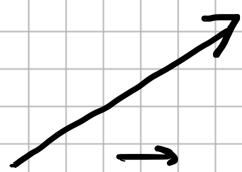
vector

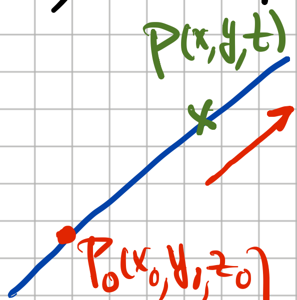


$|\vec{u}| = \sqrt{a_1^2+a_2^2+a_3^2}$ direction $\dfrac{\vec{u}}{|\vec{u}|}$

$\vec{u} = \langle a_1, a_2, a_3 \rangle$

$\vec{u} \pm \vec{v}$, multiplication $\alpha \vec{u}$, $\vec{u} \cdot \vec{v}$ $\vec{u} \times \vec{v}$

line

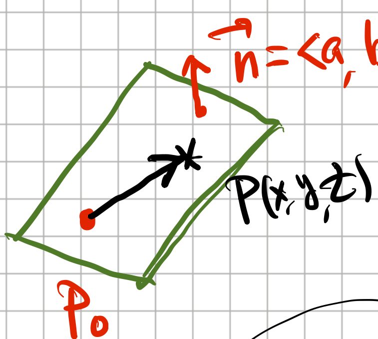


$\vec{v} = \langle a, b, c \rangle$

$\overrightarrow{P_0P} \parallel \vec{v}$

$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

plane



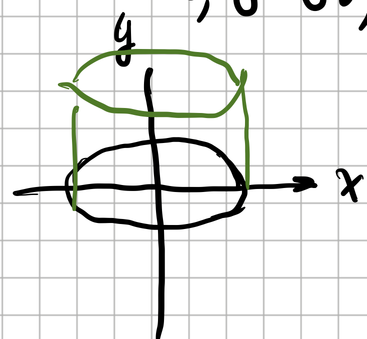
$\vec{n} = \langle a, b, c \rangle$

$\overrightarrow{P_0P} \perp \vec{n}$

$0 = \langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle$

cylinder

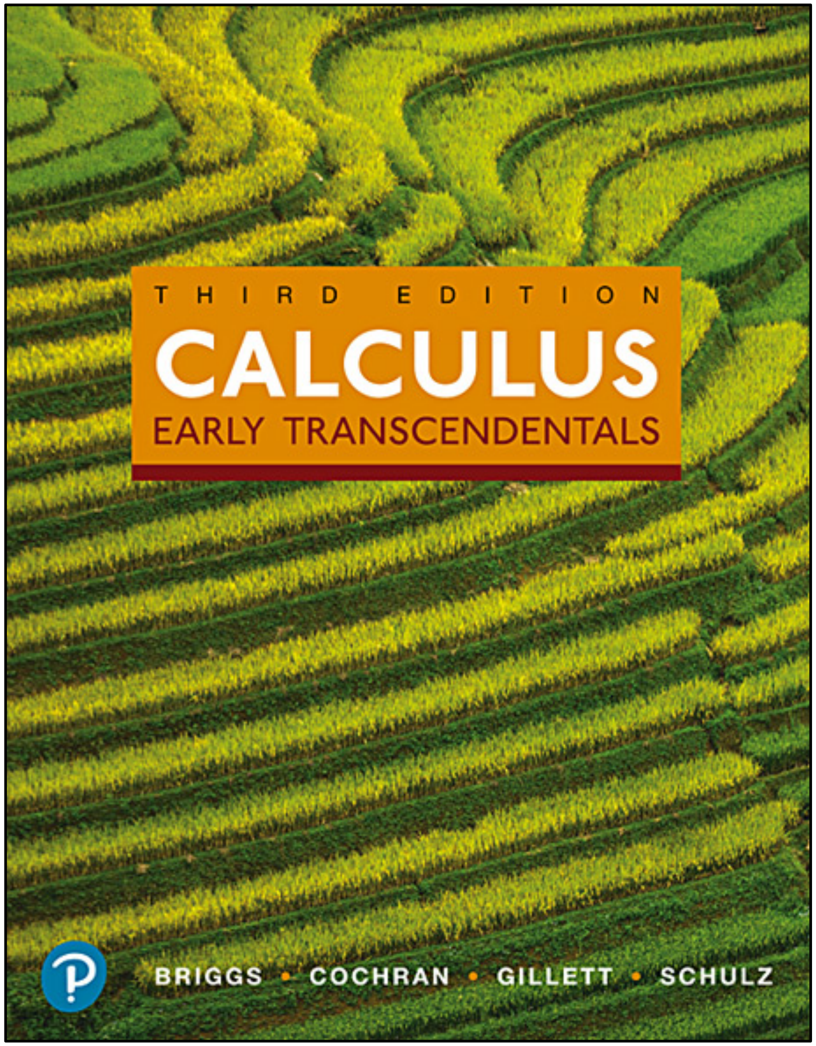
$x^2 + 2y^2 = 1$



quadric surfaces   6   **traces**

Calculus Early Transcendentals

Third Edition



Chapter 14

Vector-Valued Functions

§14.1 Vector-Valued Functions — Lesson 5

§14.2 Calculus of Vector-Valued Functions — Lesson 6

§14.3 Motion in Space — Lesson 7 (§14.2–§14.3)

§14.4 Length of Curves

§14.5 Curvature and Normal Vectors — Lesson 8 (§14.4–§14.5)

Section 14.1 Vector-Valued Functions

$y = f(x)$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$= x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

$a \leq t \leq b$

analysis (calculus) §14.2

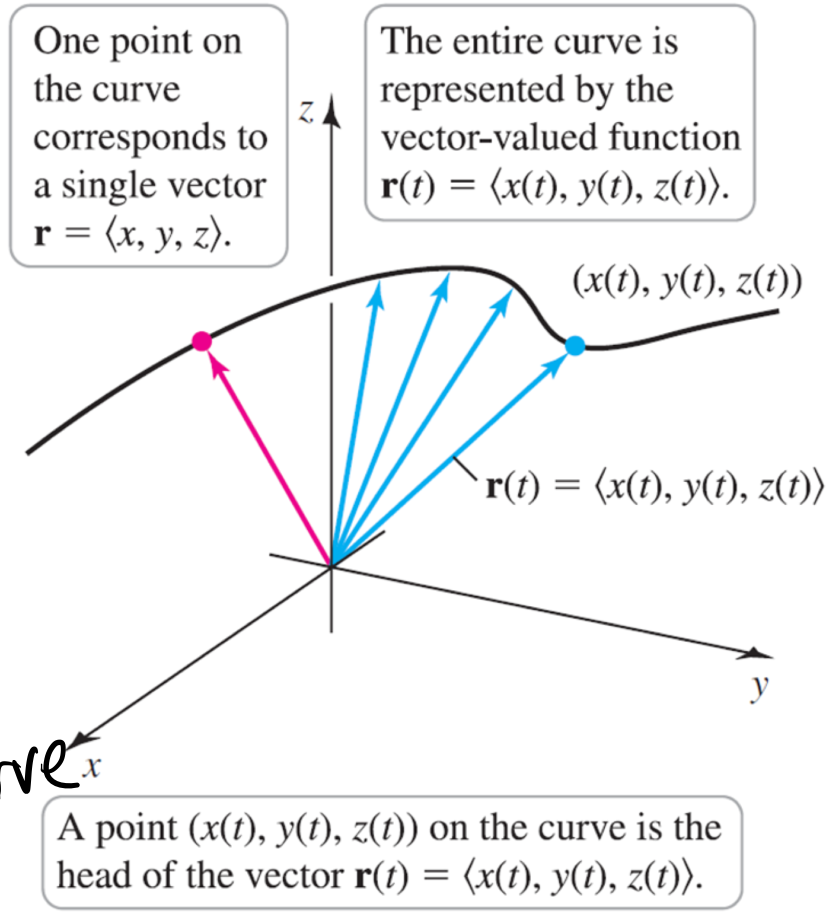
geometry §14.4-5

physics §14.3

Curves in Space

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $t \in [a,b]$

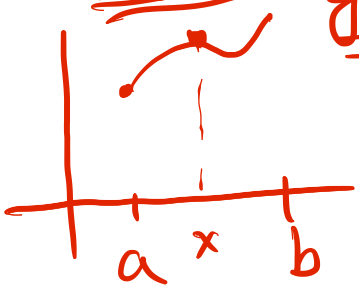
Figure 14.1



Curves in Plane

$y = f(x)$, $x \in [a,b]$

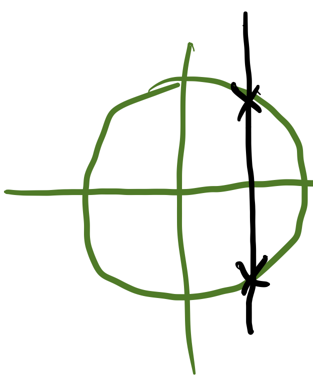
$(x, f(x))$ graph



$x^2 + y^2 = 1$

$f(x,y)$

level curve



Example 1 Find a vector-valued function for the line that passes through the points $P(2,-1,4)$ and $Q(3,0,6)$. $\vec{PQ} = \langle 1, 1, 2 \rangle$

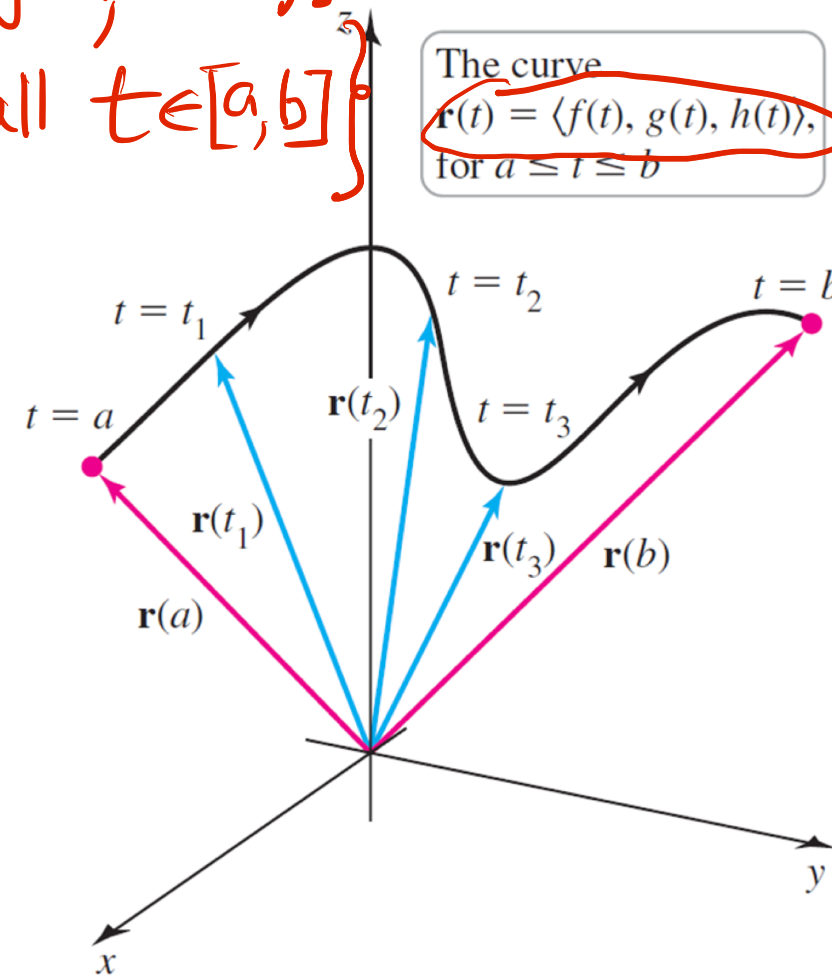
Figure 14.2

$\vec{r}(t) = \langle 2+t, -1+t, 4+2t \rangle$

Curve $= \{(x(t), y(t), z(t)) :$ for all $t \in [a,b]\}$

$t=0 \to P$ $t=1 \to Q$

parametrization of the curve



Orientation of Curves

Figure 14.3 (a & b)

