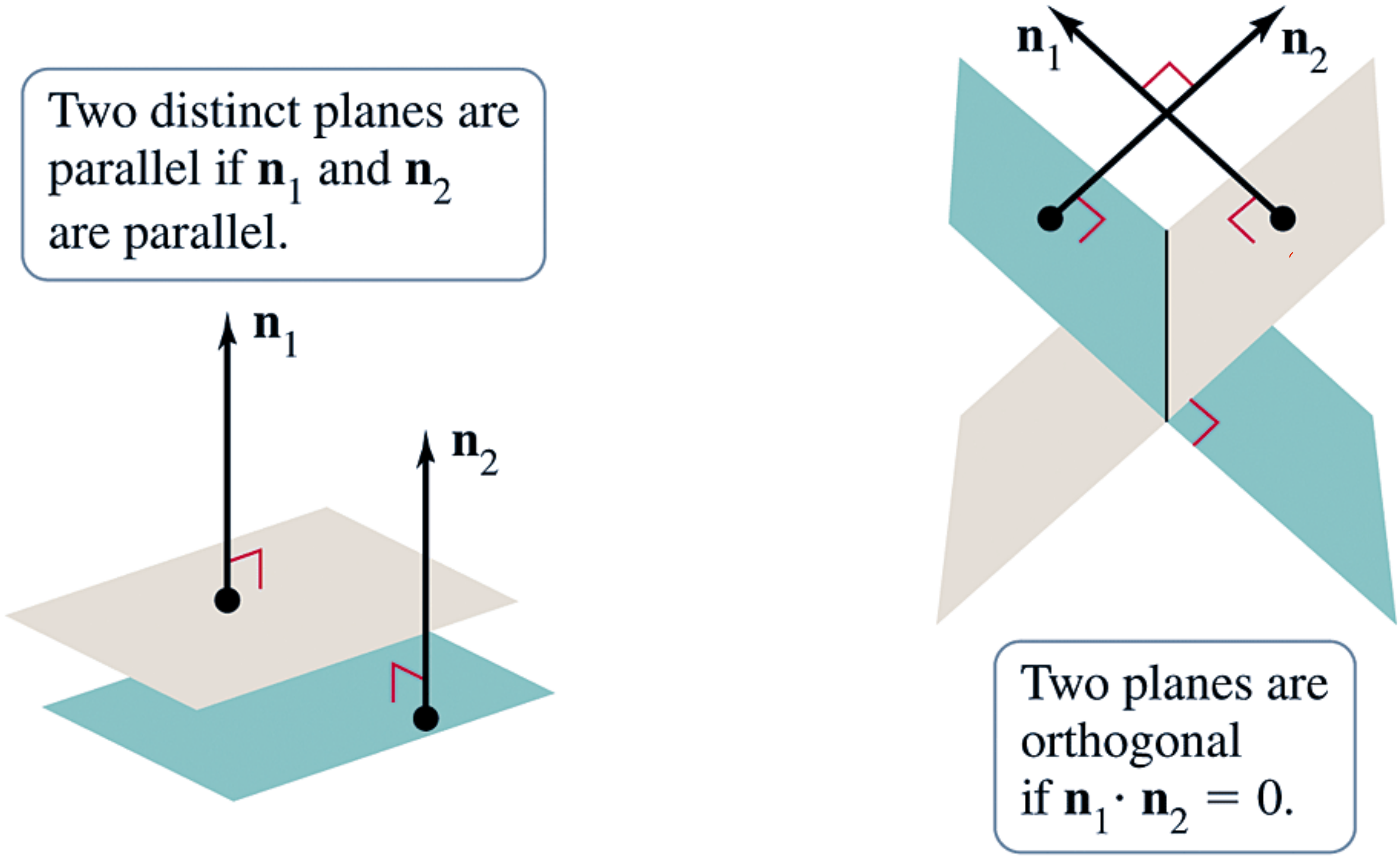
$$\begin{cases} xy\text{-plane}: & z = 0 \\ xz\text{-plane}: & y = 0 \\ yz\text{-plane}: & x = 0 \end{cases}$$

$$\begin{cases} 2x - 3y - z = 6 \\ y = 0,\ z = 0 \end{cases} \Longleftrightarrow x$$



Two planes are parallel or orthogonal

Figure 13.77

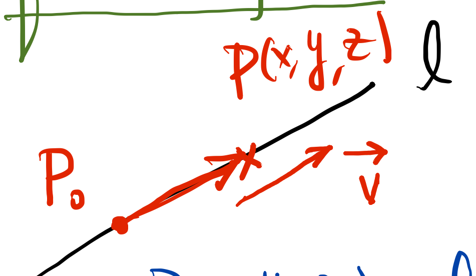


Definition Parallel and Orthogonal Planes

Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scalar multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is zero).

- Review of Lesson 2 $\overrightarrow{P_0P} \parallel \vec{v}$

equation of line



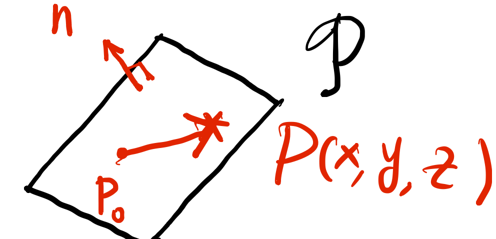
$\langle x-x_0, y-y_0, z-z_0\rangle = t\langle a, b, c\rangle$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

Given a pt $P_0(x_0, y_0, z_0)$ on ℓ

a vector $\vec{v} = \langle a, b, c\rangle \parallel \ell$

equation of plane



Given a pt $P_0(x_0, y_0, z_0)$ on $\mathcal{P}$

a vector $\vec{n} = \langle a, b, c\rangle \perp \mathcal{P}$

$0 = \langle a, b, c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle$ $\quad \vec{n} \perp \overrightarrow{P_0P}$

$= a(x-x_0) + b(y-y_0) + c(z-z_0)$

Example 9 Find an equation of the line of intersection of the planes: $\begin{cases} Q: x+2y+z=5, \\ R: 2x+y-z=7, \end{cases}$

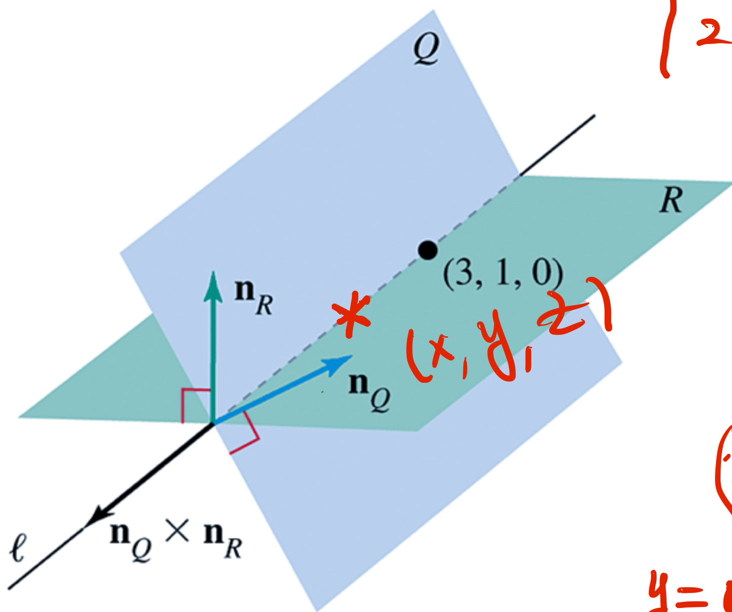
Figure 13.78

$\vec{n}_Q = \langle 1, 2, 1 \rangle$

$\vec{n}_R = \langle 2, 1, -1 \rangle$

$\vec{v} = \vec{n}_Q \times \vec{n}_R$

$= \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$



$\mathbf{n}_Q \times \mathbf{n}_R$ is a vector perpendicular to $\mathbf{n}_Q$ and $\mathbf{n}_R$.
Line ℓ is perpendicular to $\mathbf{n}_Q$ and $\mathbf{n}_R$.
Therefore, ℓ and $\mathbf{n}_Q \times \mathbf{n}_R$ are parallel to each other.

$\begin{cases} x+z=5-2y \\ 2x-z=7-y \end{cases}$

$3x = 12-3y \Longrightarrow x = 4-y$

$z = 5-x-2y$
$= 5-(4-y)-2y$
$= 1-y$

$(4-y, y, 1-y)$ on line

$y=0$ $\quad P = (4, 0, 1)$

$y=1$ $\quad Q = (3, 1, 0)$

$\vec{v} = \vec{PQ} = \langle -1, 1, -1 \rangle$

$\begin{cases} x = 4-t \\ y = t \\ z = 1-t \end{cases}$

Examples

- #69 pairs of planes: parallel, orthogonal, or identical

$$
\begin{cases} Q: 3x-2y+z=12 \\ R: -x+\frac{2}{3}y-\frac{1}{3}z=0 \\ S: -x+2y+7z=1 \\ T: \frac{3}{2}x-y+\frac{1}{2}z=6 \end{cases}
$$

$\vec{n}_Q = \langle 3,-2,1\rangle$

$\vec{n}_R = \langle -1, \frac{2}{3}, -\frac{1}{3}\rangle = -\frac{1}{3}\langle 3,-2,1\rangle = -\frac{1}{3}\vec{n}_Q$

$\vec{n}_S = \langle -1, 2, 7\rangle$

$\vec{n}_T = \langle \frac{3}{2}, -1, \frac{1}{2}\rangle = \frac{1}{2}\langle 3,-1,1\rangle = \frac{1}{2}\vec{n}_Q$

$Q \parallel R \parallel T$

$Q = T$

$Q \perp S$?

