Chapter 14 Vector-Valued Functions (4 lectures)

§14.1 Vector-Valued Functions

- vector-valued function

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

aspect of analysis

- domain where $\vec{r}(t)$ is well-defined

  find the domain of $\vec{r}(t) = \langle \dfrac{t-2}{t+2}, \sin t, \ln(9-t^2) \rangle$

  #38, 40 (p874)

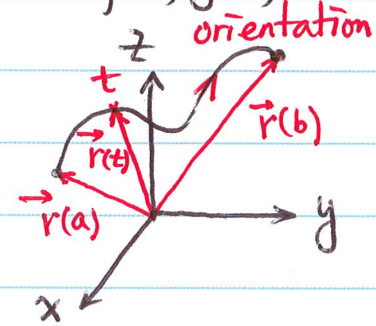
- limit $\lim\limits_{t \to a} \vec{r}(t) = \langle \lim\limits_{t \to a} x(t), \lim\limits_{t \to a} y(t), \lim\limits_{t \to a} z(t) \rangle$

- $\lim\limits_{t \to 0} \langle 1+t^3, t e^{-t}, \dfrac{\sin t}{t} \rangle =$

- continuity $\vec{r}(t)$ is continuous at $t=a \iff \lim\limits_{t \to a} \vec{r}(t) = \vec{r}(a)$

aspect of geometry

- curves in space $\quad C = \left\{ \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \;\middle|\; t \in I = [a, b] \right\}$



How to represent curves in plane and space?

(1) graph $\quad$ (2) level curve $\quad$ (3) parametric curve

examples (1) line segment from $P(2, -1, 4)$ to $Q(3, 0, 6)$

(2) a spiral graph $\vec{r}(t) = \langle 4\cos t, \sin t, \frac{t}{2\pi} \rangle$

$0 \leq t \leq \pi$ and $-\infty < t < \infty$

(3) Roller coaster curve $\vec{r}(t) = \langle \cos t, \sin t, 0.4 \sin 2t \rangle$

(4) slinky curve $\vec{r}(t) = \langle (3 + \cos 15t)\cos t, (3 + \cos 15t)\sin t, \sin 15t \rangle$

$0 \leq t \leq \pi$

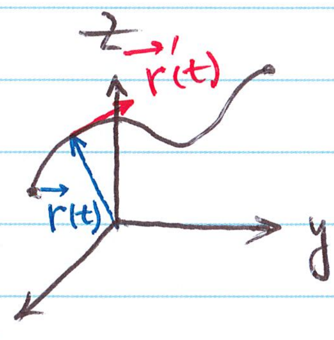
§14.2 Calculus of Vector-Valued Functions

- derivative $\quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \overset{?}{=} \langle f'(t), g'(t), h'(t) \rangle$$

#11, 16

- tangent vector



$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \quad \sim \text{unit tangent vector}$$

#24, 28

- derivative rules

(1) $\frac{d}{dt}\vec{c} = \vec{0}$ (constant $\vec{c}$) ; (2) $\left(\vec{u}(t) + \vec{v}(t)\right)' = \vec{u}\,'(t) + \vec{v}\,'(t)$

(3) $[f(t)\vec{u}(t)]' = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$; (4) $\frac{d}{dt}\vec{u}(f(t)) = \vec{u}\,'(f(t))\,f'(t)$

(5) $[\vec{u}\cdot\vec{v}]' = \vec{u}\,'\cdot\vec{v} + \vec{u}\cdot\vec{v}\,'$; (6) $[\vec{u}\times\vec{v}]' = \vec{u}\,'\times\vec{v} + \vec{u}\times\vec{v}\,'$

#34, 38, 48

- higher-order derivatives

#56

- $\underline{\text{integral}}$ $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

#71, 77

§14.3 Motion in Space

- $\underline{\text{position}}$ $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
- $\underline{\text{velocity}}$ $\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

  $\underline{\text{speed}}$ $|\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$
- $\underline{\text{acceleration}}$ $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

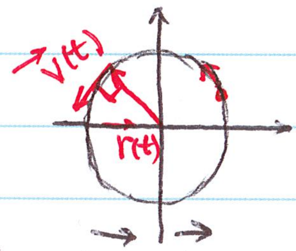
#10, 49

- $\underline{\text{straight-line motion}}$ $\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$, $t \geq 0$
- $\underline{\text{circular motion}}$ $\vec{r}(t) = \langle A\cos t, A\sin t \rangle$, $t \in [0, 2\pi]$

<u>Theorem</u> (motion with constant $|\vec{r}|$)

$\vec{r}$ is a path on which $|\vec{r}|$ is const. (motion on a circle or sphere center at the origin.)