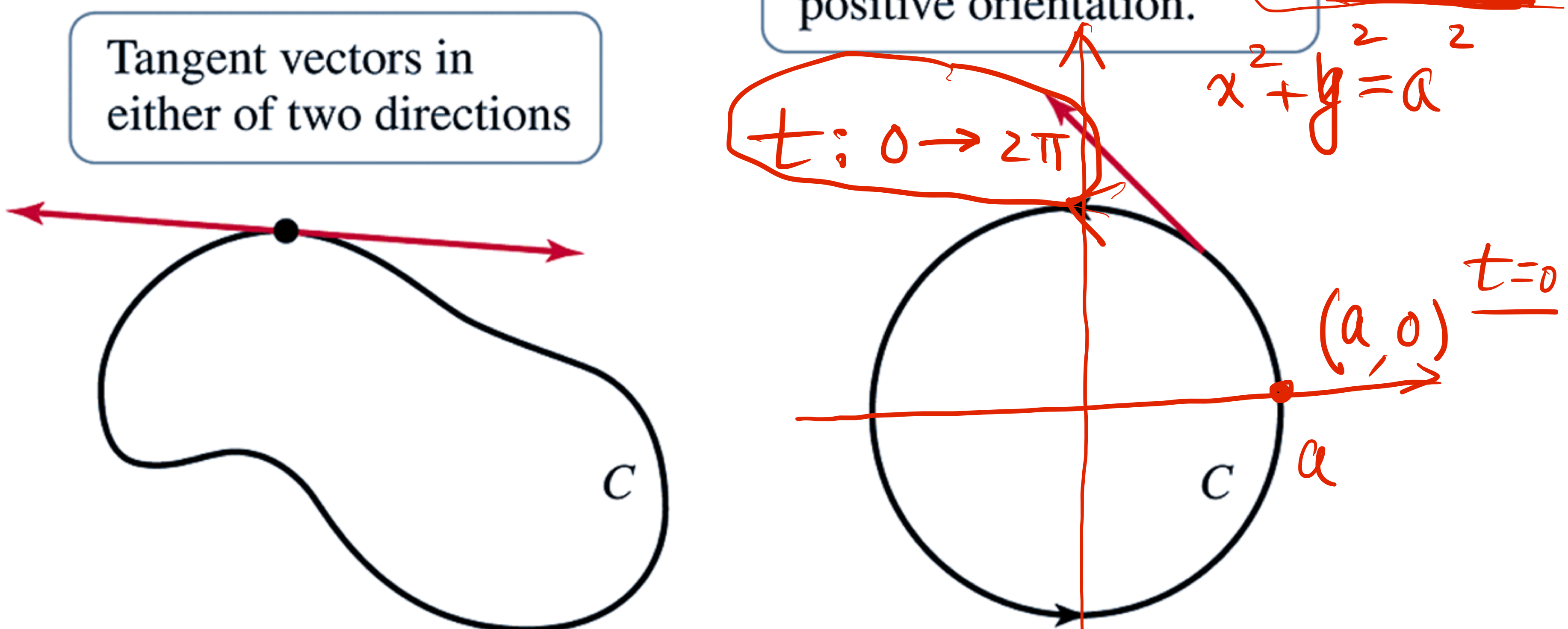
$$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$$

$$\vec{r}(t) = \langle a\cos t,\ a\sin t \rangle \qquad 0 \le t \le 2\pi$$

$$x^2 + y^2 = a^2$$

$t: 0 \to 2\pi$

$(a, 0)$ $t = 0$



Unparameterized curve

Parameterized curve

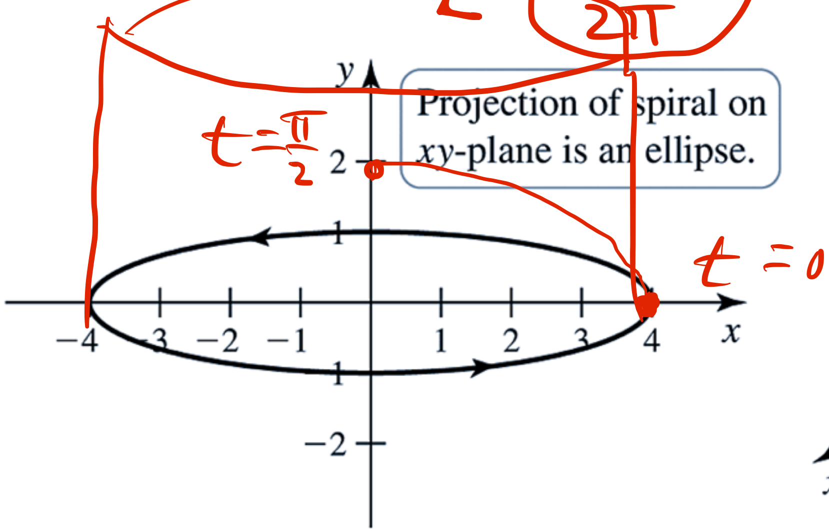
Example 2 (a spiral) Graph the curve described by the equation

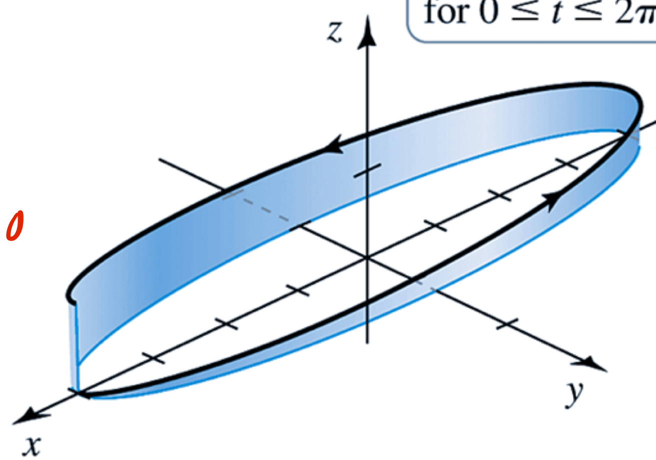
$\vec{r}(t) = \langle 4\cos t, \sin t, \frac{t}{2\pi} \rangle$ where (a) $0 \le t \le 2\pi$, (b) $-\infty < t < +\infty$

Figure 14.4 (a & b)

curves in plane $\begin{cases} x = 4\cos t \\ y = \sin t \end{cases} \Rightarrow \left(\frac{x}{4}\right)^2 + y^2 = 1$

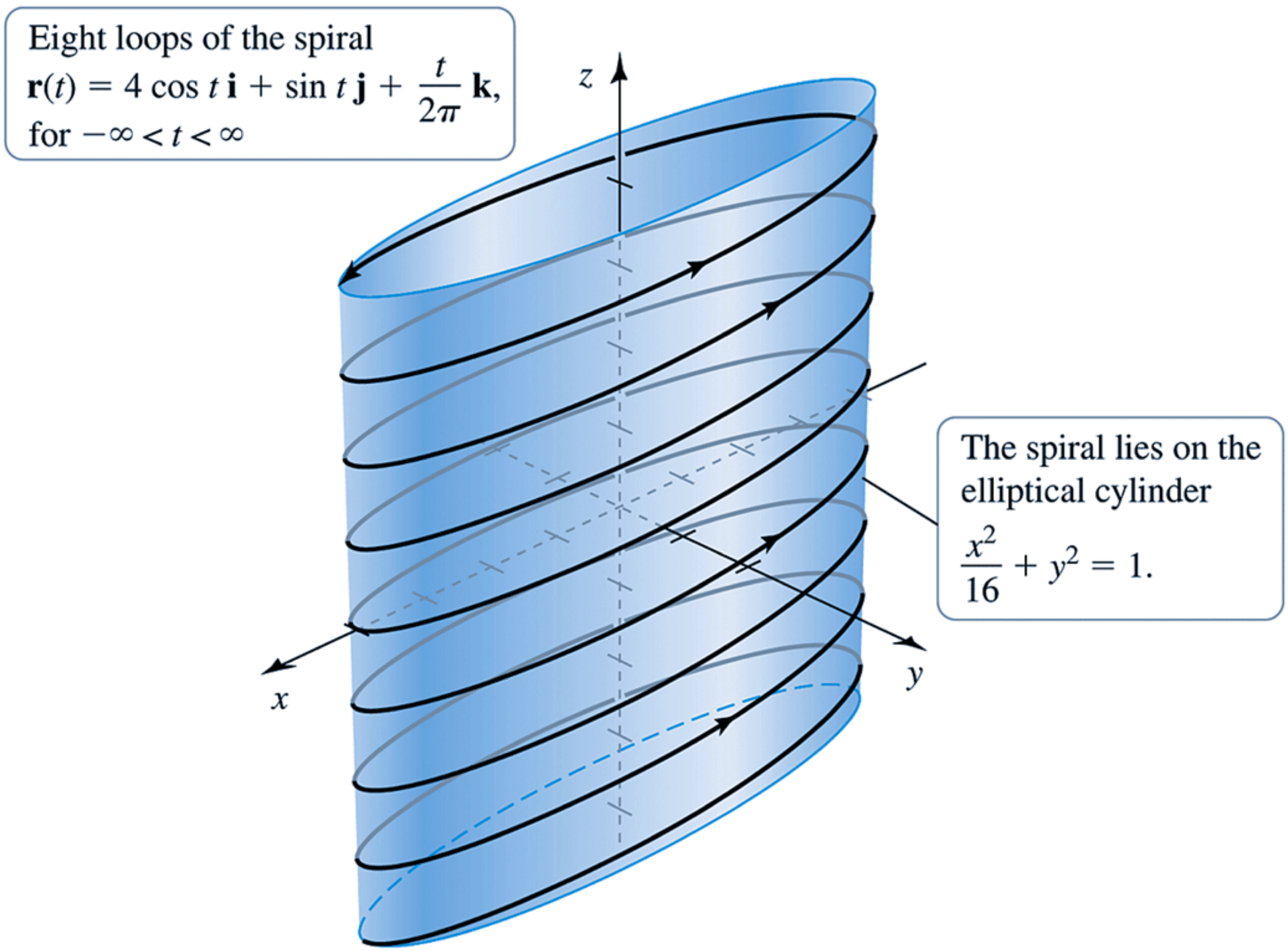
$z = \frac{t}{2\pi}$





One loop of the spiral
$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{t}{2\pi}\mathbf{k},$
for $0 \le t \le 2\pi$