$\vec{n}_Q \cdot \vec{n}_S = \langle 3,-2,1\rangle \circ \langle -1,2,7\rangle$
$= -3-4+7=0$

$\Longrightarrow Q \perp S$

- Find the intersection pt of the plane and the line:

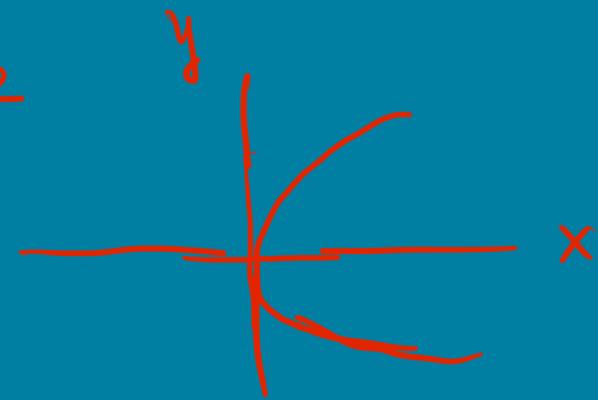
plane: $3x+2y-4z=-3$; line: $\begin{cases} x=-2t+5 \\ y=3t-5 \\ z=4t-6 \end{cases}$

$-3 = 3(-2t+5)+2(3t-5)-4(4t-6)$
$= -16t+29$

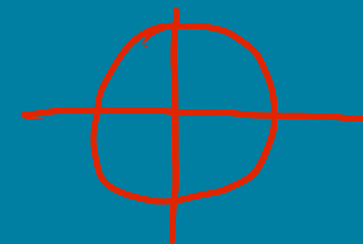
$\Rightarrow t = \frac{32}{16} = 2$ $\quad \begin{cases} x=-4+5=1 \\ y=1 \\ z=2 \end{cases}$

$(1,1,2)$

$x = y^2$



$x^2+y^2=1$



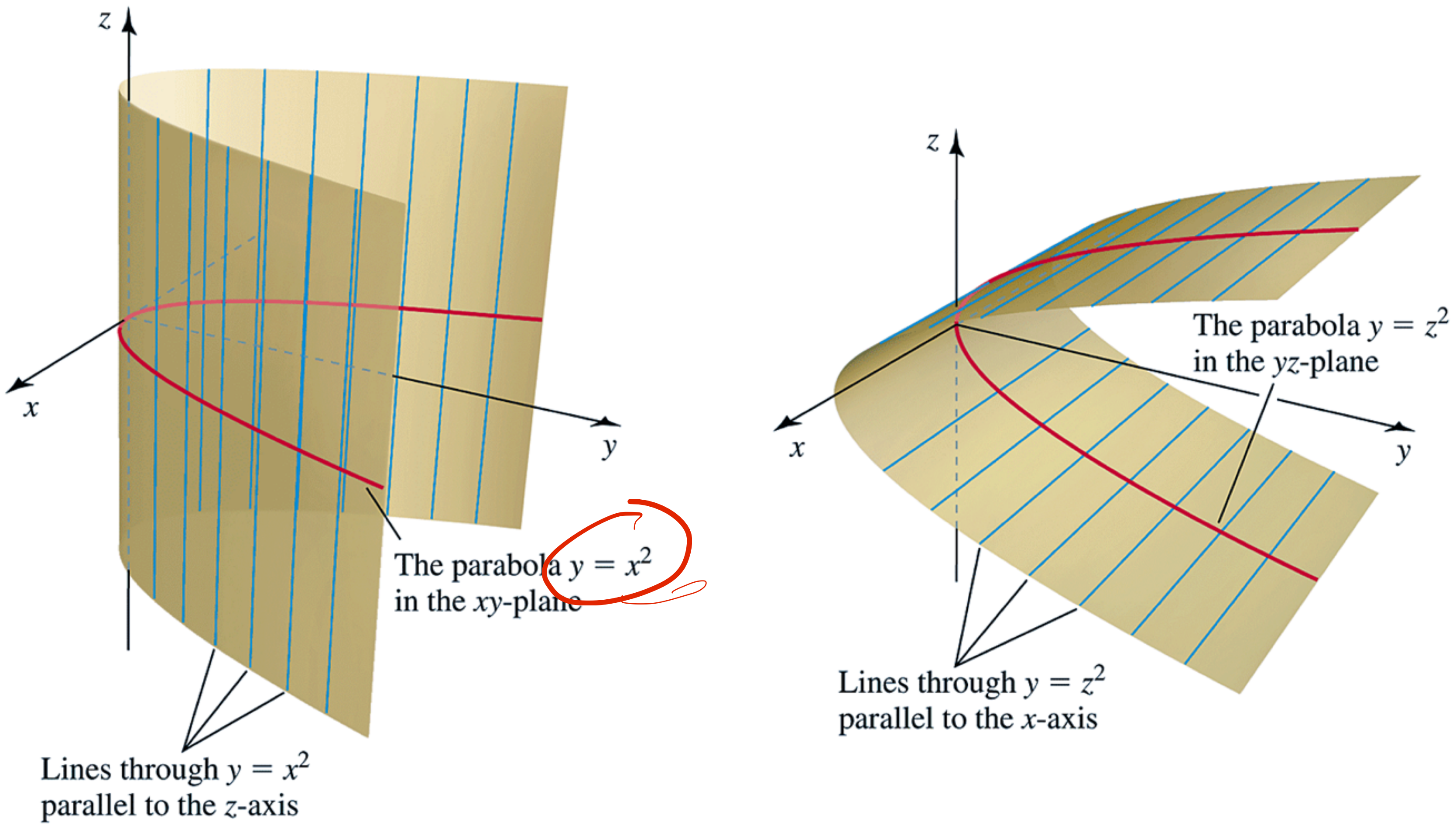
Section 13.6 Cylinders and Quadric Surfaces

Cylinder a surface that consists of all lines that parallel to a given line and pass through a given curve.

Quadric Surfaces

$$Ax^2+By^2+Cz^2+ \cdots + Exz + Fyz + Gx + Hy + Iz + J = 0$$

Figure 13-79


z
x
y
The parabola $y = x^2$
in the xy-plane
Lines through $y = x^2$
parallel to the z-axis
z
x
y
The parabola $y = z^2$
in the yz-plane
Lines through $y = z^2$
parallel to the x-axis

How to sketch cylinders and quadric surfaces?

Definition Trace

A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the ***xy*-trace**, the ***yz*-trace**, and the ***xz*-trace** (Figure 13.80).

$z = 0$ $\quad x = 0$ $\quad y = 0$

$z = z_0$ $\quad x = x_0$ $\quad y = y_0$

intersections with coordinate axes

x-axis: $y = z = 0$

y-axis: $x = z = 0$

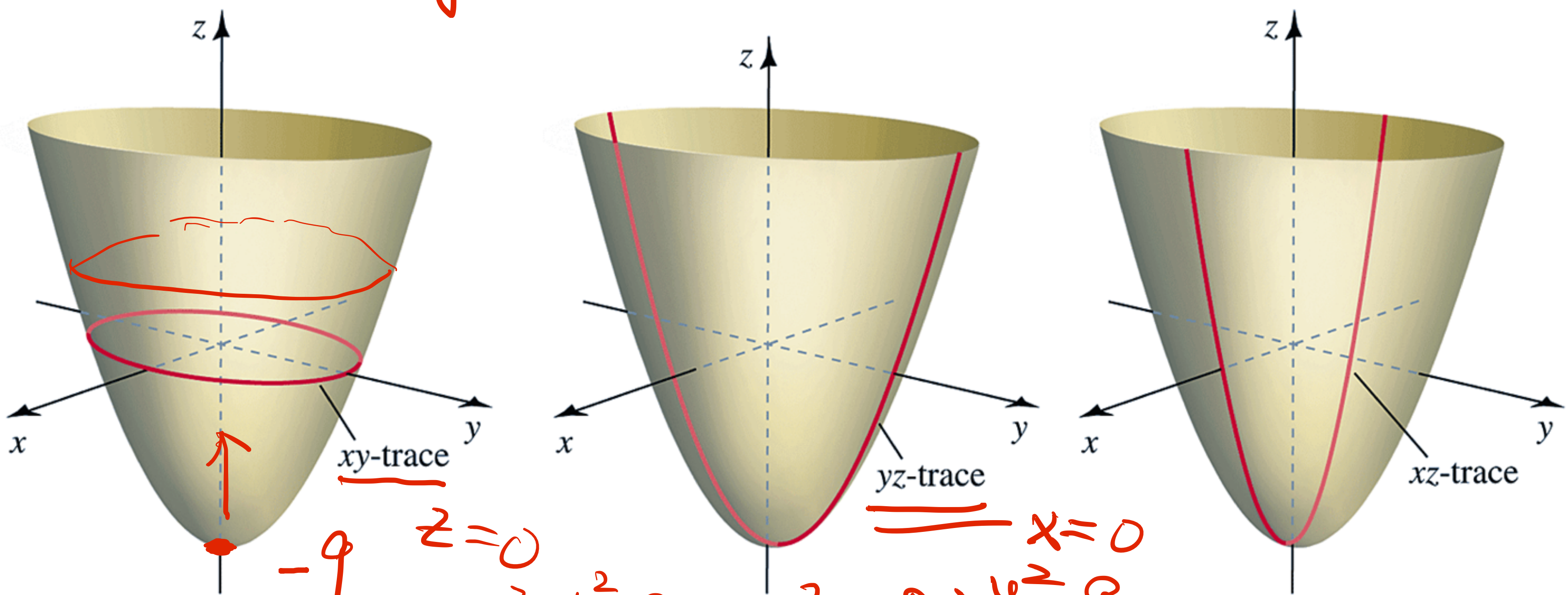
z-axis: $x = y = 0$

$$z = x^2 + y^2 - 9$$

Figure 13.80

$$z = 1 = x^2 + y^2 - 9$$

$$\Rightarrow x^2 + y^2 = 10$$



-9

$z = 0$

$= x^2 + y^2 - 9$

$x = 0$

$z = 0 + y^2 - 9$

$$\Rightarrow x^2 + y^2 = 3^2$$

Example 1 (b) Sketch the graph of the cylinder $x - \sin z = 0$. Identify the axis to which the cylinder is parallel.