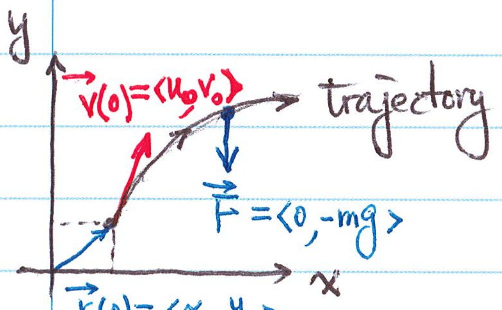
$$\Longrightarrow \vec{r}\cdot\vec{v} = 0$$



<u>Proof</u> $|\vec{r}|^2 = \text{const.}$

$$\Longrightarrow 0 = \frac{d}{dt}|\vec{r}|^2 = \frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\vec{r}\cdot\vec{v}.$$

- <u>two-dimensional motion in a gravitational field</u>



$g = 9.8\ m/s^2 = 32\ ft/s^2$

<u>Newton's second law</u>

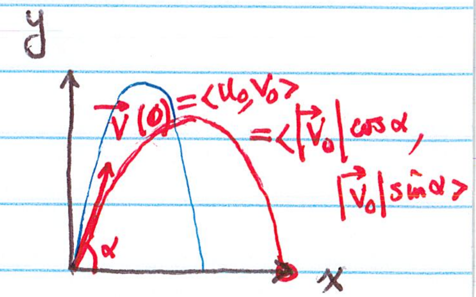
$$m\vec{a}(t) = \vec{F} = \langle 0, -mg \rangle$$

$$\Longrightarrow \vec{a}(t) = \langle 0, -g \rangle$$

$$\begin{cases} \vec{v}'(t) = \vec{a}(t) \\ \vec{v}(0) = \langle u_0, v_0 \rangle \end{cases} \Longrightarrow \vec{v}(t) = \langle u_0, -gt + v_0 \rangle$$

$$\begin{cases} \vec{r}'(t) = \vec{v}(t) \\ \vec{r}(0) = \langle x_0, y_0 \rangle \end{cases} \Longrightarrow \vec{r}(t) = \langle u_0 t + x_0, -\tfrac{1}{2}gt^2 + v_0 t + y_0 \rangle$$

Assume that $\vec{r}(0) = \langle 0, 0 \rangle$, $\vec{r}(t) = \langle x(t), y(t) \rangle$



- <u>time of flight</u>

  it returns to the ground $\quad (0 = y(t))$

  $$\Rightarrow 0 = y(t) = -\frac{1}{2} g t^2 + |\vec{v}_0| \sin\alpha \, t$$

  $$\Rightarrow \boxed{T = 2|\vec{v}_0| \sin\alpha / g}$$

- <u>range of the object</u>

  $$\boxed{x(T) = |\vec{v}_0| \cos\alpha \, T = \frac{|\vec{v}_0|^2 \sin 2\alpha}{g}}$$

- <u>maximum height of the object</u> $\quad (0 = y'(t))$

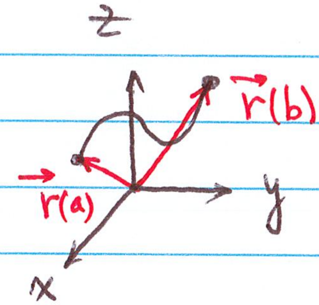
  $$0 = y'(t) = -gt + |\vec{v}_0| \sin\alpha \Rightarrow t = \frac{|\vec{v}_0| \sin\alpha}{g} = \frac{T}{2}$$

  $$\Rightarrow \boxed{y\left(\frac{T}{2}\right) = \frac{|\vec{v}_0|^2 \sin^2\alpha}{2g}}$$

- <u>three-dimensional motion in a gravitational field</u> $\quad m\vec{a} = \vec{F} = \langle 0, 0, -mg \rangle$

  $$\vec{a} = \langle 0, 0, -g \rangle$$

§14.4 Length of Curves



curve C : $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$a \le t \le b$

- arc length

$$L = \int_a^b |\vec{v}(t)|\, dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt$$