Figure 14.4 (c)



Eight loops of the spiral
$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{t}{2\pi}\mathbf{k}$,
for $-\infty < t < \infty$
z
x
y
The spiral lies on the
elliptical cylinder
$\frac{x^2}{16} + y^2 = 1.$

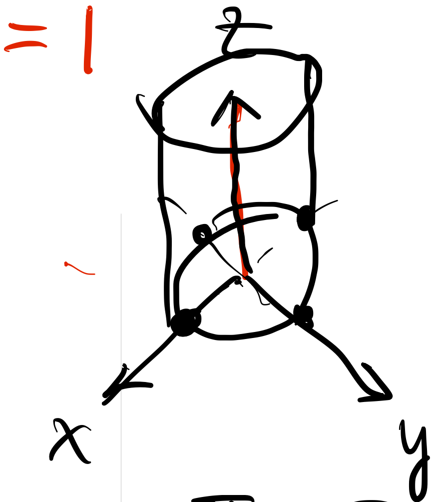
Example 3 (roller coaster curve) Graph the curve

$\vec{r}(t) = \langle \cos t, \sin t, 0.4\sin 2t \rangle$ for $0 \le t \le 2\pi$.

$\cos t = x$, $\sin t = y$, $0.4\sin 2t = z$

$x^2 + y^2 = 1$

Figure 14.5 (a)



$\vec{r}(0) = \langle 1, 0, 0 \rangle$

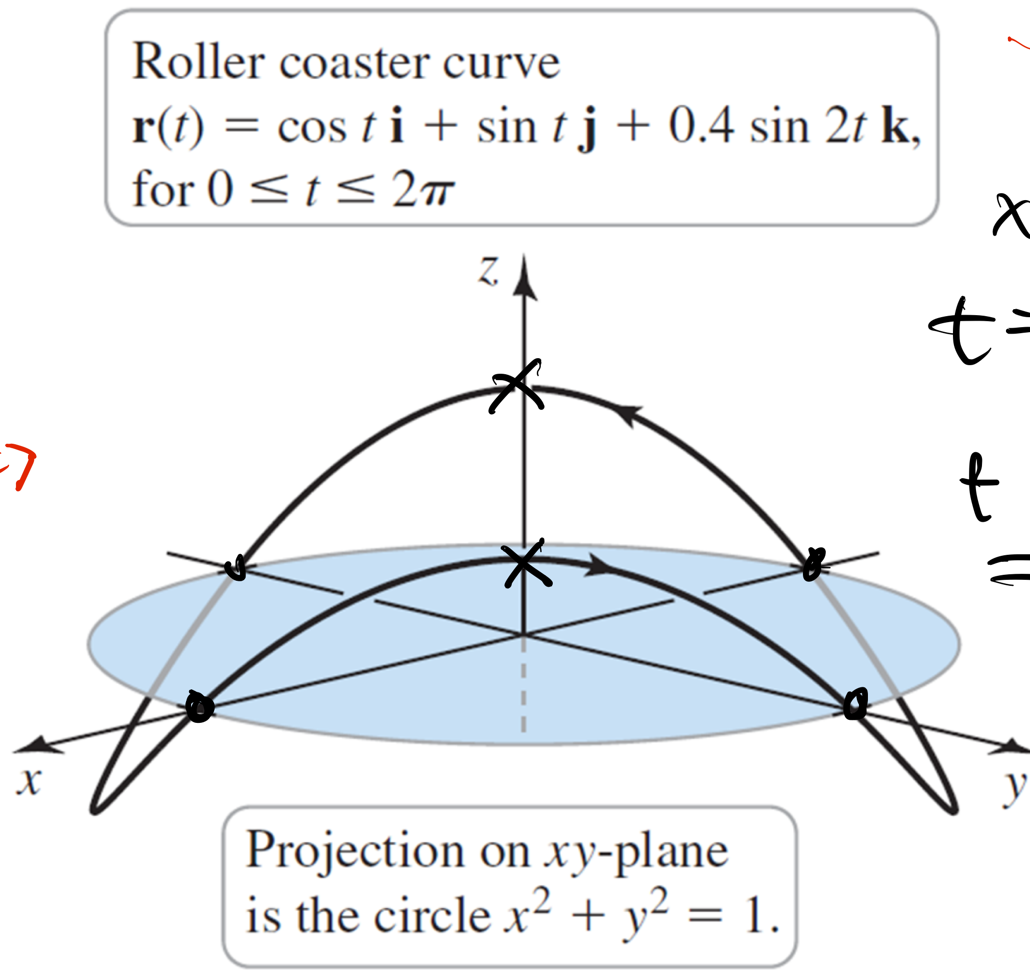
$\vec{r}\left(\frac{\pi}{2}\right) = \langle 0, 1, 0 \rangle$

$\vec{r}(\pi) = \langle -1, 0, 0 \rangle$

$\vec{r}\left(\frac{3\pi}{2}\right) = \langle 0, -1, 0 \rangle$

$\vec{r}\left(\frac{\pi}{4}\right) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0.4 \rangle$

$\vec{r}\left(\frac{3\pi}{4}\right) = \langle$



$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

$t = \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}, \ldots$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0.4\right)$

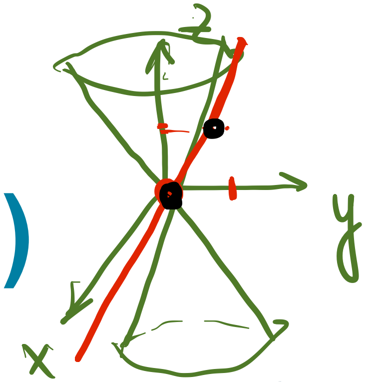
Review of Lesson 5

Vector-valued function

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad \text{for } a \le t \le b$$

- curve in space
- $\lim_{t \to t_0} \vec{r}(t) = \langle \lim_{t \to t_0} f(t), \lim g(t), \lim h(t) \rangle$

Figure 14.5 (b)



$$\vec{r}(t) = \langle t\cos t, t\sin t, t \rangle$$

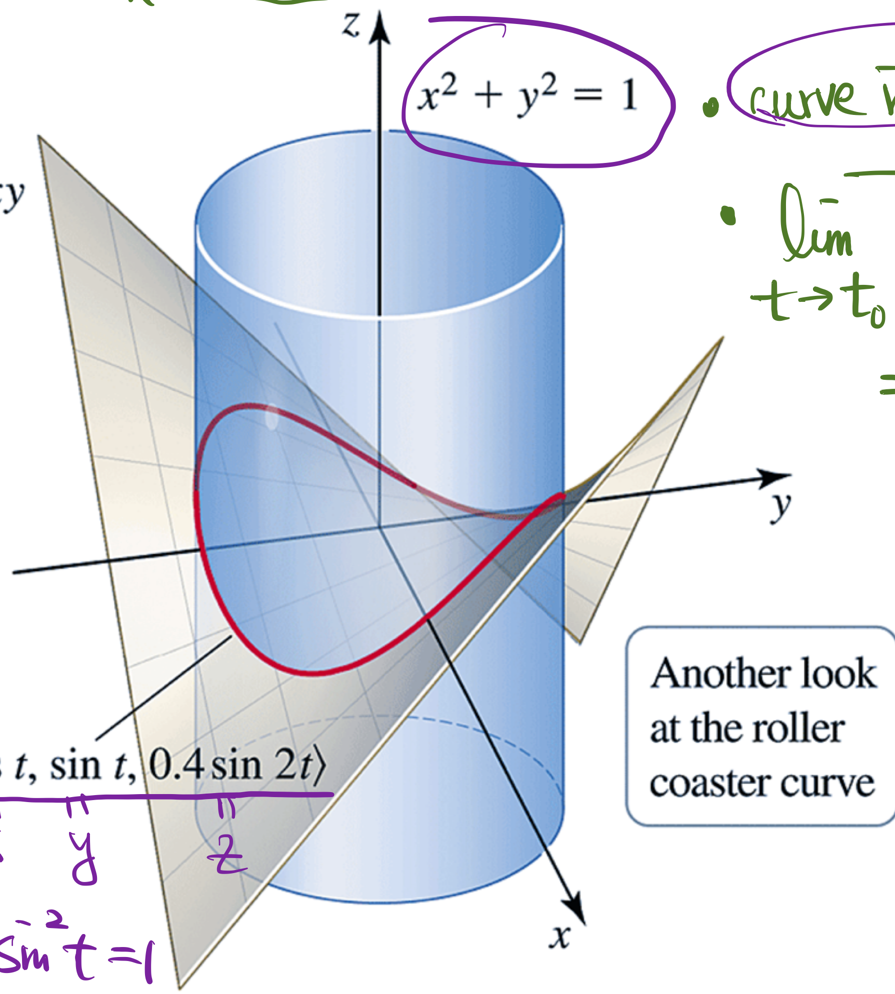
$x = t\cos t$, $y = t\sin t$, $z = t$

$$x^2 + y^2 = (t\cos t)^2 + (t\sin t)^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = z^2$$

$$\Rightarrow \boxed{x^2 + y^2 = z^2}$$

$t = 0$ $\langle 0, 0, 0 \rangle$

$t = \frac{\pi}{2}$ $\langle 0, \frac{\pi}{2}, \frac{\pi}{2} \rangle$



$x = \cos t$, $y = \sin t$, $z = 0.4\sin 2t$

$$\cos^2 t + \sin^2 t = 1$$

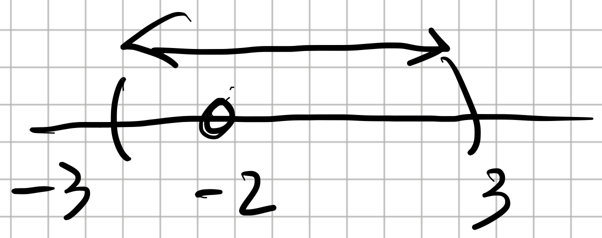
Aspect of Analysis

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

domain the set of t at where $f(t)$, $g(t)$, and $h(t)$ is well defined

example $\vec{r}(t) = \langle \frac{t-2}{t+2}, \sin t, \ln(9-t^2) \rangle$

$t \neq -2$ $\qquad 9 - t^2 > 0 \iff t^2 < 9 \iff -3 < t < 3$



limit $\lim_{t\to a} \vec{r}(t) = \vec{L} \iff \lim_{t\to a} \left| \vec{r}(t) - \vec{L} \right| = 0$