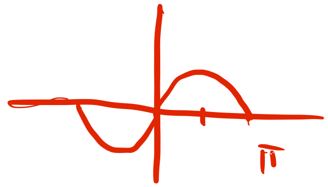
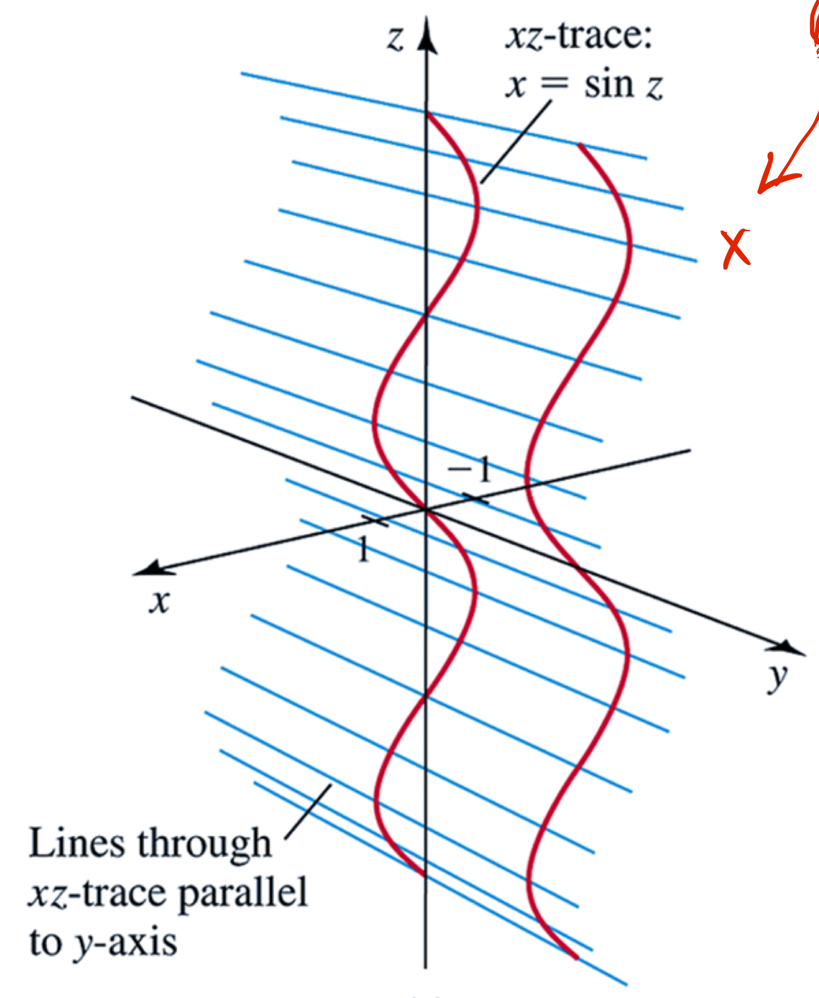
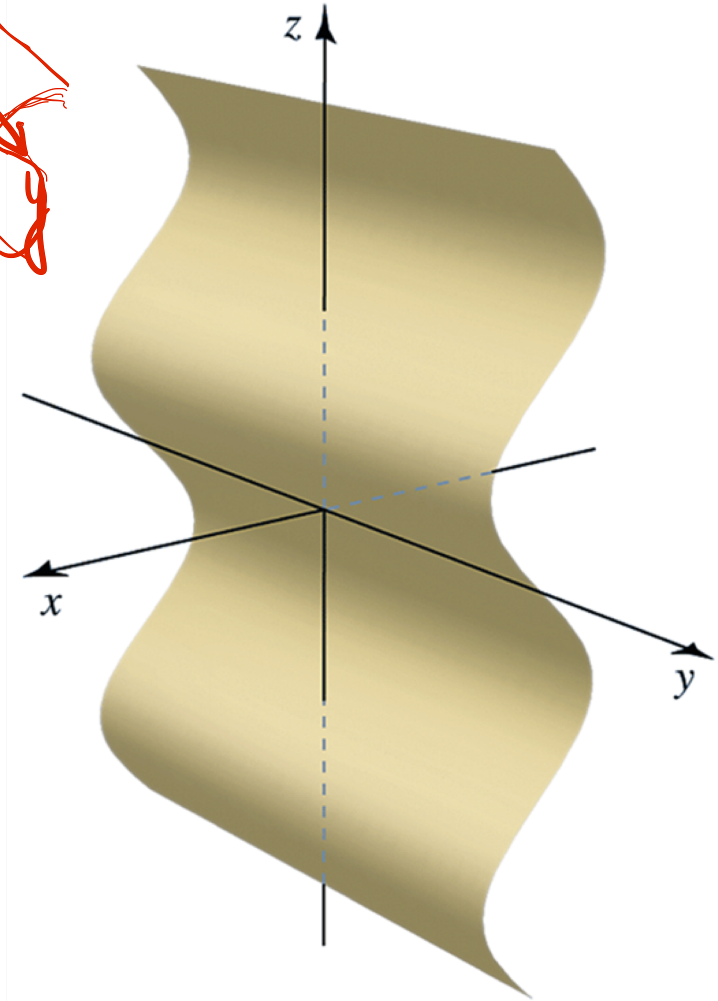


Figure 13.81





cylinder $x^2+4y^2=16$

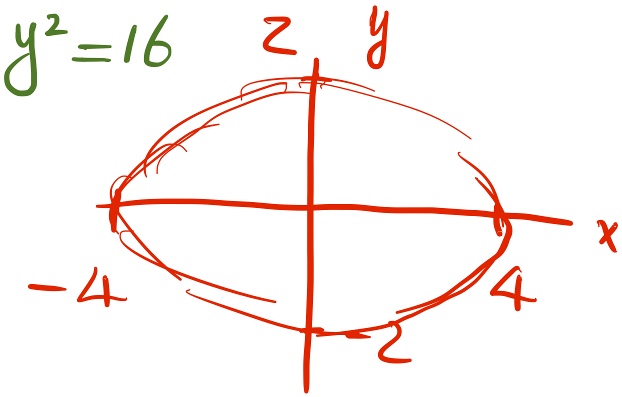


Figure 13.82 (a)

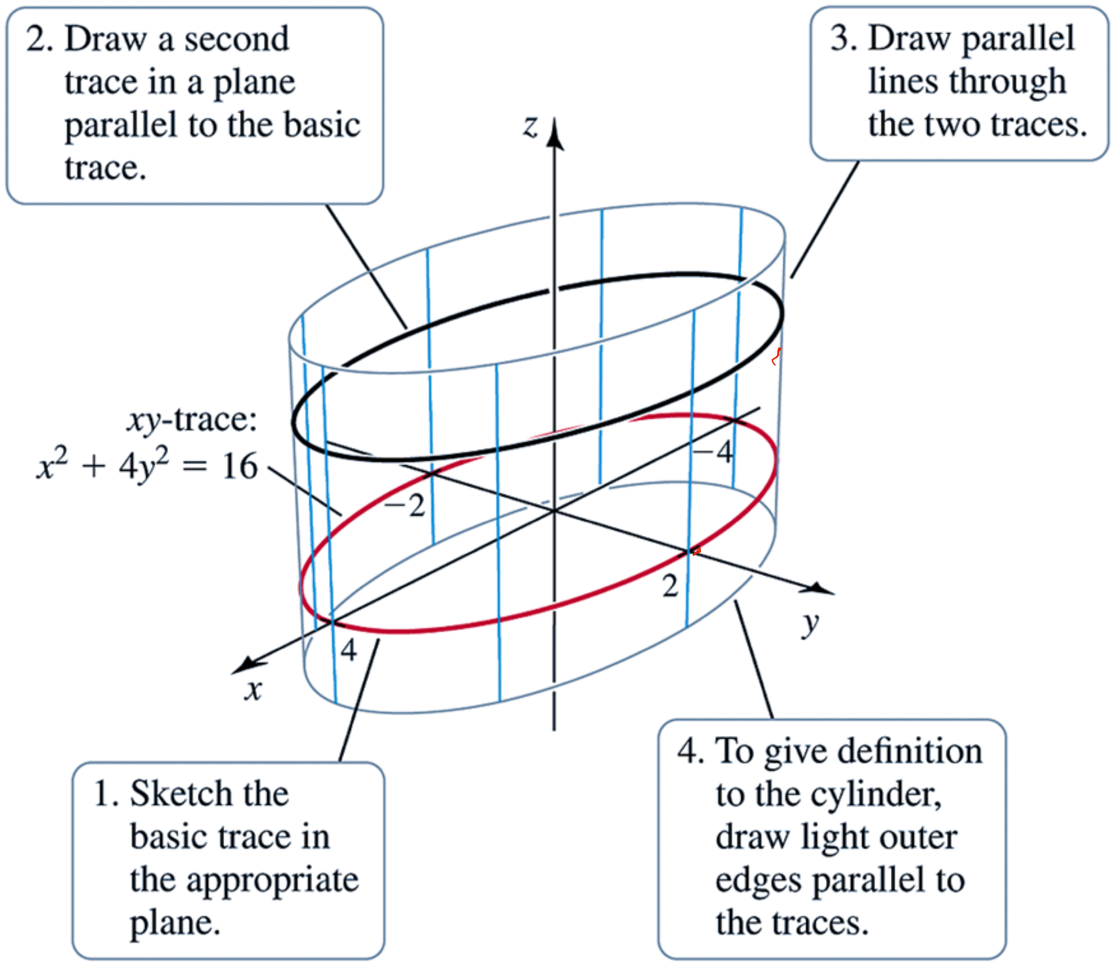
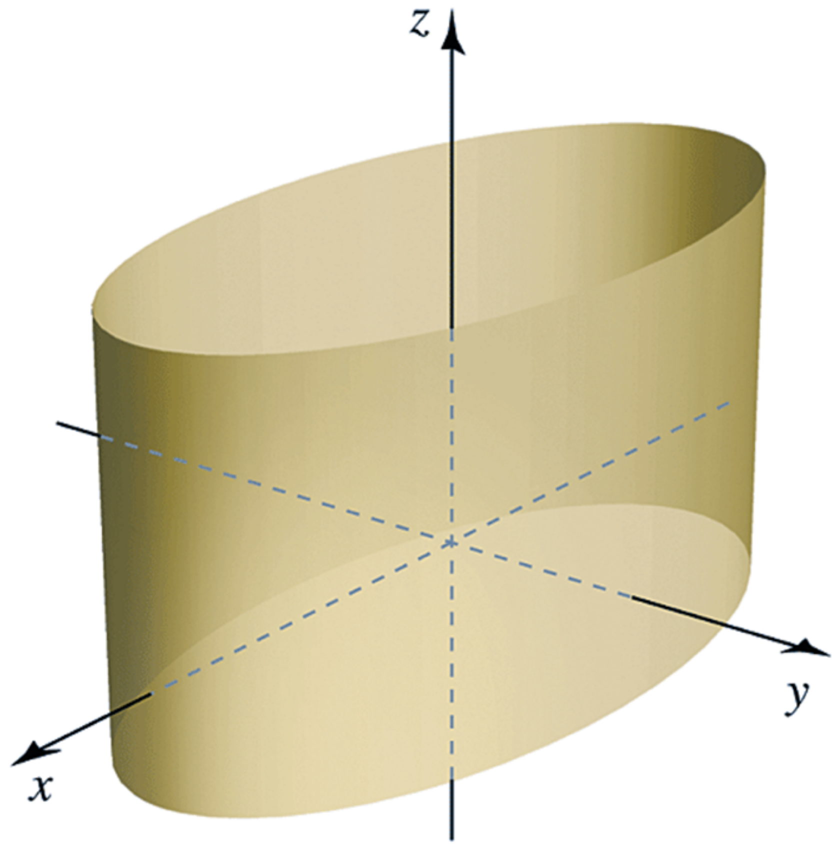


Figure 13.82 (b)



Elliptic cylinder

Example 2 The surface defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ is an ellipsoid.