#14, 24

C : unit circle $x^2+y^2=1$ $\begin{cases} \vec{r}_1 = \langle \cos t, \sin t \rangle, & t \in [0, 2\pi] \text{ — arc length} \\ \vec{r}_2 = \langle \cos 2t, \sin 2t \rangle, & t \in [0, \pi] \text{ — } \cancel{\text{arc length}} \end{cases}$

- arc length as a parameter

$$s(t) = \int_a^t |\vec{v}(u)|\, du \iff \frac{ds}{dt} = |\vec{v}(t)| \text{ — speed.}$$

Example 3 helix $\vec{r}(t) = \langle 2\cos t, 2\sin t, 4t \rangle$ for $t \ge 0$

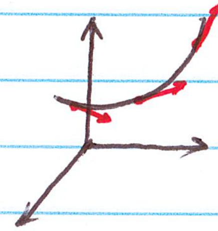
(a) Find the arc length function $s(t)$

(b) use arc length as the parameter to describe the helix.

$$\begin{cases} s = s(t) \Longrightarrow t = t(s) \\ \vec{r}(t) \Longrightarrow \vec{p}(t) = \vec{r}\big(t(s)\big) \text{ — arc length parametrizatio[n]} \end{cases}$$

§14.5 Curvature and Normal Vectors

- **curvature** (how a curve turns or bends)



(1) $\vec r(t)$ is smooth $\Leftrightarrow$ $\vec r\,'(t)$ is cont. & $\vec r\,'(t) \neq \vec 0$

(2) $C\cancel{\{\vec r(t) \mid t\in[a,b]\}}$ is a smooth curve

$\Leftrightarrow$ C has a smooth parametrization

(3) unit tangent vector $\vec T(t) = \dfrac{\vec r\,'(t)}{|\vec r\,'(t)|}$

(4) curvature of a curve

$$\kappa = \left|\frac{d\vec T}{ds}\right|$$

measurement of how quickly the curve changes direction

$\vec T(s) = \vec T(s(t))$

$\Rightarrow \dfrac{d\vec T}{dt} = \dfrac{d\vec T}{ds}\cdot\dfrac{ds}{dt}$ $\longrightarrow$

$$= \left|\frac{d\vec T}{dt}\Big/\frac{ds}{dt}\right| = \frac{1}{|\vec v|}\left|\frac{d\vec T}{dt}\right|$$

$$\boxed{\kappa = \frac{|\vec T\,'(t)|}{|\vec r\,'(t)|}} = \frac{|\vec v\times\vec a|}{|\vec v|^3}$$

$\boxed{\begin{array}{l}\vec v = \vec r\,'(t)\\ \vec a = \vec r\,''(t)\end{array}}$

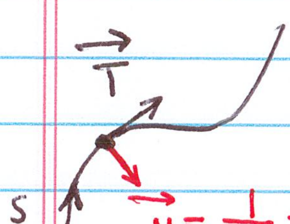
#12, 14, 16, 22

- **principal unit normal vector** at $\kappa \neq 0$

$1 = |\vec T| \Rightarrow \vec T\cdot\vec T = 1$

$\Rightarrow 0 = \dfrac{d}{dt}(\vec T\cdot\vec T) = 2\vec T\cdot\vec T\,'$

$\Rightarrow \vec T \perp \vec T\,'$



$\vec N = \dfrac{1}{\kappa}\dfrac{d\vec T}{ds} = \dfrac{\vec T\,'(t)}{|\vec T\,'(t)|}$

turning direction

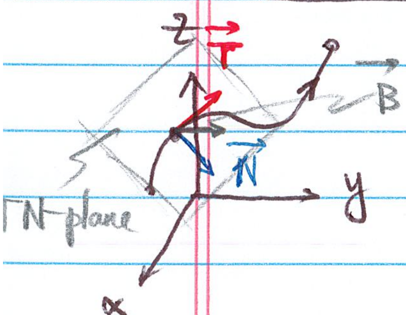
<u>Example 5</u> the helix $\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle$

$t \in (-\infty, +\infty)$, $a > 0$, $b > 0$

$\vec{N} = ?$

- <u>unit binormal vector</u> $\vec{B} = \vec{T} \times \vec{N}$ (Fig. 14.38)

- <u>torsion</u> $\tau = -\dfrac{d\vec{B}}{ds} \cdot \vec{N}$ the rate at which the curve twists out of the TN-plane.



$$= \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{|\vec{r}' \times \vec{r}''|^2}.$$

Ex. 8 and 9.