$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$

example $\lim_{t\to 0} \langle 1+t^3, te^{-t}, \frac{\sin t}{t} \rangle$

$= \langle 1, 0, 1 \rangle$

$f(x)$ is cont at x_0

$\iff \lim_{x\to x_0} f(x) = f(x_0)$

$\Updownarrow$

$f(\lim_{x\to x_0} x)$

continuity $\vec{r}(t)$ is continuous at $t = t_0$

$\iff \lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$

Example 5 (a) Evaluate $\lim_{t\to 2}\vec{r}(t) = \lim_{t\to 2}\langle \cos \pi t, \sin \pi t, e^{-t}\rangle = \langle 1, 0, e^{-2}\rangle$

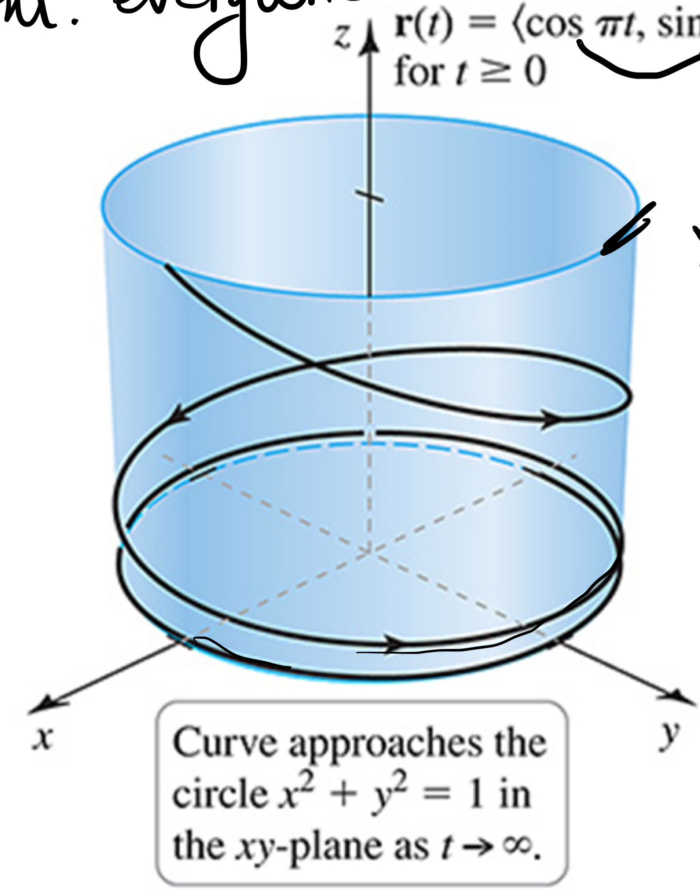
(b) Evaluate $\lim_{t\to \infty}\vec{r}(t) = \lim_{t\to \infty}\langle \cos \pi t, \sin \pi t, e^{-t}\rangle = \langle \text{DNE}, \text{DNE}, 0\rangle$

$= \text{DNE}$

Figure 14.7

(c) at what pts is $\vec{r}(t)$ continuous?

It is cont. everywhere. $\lim_{t\to t_0}\vec{r}(t) = \vec{r}(t_0)$



$x^2+y^2=1$

Section 14.2 Calculus of Vector-Valued Functions

- derivative
- integral

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $t \in [a, b]$

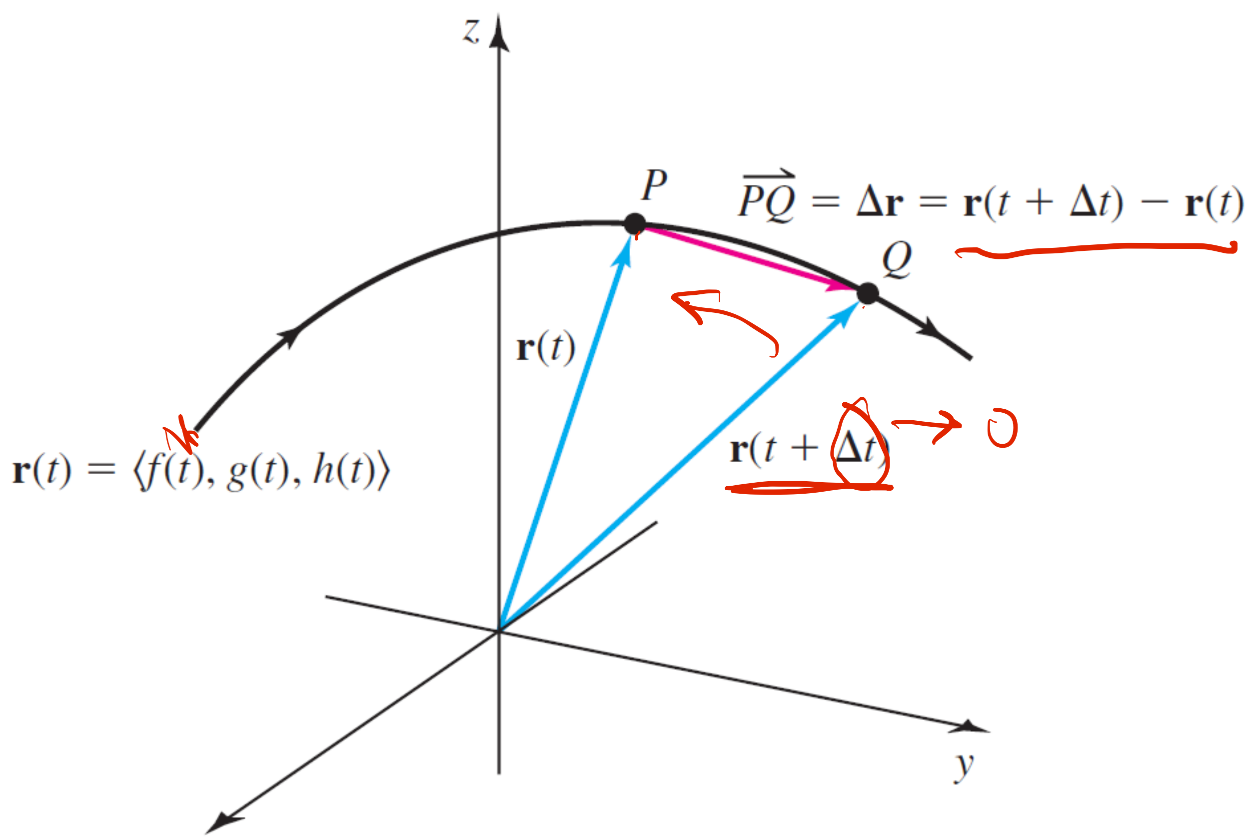
$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{(t+\Delta t) - t}$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, *, * \right\rangle$$

Pearson

$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Figure 14.8 (a)



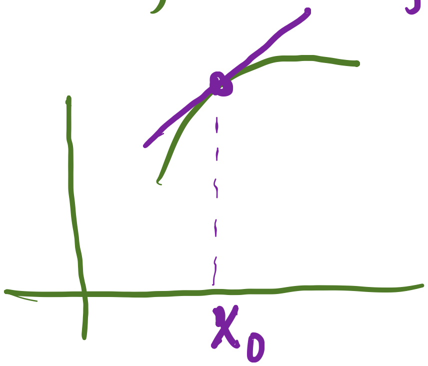
(a)

Line tangent to the curve at $P_0(t_0)$

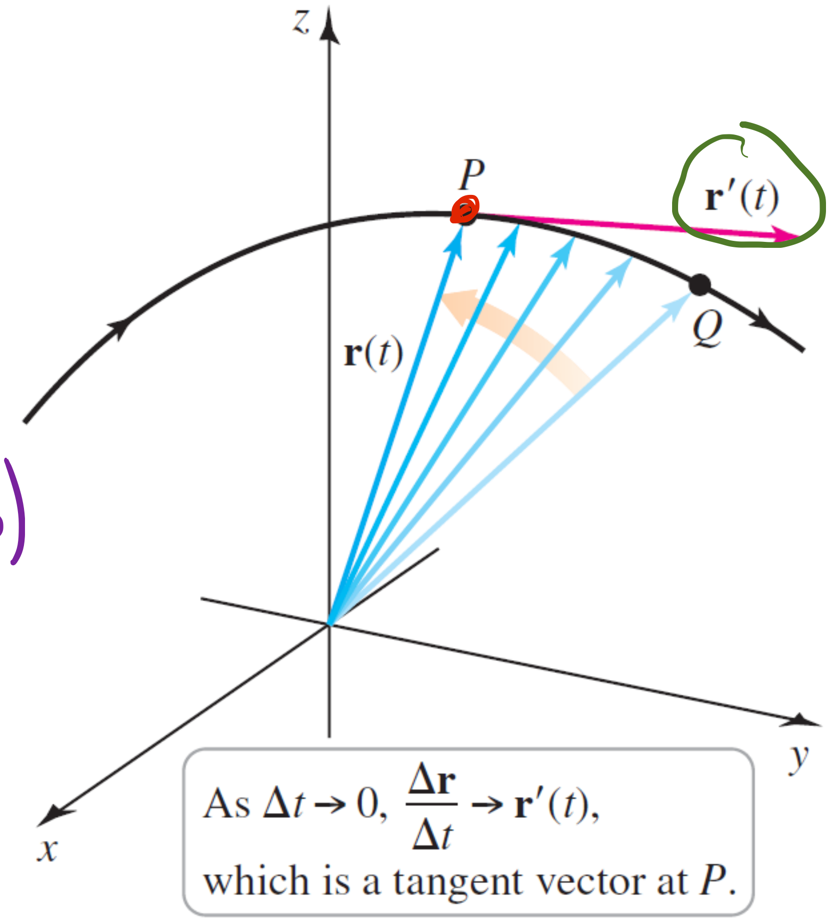
$$\vec{R}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0)$$

Figure 14.8 (b)

$y = f(x)$ $\quad f'(x_0)$



$$y = f(x_0) + f'(x_0)(x - x_0)$$



Definition Derivative and Tangent Vector

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$ where *f*, *g*, and *h* are differentiable functions on (*a*, *b*). Then **r** has a **derivative** (or is **differentiable**) on (*a*, *b*) and