Graph it with $a=3$, $b=4$, $c=5$.

Figure 13.83 (a)

intersections

$y=z=0$ $\quad \frac{x^2}{a^2}=1 \Rightarrow x=\pm a$

$x=z=0$ $\quad y^2/b^2=1 \Rightarrow y=\pm b$

$x=y=0$ $\quad z^2/c^2=1 \Rightarrow z=\pm c$

xy-trace $z=0$

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

$z=1$ $\quad \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 - \frac{1}{5^2} = \frac{5^2-1}{5^2}$

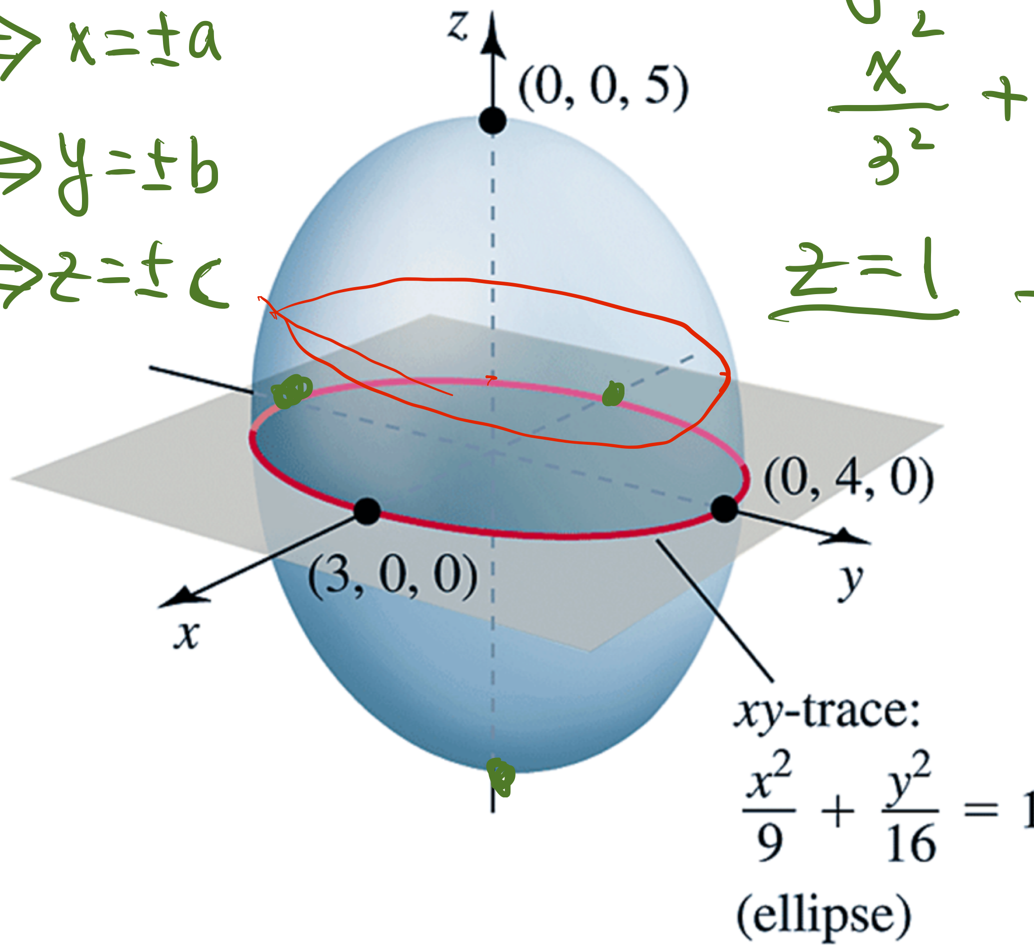


Figure 13.83 (b)

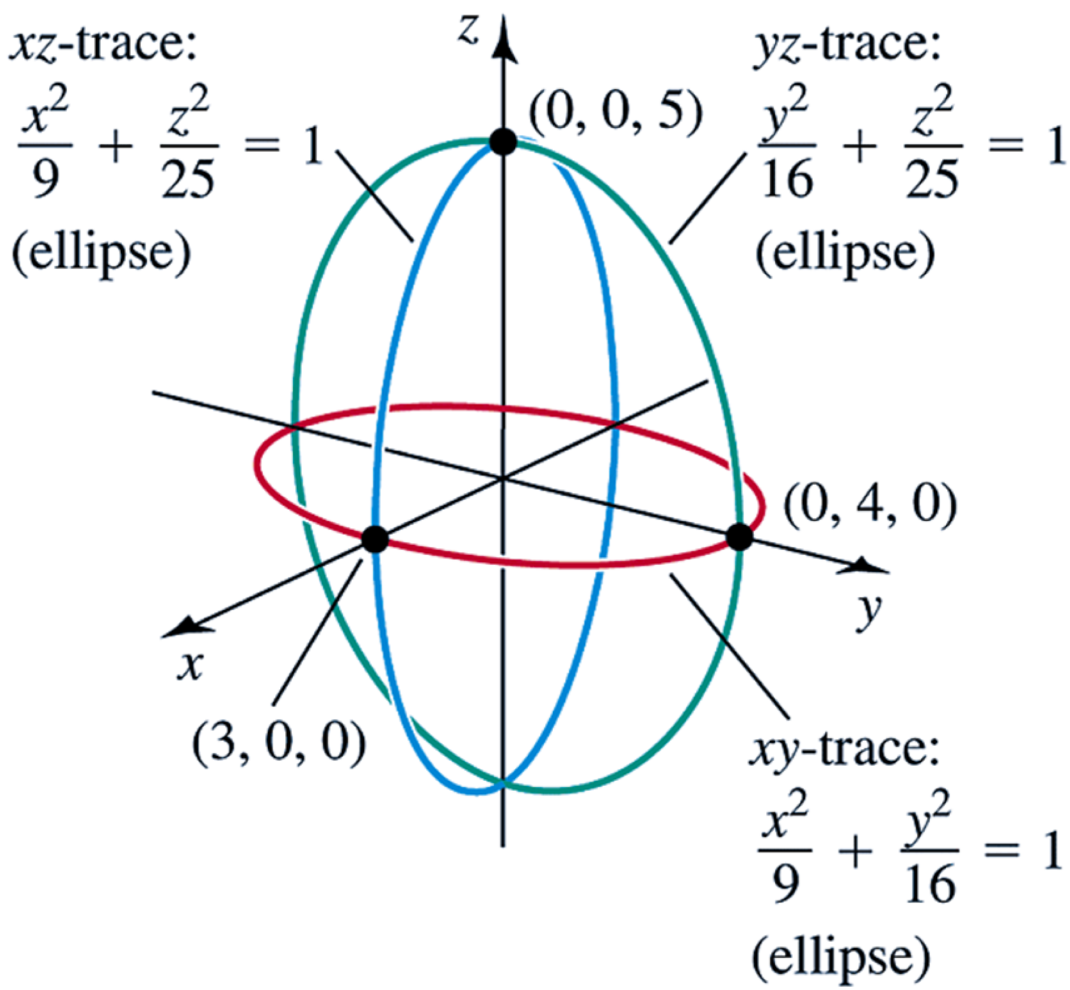
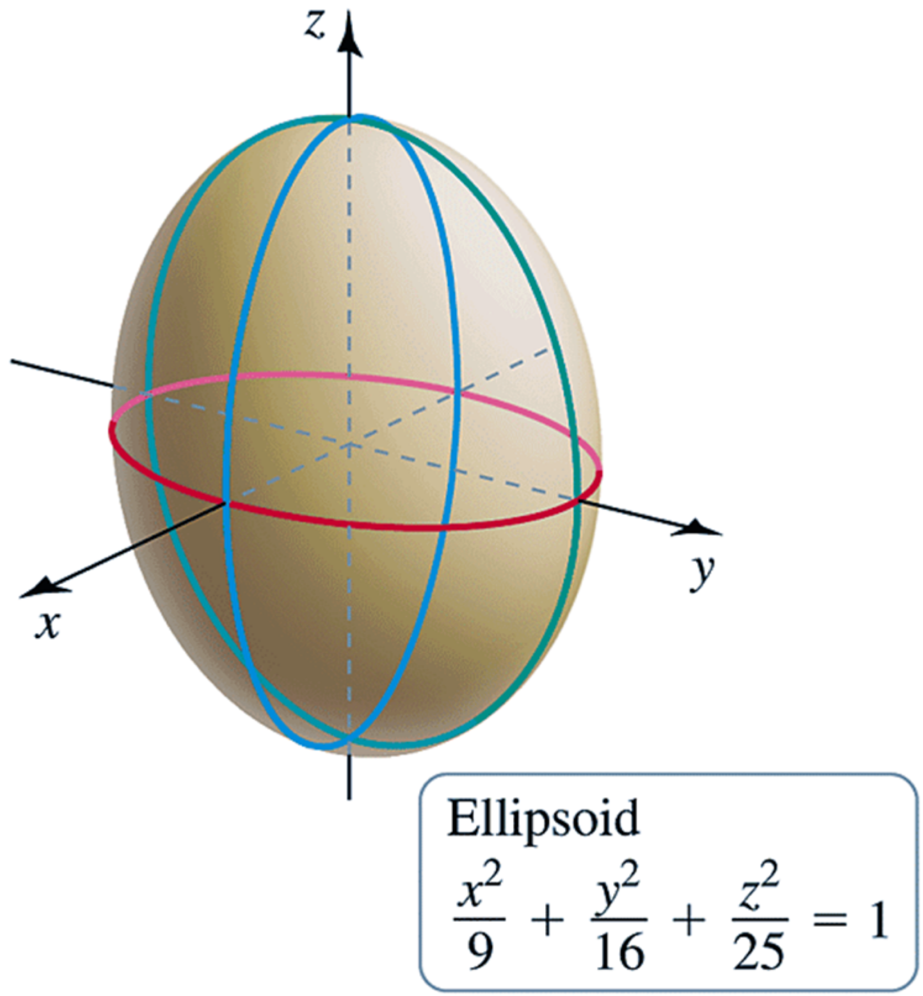


Figure 13.83 (c)



Example 3 the surface defined by the equation $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is an elliptic paraboloid. Graph it with $a=4$, $b=2$.