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\ \mathbf{j} + h'(t)\mathbf{k}.$$

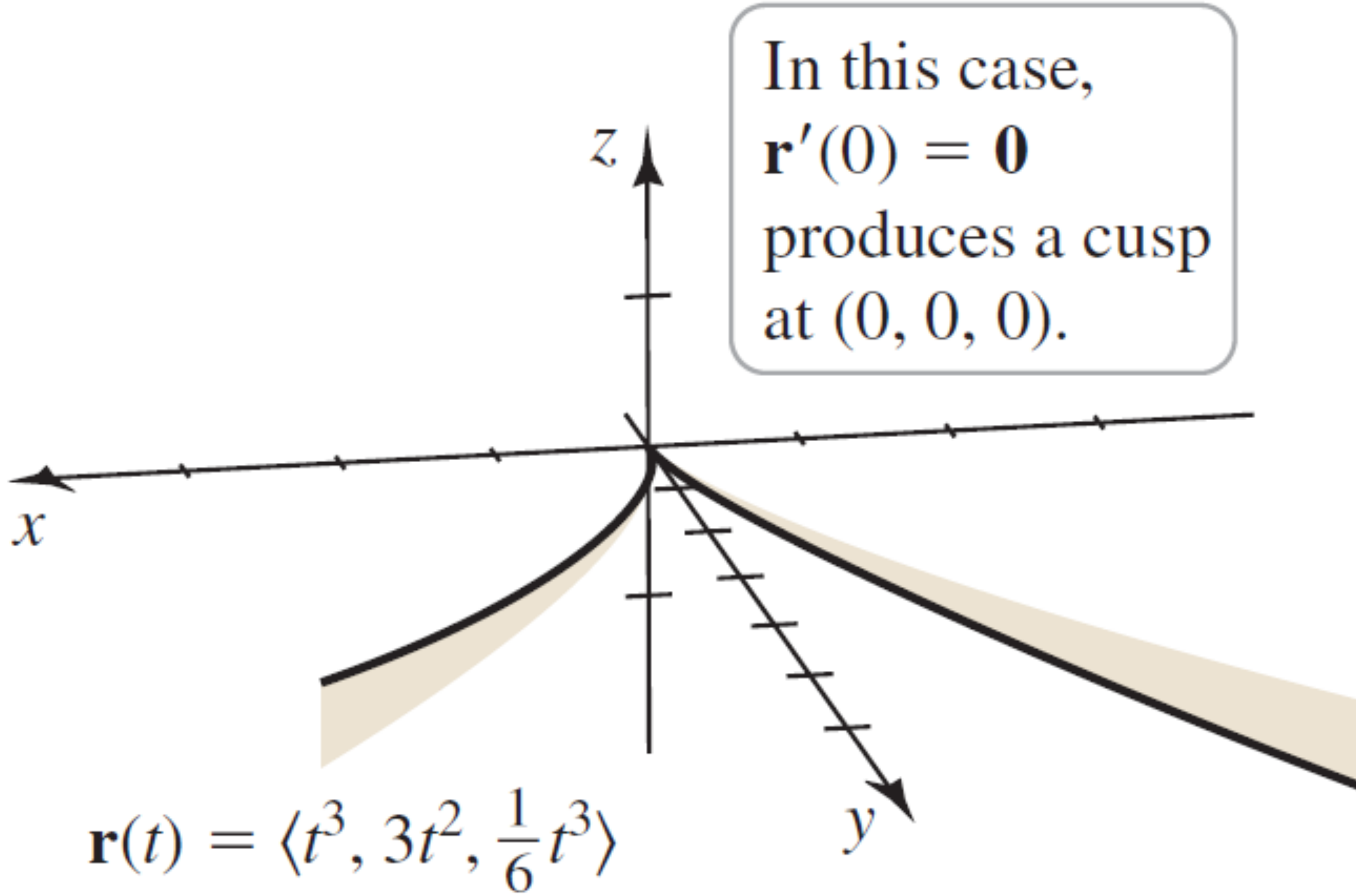
Provided $\mathbf{r}'(t) \neq \mathbf{0},\ \mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

Example 1 Compute derivatives

(a) $\vec{r}(t) = \langle t^3, 3t^2, t^3/6 \rangle$, (b) $\vec{r}(t) = e^{-t}\vec{i} + 10\sqrt{t}\,\vec{j} + 2\cos 3t\,\vec{k}$

Figure 14.9

$\vec{r}\,'(t) = \langle 3t^2, 6t, \frac{1}{2}t^2 \rangle$ $\vec{r}\,'(t) = -e^{-t}\vec{i} + 5t^{-\frac{1}{2}}\vec{j} - 6\sin 3t\,\vec{k}$



Definition Unit Tangent Vector

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$ be a smooth parameterized curve, for $a \le t \le b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

① ②

Example 2 Find $\vec{T}(t)$

(a) $\vec{r}(t) = \langle t^2, 4t, 4\ln t \rangle$ for $t > 0$

$\vec{r}\,' = \langle 2t, 4, \frac{4}{t} \rangle$

$|\vec{r}\,'| = \left(4t^2 + 4^2 + \frac{4^2}{t^2}\right)^{\frac{1}{2}}$

$= \frac{2}{|t|}\left(t^4 + 4t^2 + 4\right)^{\frac{1}{2}}$

$= \frac{2}{|t|}\left(t^2 + 2\right)$

(b) $\vec{r}(t) = \langle 10, 3\cos t, 3\sin t \rangle$, for $0 \le t \le 2\pi$.

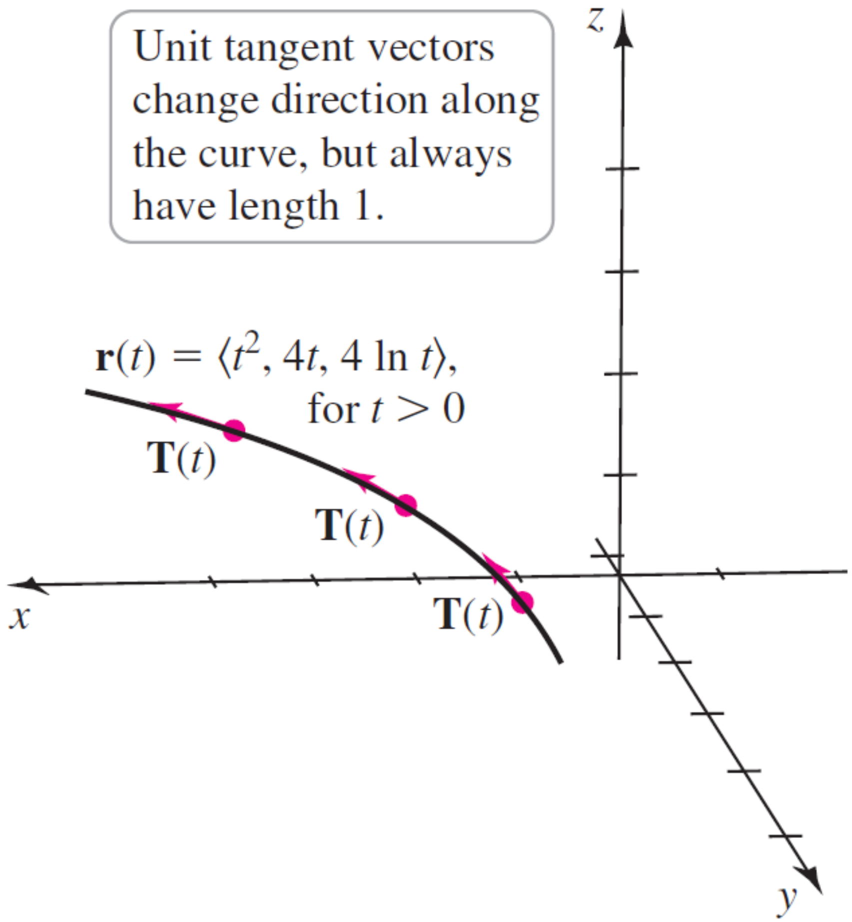
$\vec{r}\,' = \langle 0, -3\sin t, 3\cos t \rangle$

$|\vec{r}\,'| = \left(9(\sin^2 t + \cos^2 t)\right)^{\frac{1}{2}}$

$= 3$

$\vec{T} = \frac{1}{3}\vec{r}\,'(t) = \langle 0, -\sin t, \cos t \rangle$

Figure 14.10


Unit tangent vectors change direction along the curve, but always have length 1.
$\mathbf{r}(t) = \langle t^2, 4t, 4 \ln t \rangle$, for $t > 0$
$\mathbf{T}(t)$
$\mathbf{T}(t)$
$\mathbf{T}(t)$
z
x
y

Theorem 14.1 Derivative Rules

Let **u** and **v** be differentiable vector-valued functions and let *f* be a differentiable scalar-valued function, all at a point *t*. Let **c** be a constant vector. The following rules apply.

1. $\frac{d}{dt}(\mathbf{c}) = \mathbf{0}$ — Constant Rule

2. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ — Sum Rule

$(fg)' = f'g + fg'$

3. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ — Product Rule

u

$\vec{u}$

4. $\frac{d}{dt}\big(\mathbf{u}(f(t))\big) = \mathbf{u}'\big(f(t)\big)f'(t)$ — Chain Rule

5. $\frac{d}{dt}\big(\mathbf{u}(t) \cdot \mathbf{v}(t)\big) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ — Dot Product Rule

6. $\frac{d}{dt}\big(\mathbf{u}(t) \times \mathbf{v}(t)\big) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ — Cross Product Rule

Examples #34 $\vec{v}(t) = \langle e^{t}, 2e^{-t}, -e^{2t} \rangle$

$$\frac{d}{dt}\left[(4t^8 - 6t^3)\vec{v}(t)\right] = (32t^7 - 18t^2)\vec{v} + (4t^8 - 6t^3)\langle e^{t}, -2e^{-t}, -2e^{2t} \rangle$$

#48 $\vec{u}(t) = \langle 1, t, t^2 \rangle$, $\vec{v}(t) = \langle t^2, -2t, 1 \rangle$ $\quad \vec{u} \cdot \vec{v} = t^2 - 2t^2 + t^2 = 0$

$$\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = 0$$

$$= \langle 0, 1, 2t \rangle \cdot \langle t^2, -2t, 1 \rangle + \langle 1, t, t^2 \rangle \cdot \langle 2t, -2, 0 \rangle$$

$$= (-2t + 2t) + (2t - 2t) = 0$$

#56 $\vec{r}(t) = \langle e^{4t}, 2e^{-t} + 1, 2e^{-t} \rangle$

$$\vec{r}\,'(t) = \langle 4e^{4t}, -2e^{-t}, -2e^{-t} \rangle$$

$$\vec{r}\,''(t) = \left(\vec{r}\,'(t)\right)' = \langle 16e^{4t}, 2e^{-t}, 2e^{-t} \rangle$$

$$\vec{r}\,'''(t) = \left(\vec{r}\,''(t)\right)'$$

- Integration

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

indefinite integral

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$$

definite integral

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_0^\pi 1\,dt, \int_0^\pi 3\cos\frac{t}{2}\,dt, \int_0^\pi (-4t)\,dt \right\rangle = \left\langle \pi, -6\sin\frac{t}{2}\Big|_0^\pi, -2t^2\Big|_0^\pi \right\rangle$$