Figure 13.84 (a & b)

xy-trace

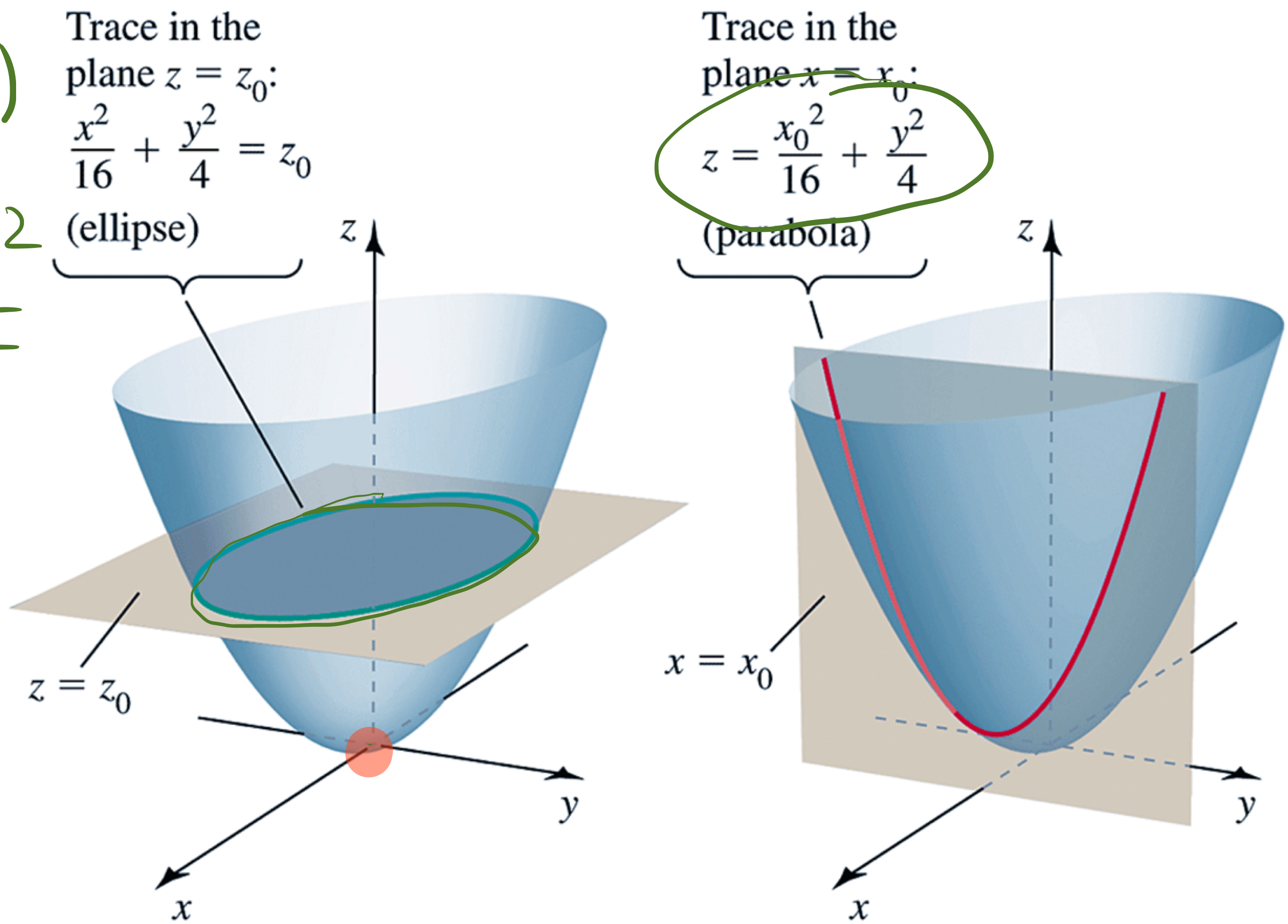
$z=0 \Rightarrow (0,0)$

$z=1 \Rightarrow$

$1 = \frac{x^2}{16} + \frac{y^2}{4}$

yz-trace

$x = x_0$



Review of Lesson 3

Cylinder $x^2 + \sin y = 0$, $f(x, z) = 0$

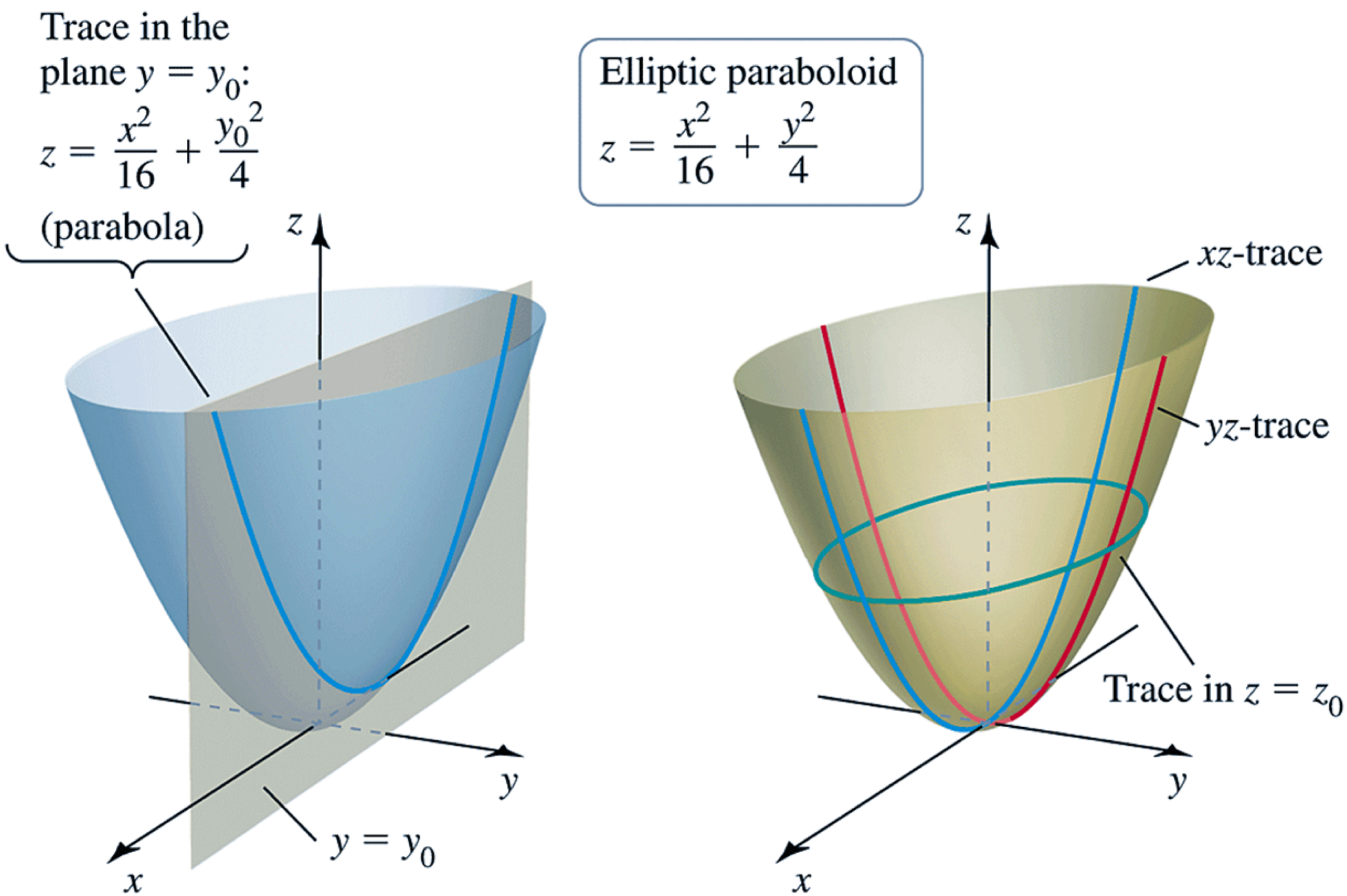
ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

elliptic paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

intercepts of coordinate axes

traces of coordinate planes

Figure 13.84 (c & d)



Lesson 4

Example 4 A hyperboloid of one sheet

Graph $\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$.