Example 7 $\int_0^\pi \langle 1, 3\cos\frac{t}{2}, -4t \rangle dt$

Example #77

$$\int_0^1 \langle te^t, t\sin t, 1 \rangle dt = \left\langle \int_0^1 te^t dt, \int_0^1 t\sin t\,dt, \int_0^1 1\,dt \right\rangle$$

$u' = \sin t,\ v = t$

$u = -\cos t,\ v' = 1$

$(uv)' = u'v + uv'$

$$\boxed{\int u'v = uv - \int uv'}$$

$$\int_0^1 t\sin t = -t\cos t\Big|_0^1 + \int_0^1 \cos t\,dt$$

$$= -\cos 1 + \sin t\Big|_0^1 = -\cos 1 + \sin 1$$

Section 14.3 Motion in Space

Newton's 2^{nd} Law $\quad \vec{F} = m\vec{a}$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

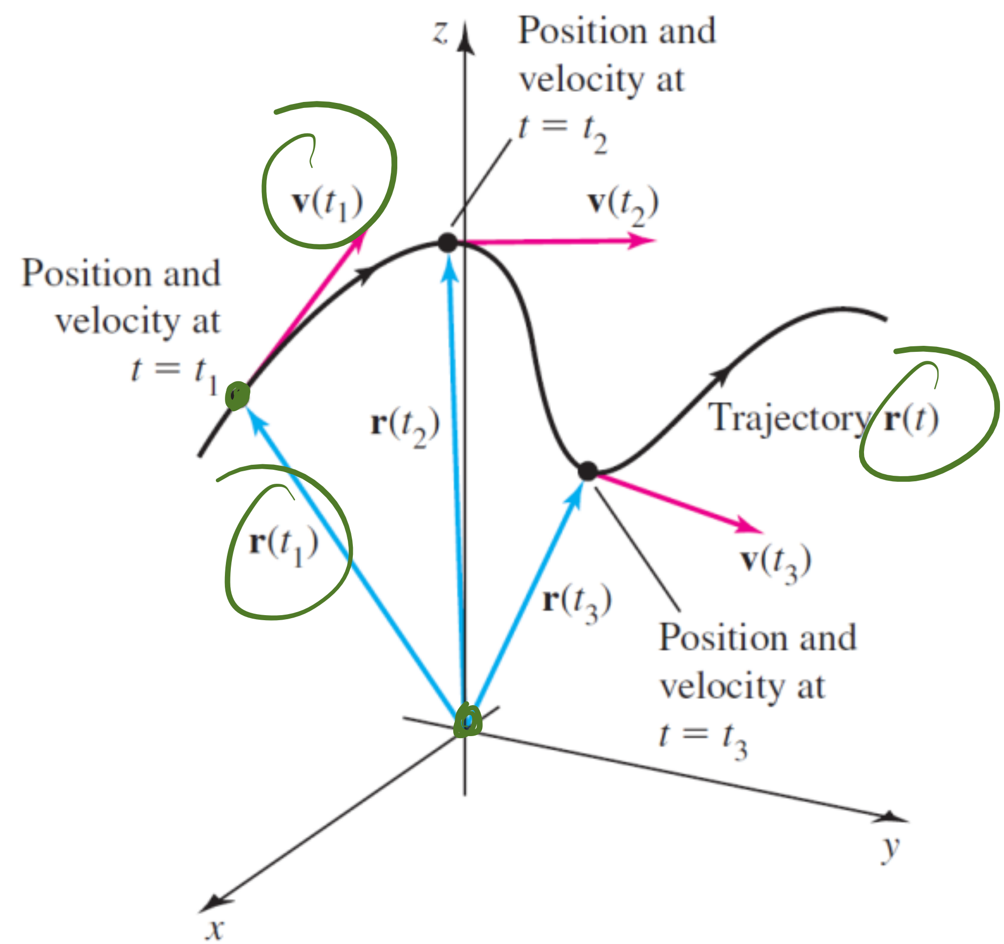
$\vec{v} = \int \vec{a}(t)\,dt + \vec{C}_1$.

$\vec{v}(t) = \vec{r}\,'(t)$, $s = |\vec{v}(t)|$

$\vec{r} = \int \vec{v}(t)\,dt + \vec{C}_2$.

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

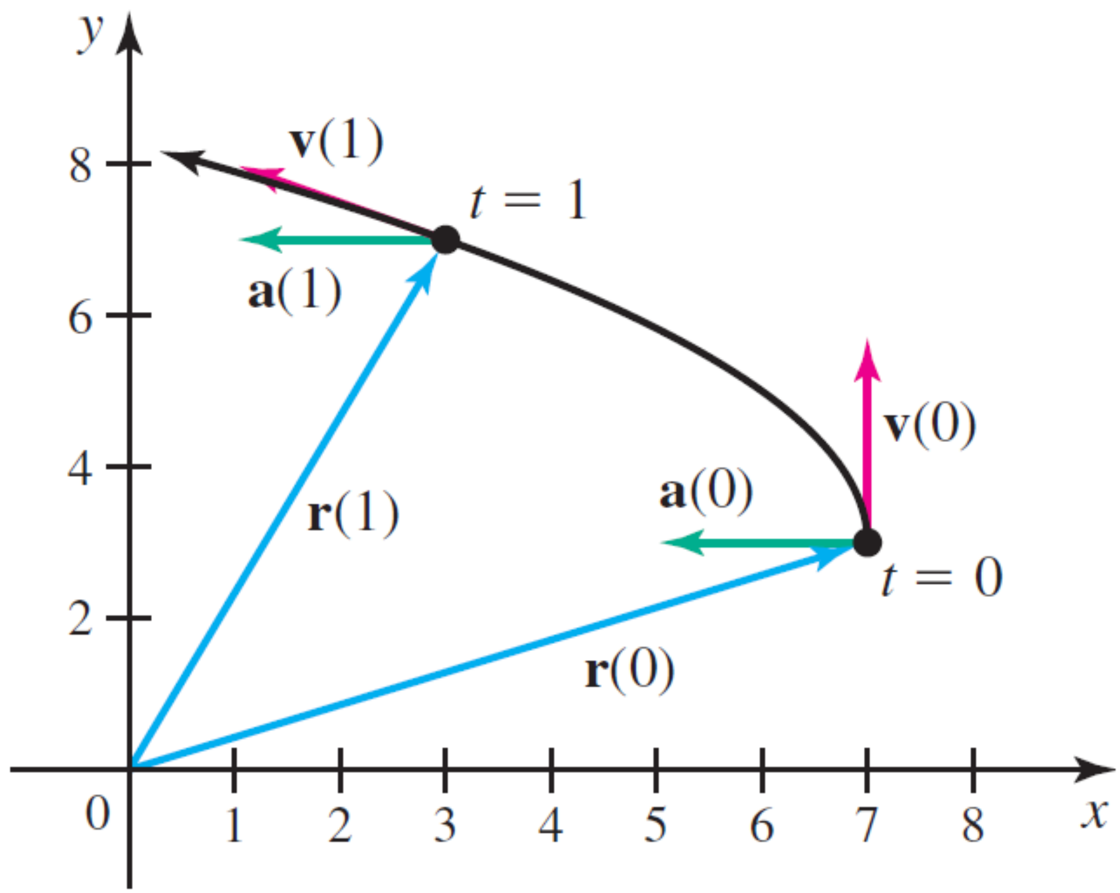
Figure 14.11



z
Position and velocity at $t = t_2$
$\mathbf{v}(t_1)$
$\mathbf{v}(t_2)$
Position and velocity at $t = t_1$
$\mathbf{r}(t_2)$
Trajectory $\mathbf{r}(t)$
$\mathbf{r}(t_1)$
$\mathbf{v}(t_3)$
$\mathbf{r}(t_3)$
Position and velocity at $t = t_3$
y
x

Difficult to visualize the acceleration vector

Figure 14.12



Definition Position, Velocity, Speed, Acceleration

Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $t \geq 0$.

The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example 1 Consider the two dimensional motion given by the position vector $\vec{r}(t)=\langle 3\cos t, 3\sin t\rangle$, $t\in[0,2\pi]$

(a) Sketch the trajectory of the object.

(b) Find the velocity and speed of the object.

$$\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}$$

$$1 = \cos^2 t + \sin^2 t = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 \Rightarrow x^2+y^2=3^2$$

Figure 14.13

(c) Find the acceleration of the object.

(d) Sketch the position, velocity, and acceleration vectors at $t=0,\frac{\pi}{2},\pi$, and $\frac{3\pi}{2}$.

(b) $\vec{v}(t) = \vec{r}\,'(t) = 3\langle -\sin t, \cos t\rangle$

$s = |\vec{v}| = 3\sqrt{\sin^2 t + \cos^2 t} = 3$

$\vec{a}(t) = \vec{v}\,'(t) = 3\langle -\cos t, -\sin t\rangle$

$= -3\langle \cos t, \sin t\rangle$

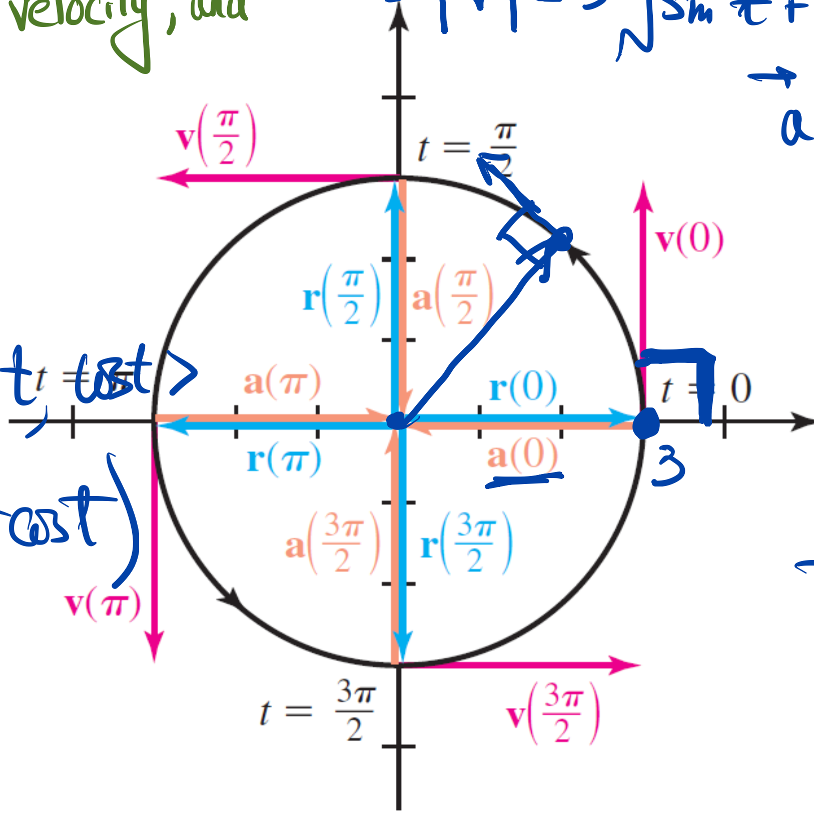
$= -\vec{r}(t)$

$\vec{r}(t)\cdot\vec{v}(t)$

$= 3\langle \cos t, \sin t\rangle \cdot 3\langle -\sin t, \cos t\rangle$

$= 9(-\cos t\sin t + \sin t\cos t)$

$= 0$



$\vec{r}(0) = 3\langle 1, 0\rangle$

$\vec{v}(0) = 3\langle 0, 1\rangle$

Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.

Questions (a) Find the interval [a, b] over which the $\vec{R}$ trajectory is the same as the $\vec{r}$ trajectory over [0, 2].

(b) Find the velocity and speed.

$$\vec{r}(t): \begin{cases} x(t) = t \\ y(t) = t^2 - 4 \\ z(t) = \frac{1}{4}t^3 - 8 \end{cases}$$

$$\vec{R}(t): \begin{cases} X(t) = t^2 = x(t^2) \\ Y(t) = t^4 - 4 = y(t^2) \\ Z(t) = \frac{1}{4}t^6 - 8 = z(t^2) \end{cases}$$

Figure 14.14

$$\boxed{\vec{R}(t) = \vec{r}(t^2)} \Rightarrow t^2 \in [0, 2] \iff 0 \le t^2 \le 2$$

$$0 \le t \le \sqrt{2}$$

$$[a, b] = [0, \sqrt{2}]$$

$$\boxed{\vec{R}(t) = \vec{r}(f(t))}$$

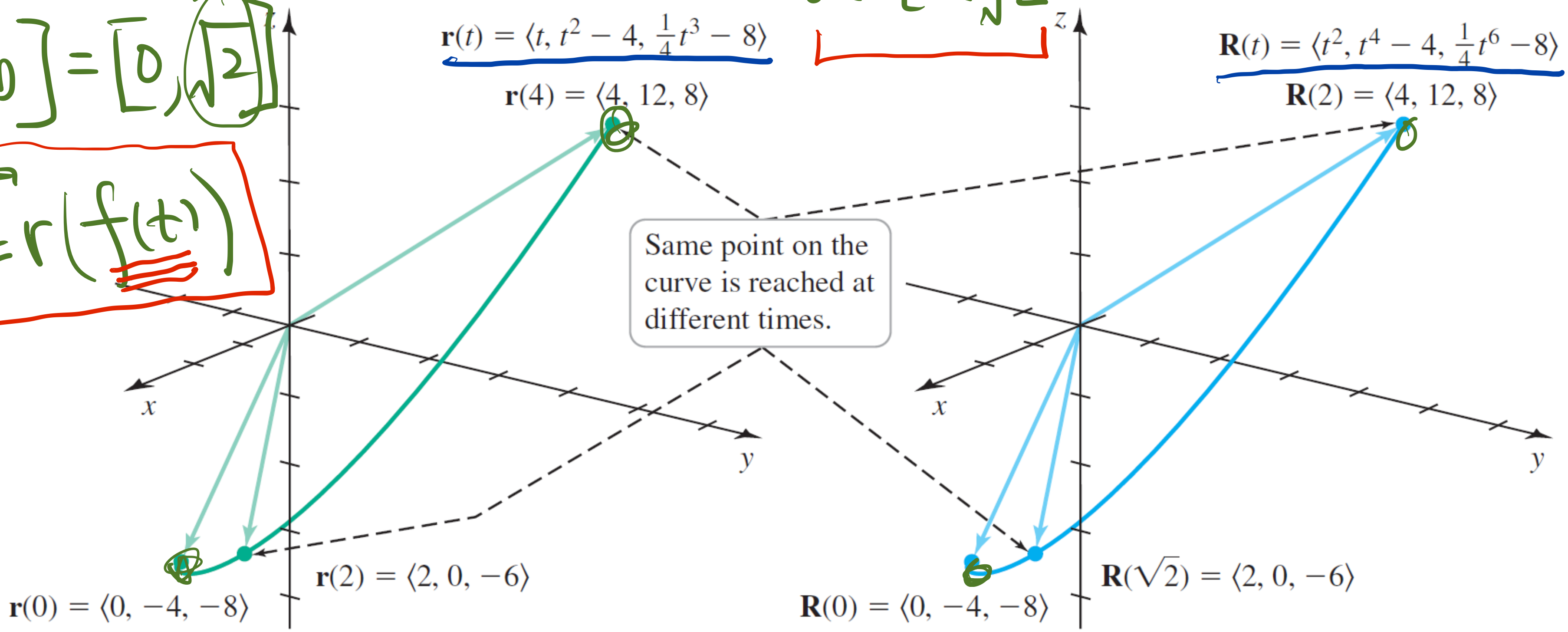
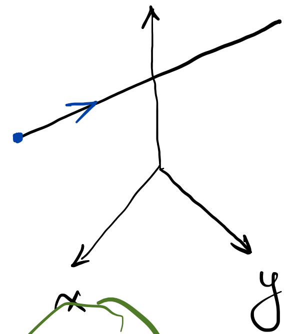


Figure 14.15

• Straight Line Motion



$$\vec r(t) = \vec r_0 + t\vec v$$

$$\vec v(t) = \vec v$$

$$\vec a(t) = \vec 0$$

#49 Given $\vec a(t) = \langle \sin t, \cos t, 1 \rangle$, $\vec v(0) = \langle 0, 2, 0 \rangle$, $\vec r(0) = \langle 0, 0, 0 \rangle$. Find $\vec v(t)$ and $\vec r(t)$.

$$\vec v(t) = \int \vec a(t)\,dt + \vec C_1$$

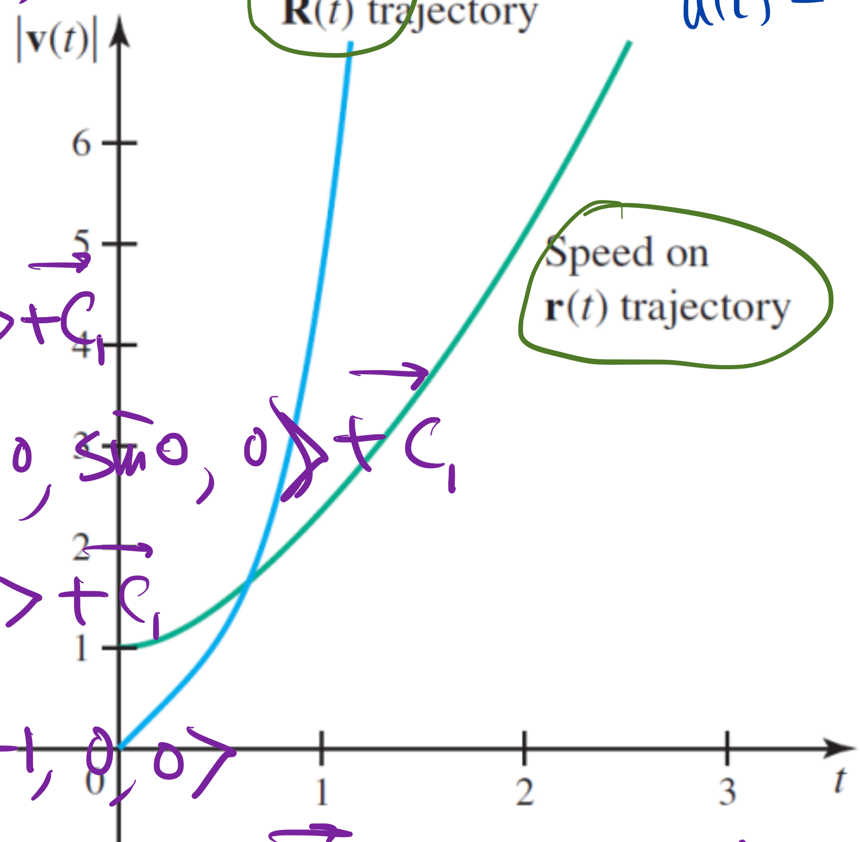
$$= \langle -\cos t, \sin t, t \rangle + \vec C_1$$

$$\langle 0, 2, 0 \rangle = \vec v(0) = \langle -\cos 0, \sin 0, 0 \rangle + \vec C_1$$

$$= \langle -1, 0, 0 \rangle + \vec C_1$$

$$\vec C_1 = \langle 0, 2, 0 \rangle - \langle -1, 0, 0 \rangle$$

$$= \langle 1, 2, 0 \rangle \Longrightarrow \vec v(t) = \langle -\cos t + 1, \sin t + 2, t \rangle$$

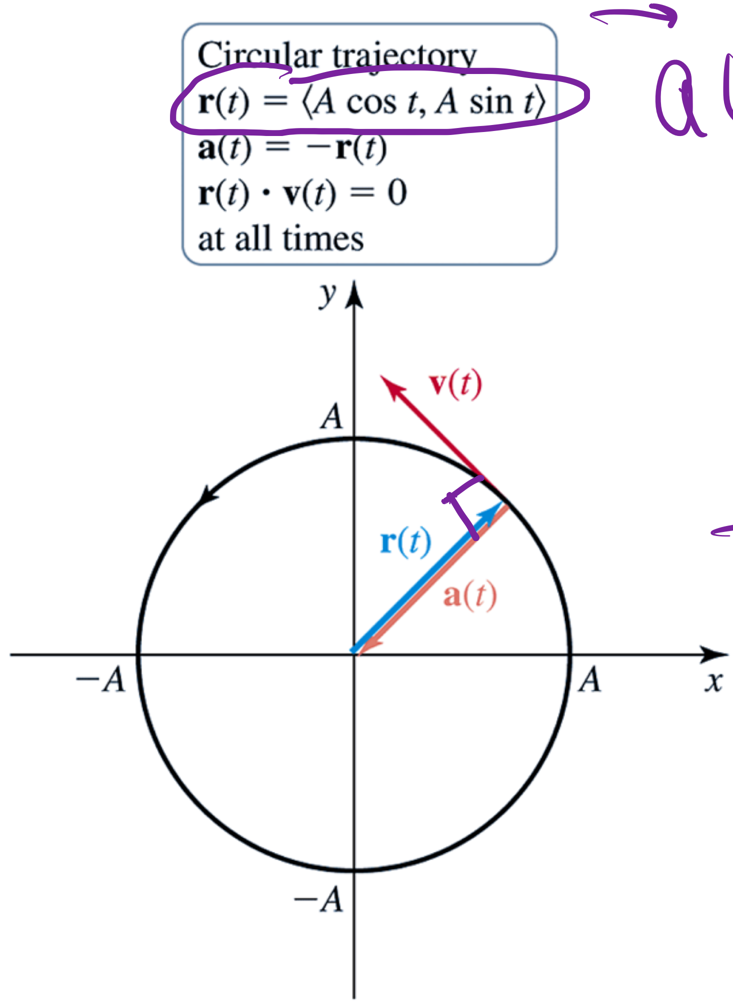


- Circular Motion

$$\begin{cases} x(t) = A\cos t \\ y(t) = B\sin t \end{cases} \Longrightarrow x^2 + y^2 = A^2$$