Figure 13.85 (a & b)

trace on $z = z_0$

$\frac{x^2}{4} + \frac{y^2}{9} = 1 + z_0^2$

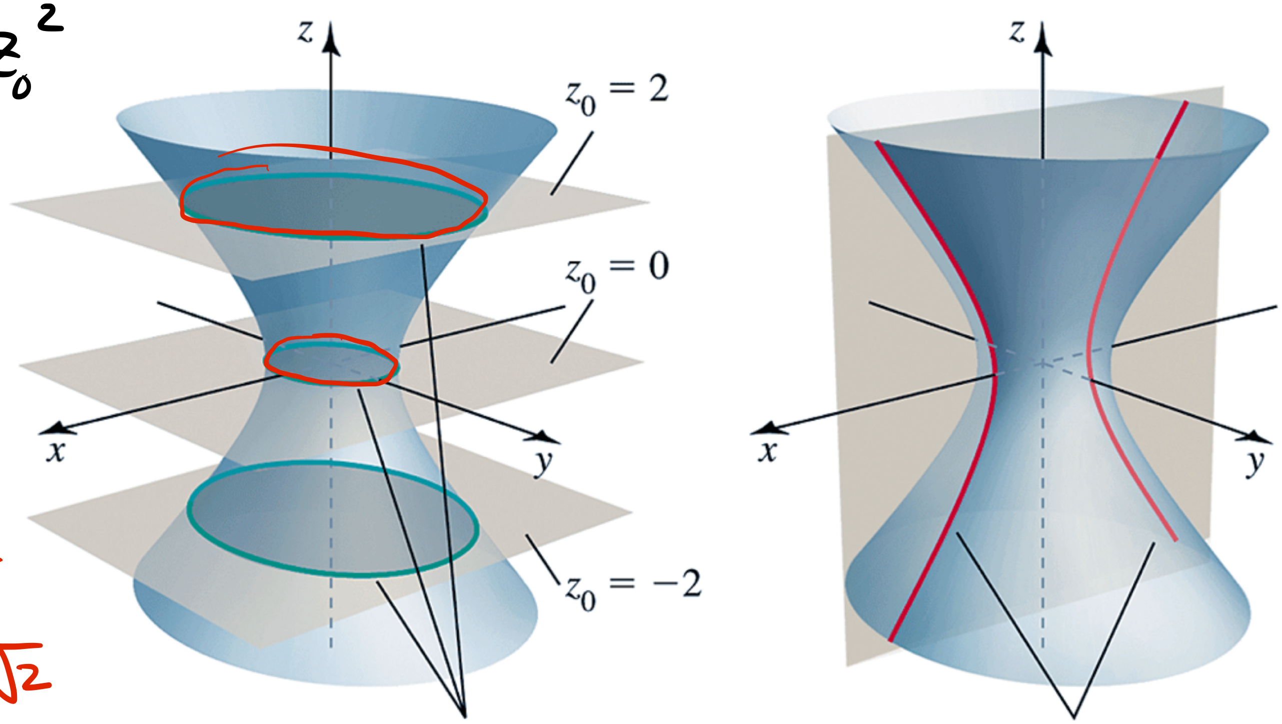
$z_0 = 0$

$z_0 = 1$

$\frac{x^2}{4} + \frac{y^2}{9} = 2$

x-intercept $\pm 2\sqrt{2}$

y-intercept $\pm 3\sqrt{2}$



$z = z_0$ traces:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 + z_0^2$$

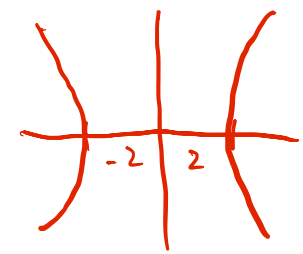
(ellipse) for $z_0 = -2, 0, 2$

xz-trace: $y = 0$

$$\frac{x^2}{4} - z^2 = 1$$

(hyperbola)

z



$\frac{x^2}{4} = 1 + z^2$

$z = 0$ $x^2 = 4$

yz-trace $x = 0$

$\frac{y^2}{9} = 1 + z^2$

z

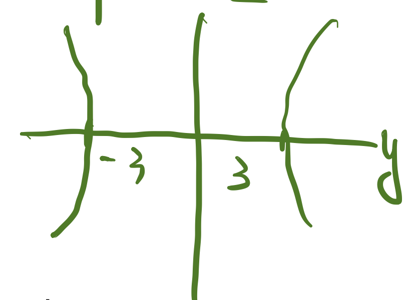
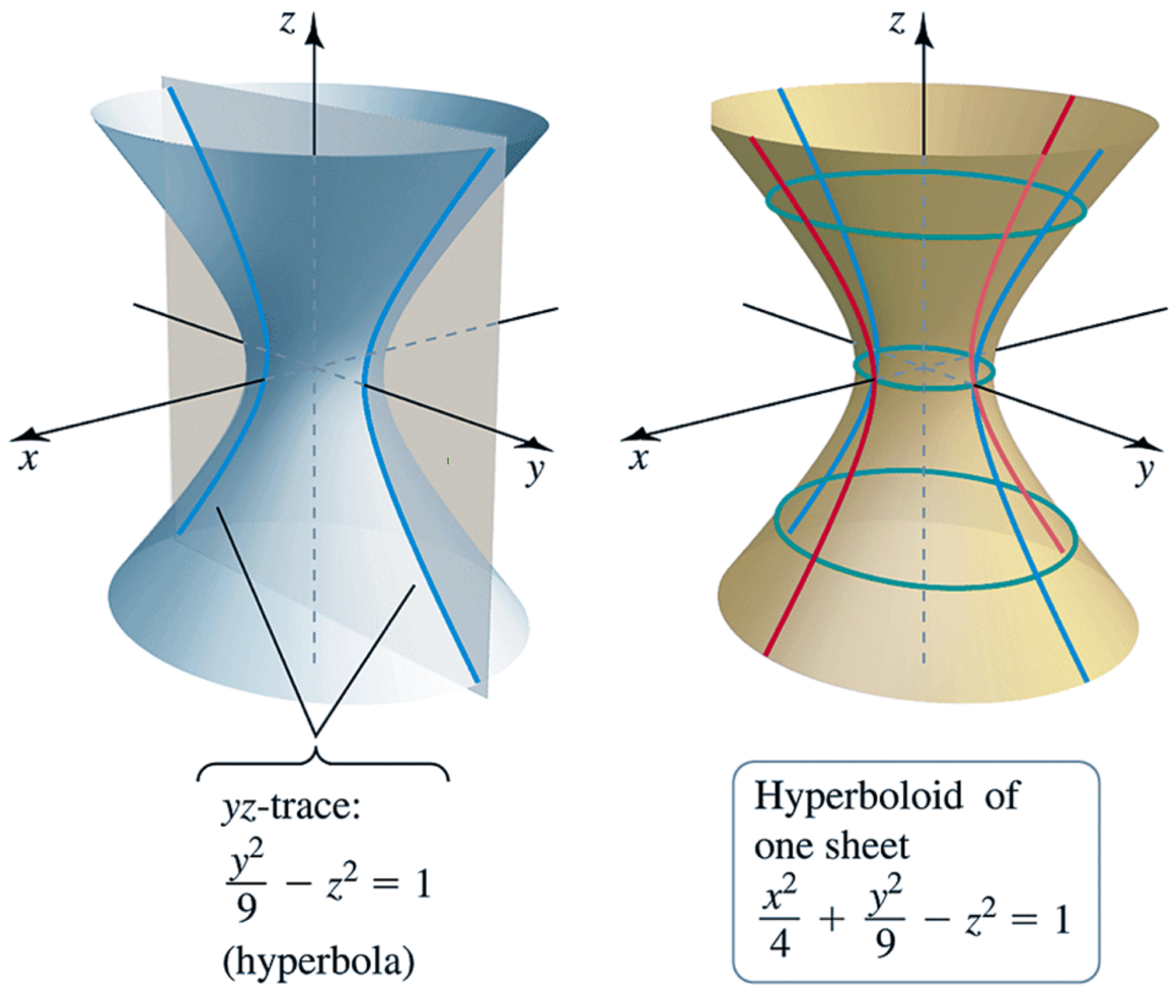


Figure 13.85 (c & d)



Example 5 (a hyperbolic paraboloid) Graph $z = x^2 - \frac{y^2}{4}$

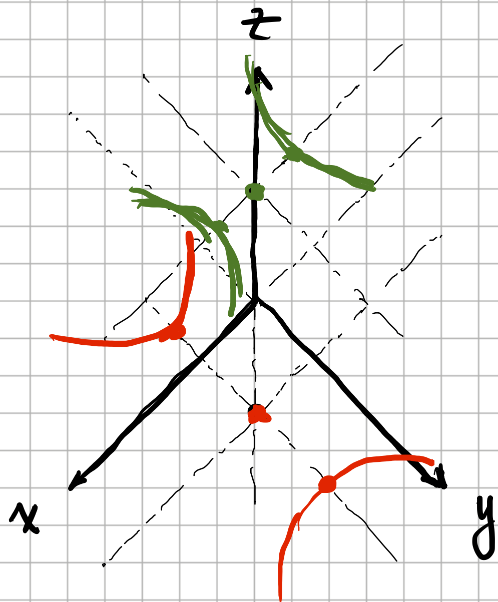
traces in the planes $z = -1, 0, 1$

xz-trace and yz-trace ($x=0$)

$y=0$

$z = x^2$

$z = -\frac{y^2}{4}$

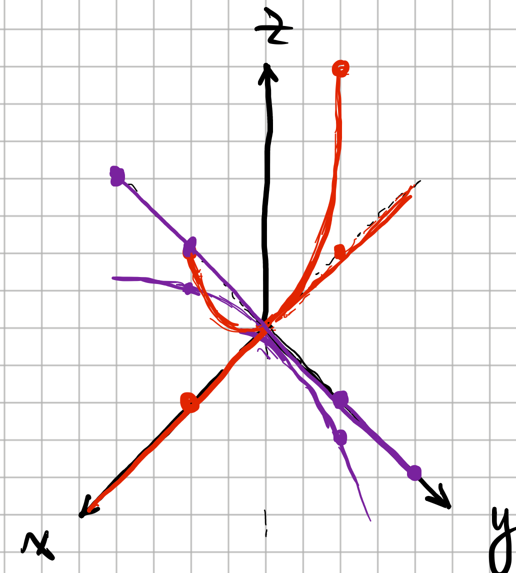


trace on the plane $z=1$

$x^2 = 1 + \frac{y^2}{4} \Rightarrow x = \pm 1$

trace on the plane $z=-1$

$\frac{y^2}{4} = 1 + x^2 \Rightarrow y = \pm 2$



Eample 5 A hyperbolic paraboloid

Graph $z = x^2 - \frac{y^2}{4}$

Figure 13.86 (a & b)

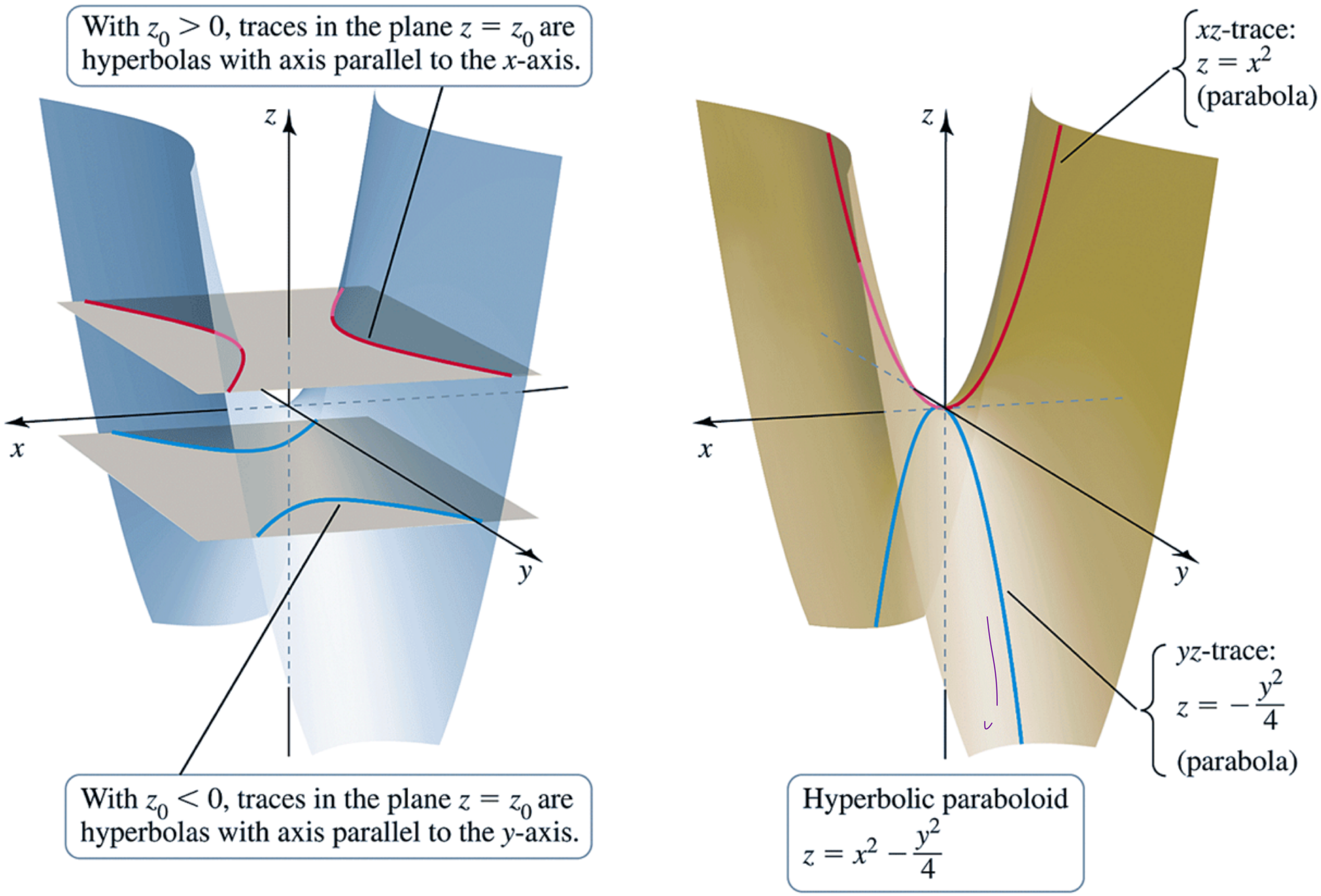
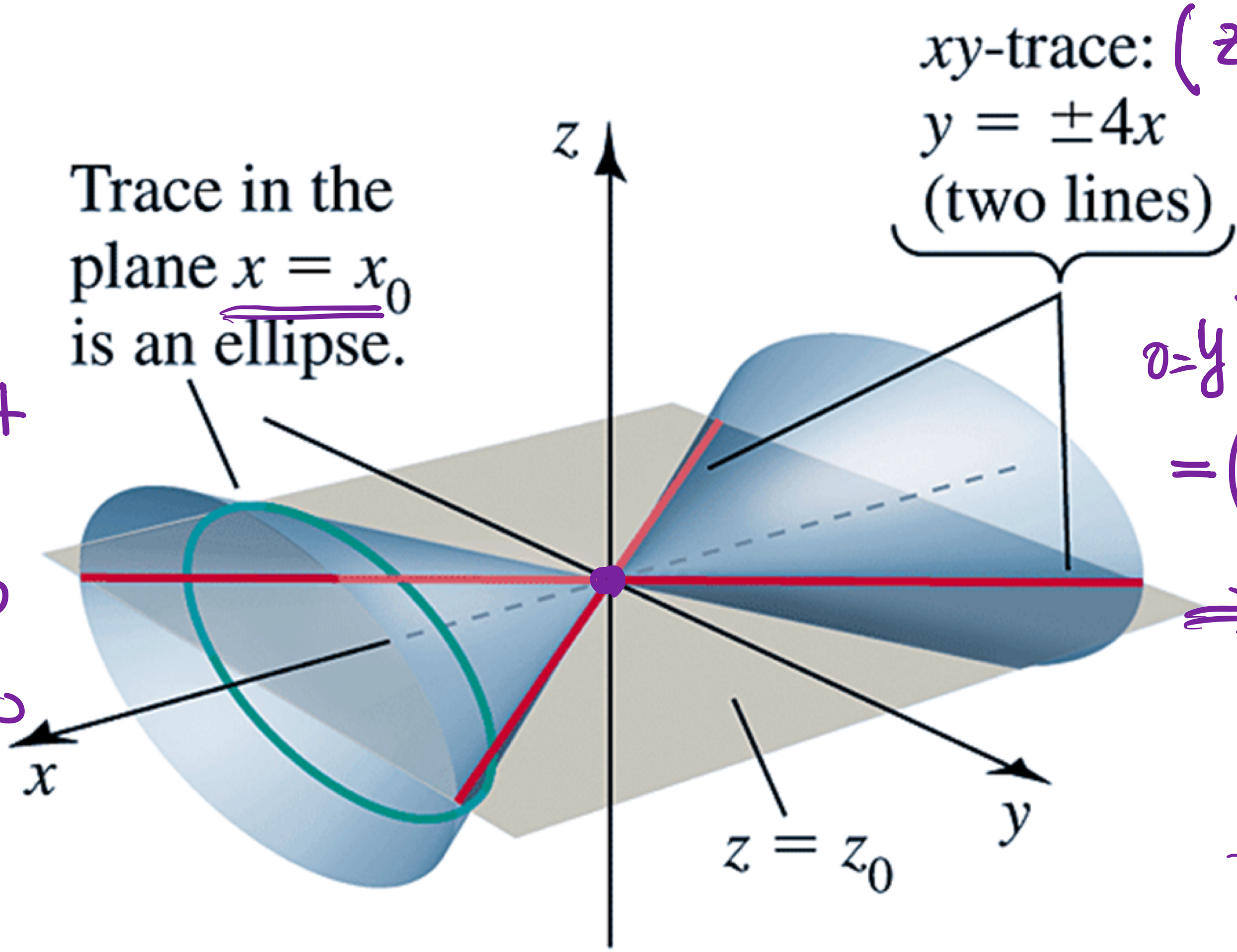


Figure 13.87 (a)

Example 6 — Elliptic cone

Graph $\frac{y^2}{4} + z^2 = 4x^2$

$a^2 - b^2 = (a+b)(a-b)$



$\frac{y^2}{4} + z^2 = 4x_0^2$

$x_0 = 1$: $\frac{y^2}{4} + z^2 = 4$

$x_0 = 0$: $\frac{y^2}{4} + z^2 = 0$

$\Rightarrow y = 0,\ z = 0$

$\frac{y^2}{4} = 4x^2$

$0 = y^2 - (4x)^2$

$= (y+4x)(y-4x)$

$\Rightarrow y + 4x = 0$

$y - 4x = 0$

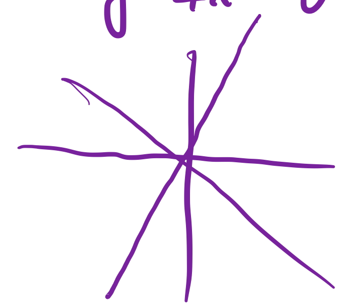
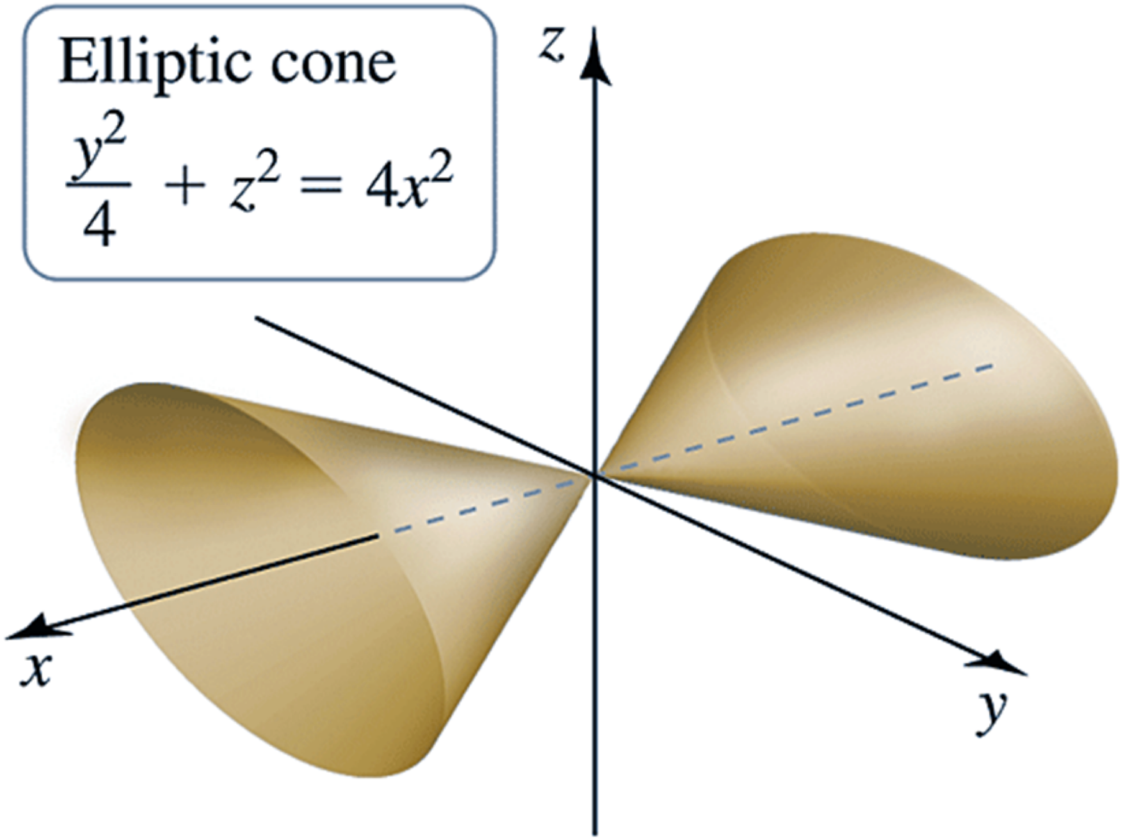


Figure 13.87 (b)



Example 7 A hyperboloid of two sheets

Graph $-16x^2-4y^2+z^2+64x-80=0$

$-16(x^2-4x+2^2)-4y^2+z^2=80-16\cdot 2^2$

$-16(x-2)^2-4y^2+z^2=16$

Figure 13.88

$\frac{z^2}{16}=1+(x-2)^2+\frac{y^2}{4}\geq 1$

$\Rightarrow z^2\geq 4^2 \Rightarrow z\geq 4$ or $z\leq -4$

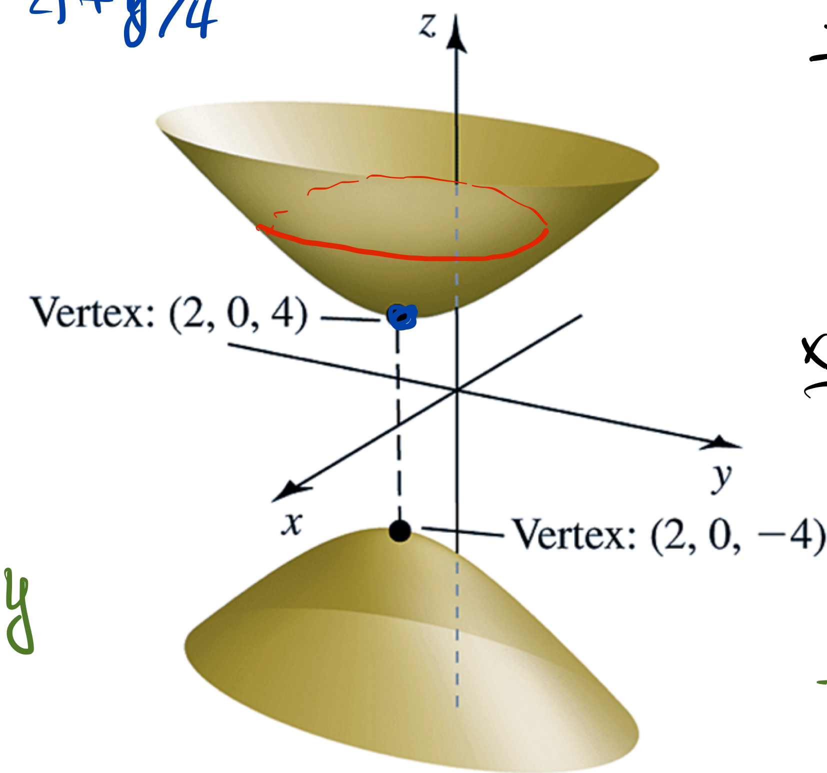
$(2,0,4)$ $\quad 1=1+(x-2)^2+y^2/4$

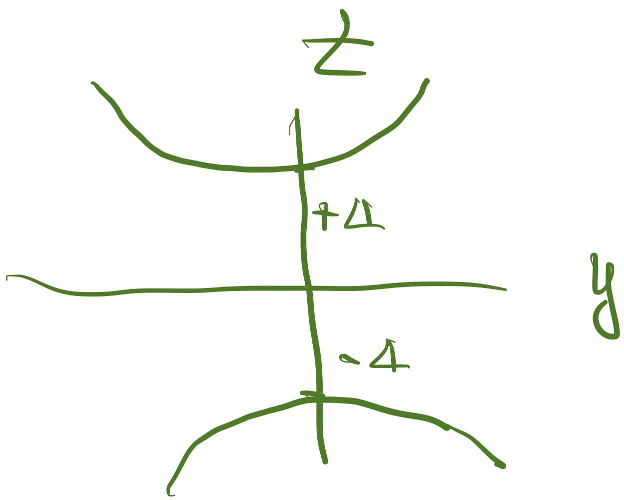
trace on $z=5$

$(x-2)^2+\frac{y^2}{4}=\frac{25}{16}-1=\frac{9}{16}$

Hyperboloid of two sheets

$$-(x-2)^2-\frac{y^2}{4}+\frac{z^2}{16}=1$$

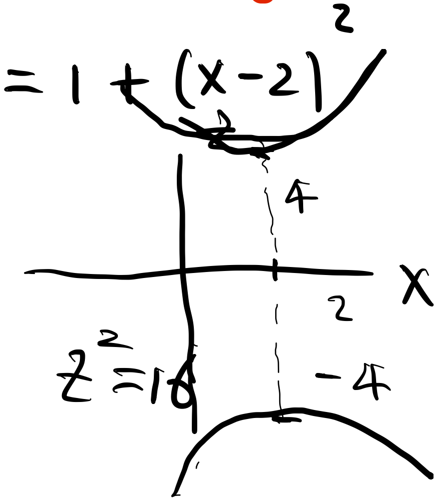




xz-trace $y=0$

$\frac{z^2}{16}=1+(x-2)^2$

$x=2 \quad z^2=16$



trace on $x=2$

$\frac{z^2}{16}=1+\frac{y^2}{4}$

Table 13.1 (1 of 3)

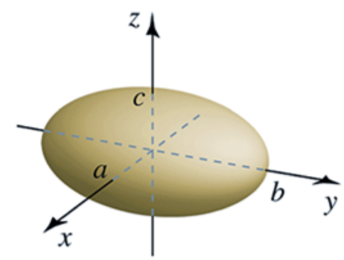
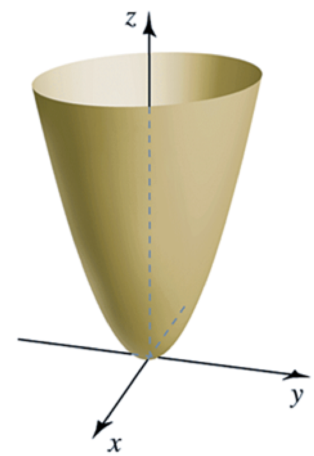
Name	Standard Equation	Features	Graph
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	All traces are ellipses.	
hyperboloid	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	one sheet	
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	two sheets	
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
hyper. paraboloid	$z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$		
cone	$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$		

Table 13.1 (2 of 3)

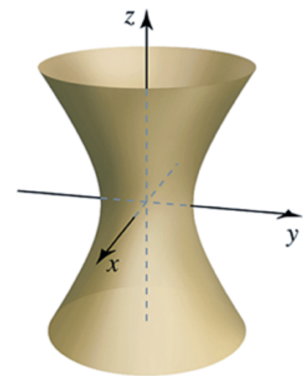
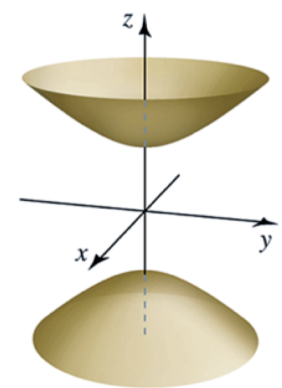
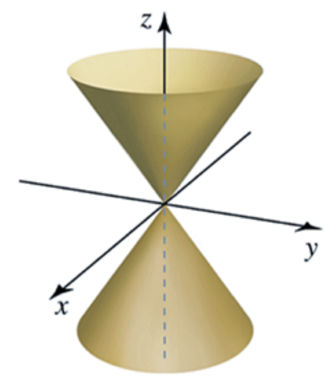
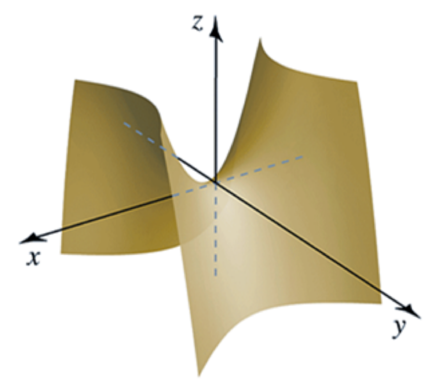
Name	Standard Equation	Features	Graph
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $\|z_0\| > \|c\|$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.	

Table 13.1 (3 of 3)

Name	Standard Equation	Features	Graph
Elliptic cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas..	

Examples identify the following quadric surfaces by name

#27 $9x^2+4z^2-36y=0$

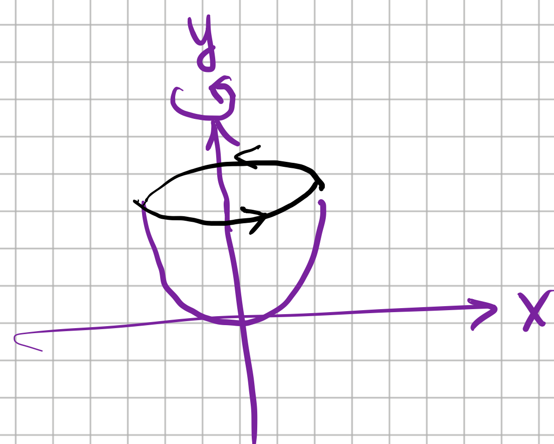
$$y=\frac{x^2}{4}+\frac{z^2}{9}$$ elliptic paraboloid

#56 $-x^2-y^2+\frac{z^2}{9}+6x-8y=26$

$$-(x^2-2\cdot3\cdot x+3^2)-(y^2+2\cdot4\cdot y+4^2)+\frac{z^2}{9}=26-9-16$$

$$-(x-3)^2-(y+4)^2+\frac{z^2}{9}=1$$ hyperboloid of two sheets

Surfaces of revolution



- $y=x^2$ revolves about y-axis

- $y=x$ revolves about x-axis