Figure 14.16

$$\vec{v}(t) = \vec{r}\,'(t) = A\langle -\sin t, \cos t\rangle$$



$$\vec{a}(t) = A\langle -\cos t, -\sin t\rangle$$

$$= -A\langle \cos t, \sin t\rangle$$

$$= -\vec{r}(t)$$

$$\vec{r} \cdot \vec{v} = 0$$

Theorem 14.2 Motion with Constant Magnitude of *r*

Let **r** describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then $\mathbf{r}\cdot\mathbf{v}=0,$ which means the position vector and the velocity Vector are orthogonal at all times for which the functions are defined.

Given $\boxed{|\vec{r}| = \text{const}}$ $\overset{?}{\Longrightarrow}$ $\vec{r} \perp \vec{v}$

$\Longleftrightarrow \vec{r}\cdot\vec{r}\,' = 0$

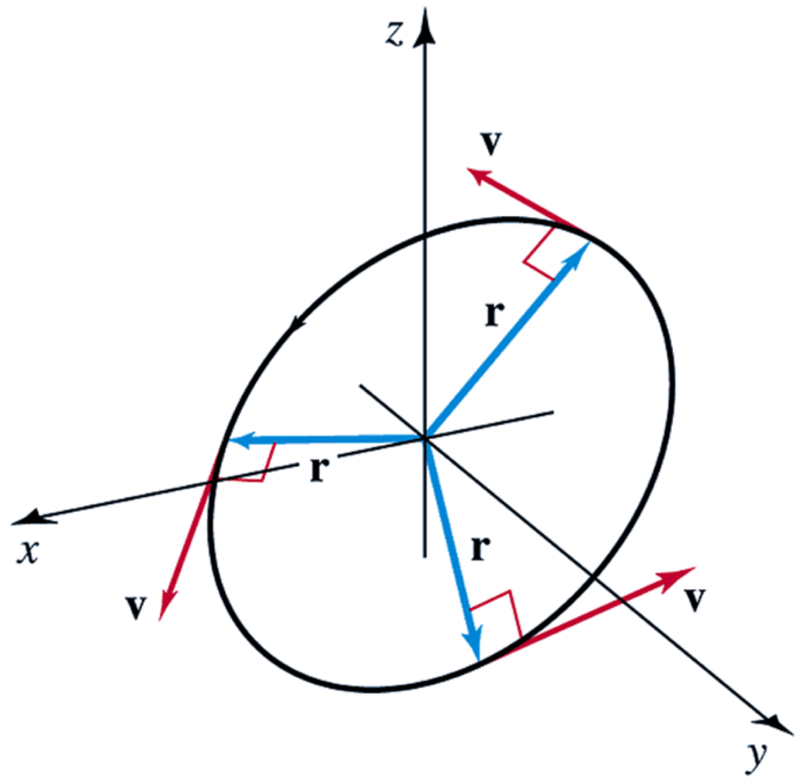
$$\frac{d}{dt}\left(\vec{r}\cdot\vec{r} = |\vec{r}|^2 = \text{const}\right)$$

$$\vec{r}\,'\cdot\vec{r} + \vec{r}\cdot\vec{r}\,' = 0$$

$$\parallel$$

$$2\vec{r}\cdot\vec{r}\,'$$

Figure 14.17



On a trajectory on which $|\mathbf{r}|$ is constant, $\mathbf{v}$ is orthogonal to $\mathbf{r}$ at all points.

Example 3 An object moves on a trajectory described by $\vec{r}(t) = \langle 3\cos t, 5\sin t, 4\cos t\rangle$, $t \in [0, 2\pi]$

(a) Show that the object moves on a sphere and find the radius of the sphere.

(b) Find the velocity and speed of the object.

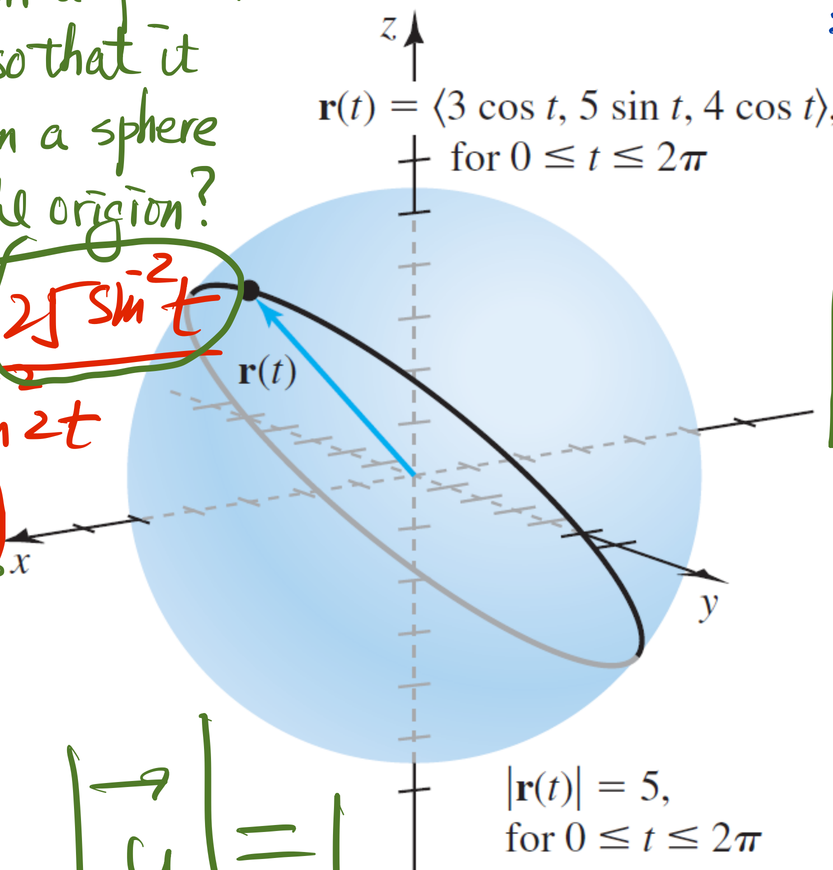
$$x^2 + y^2 + z^2 = 9\cos^2 t + 25\sin^2 t + 16\cos^2 t$$
$$= 25(\cos^2 t + \sin^2 t)$$
$$= 25 = 5^2$$
$$\rho = 5$$

Figure 14.18

(c) $\vec{r}(t) = \langle 5\cos t, 5\sin t, 5\sin 2t\rangle$.

- Show the curve does not lie on a sphere.
- How could $\vec{r}$ be modified so that it describes a curve that lies on a sphere of radius 1, centered at the origin?

$$x^2 + y^2 + z^2 = 25\cos^2 t + 25\sin^2 t + 25\sin^2 2t$$
$$= 25(1 + \sin^2 2t)$$

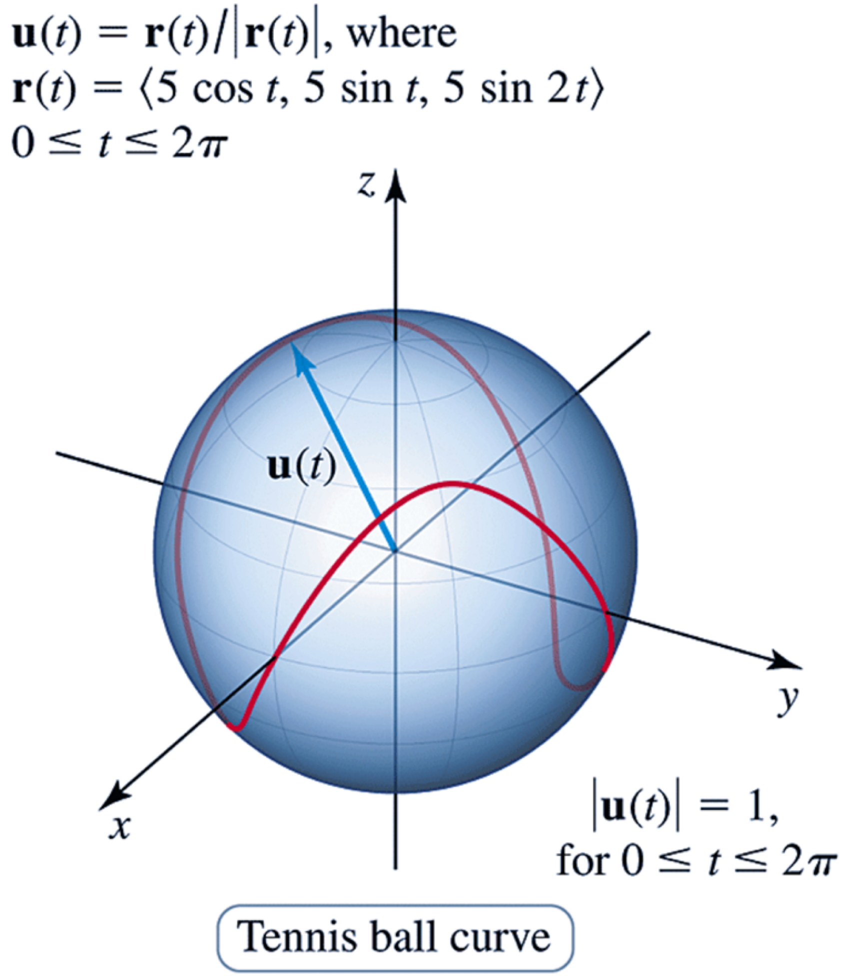


$$|\vec{r}|^2 = 5^2$$

$$\vec{u}(t) = \frac{\vec{r}(t)}{|\vec{r}(t)|}$$

$$|\vec{u}| = 1$$

Figure 14.19



$\mathbf{u}(t) = \mathbf{r}(t)/|\mathbf{r}(t)|$, where
$\mathbf{r}(t) = \langle 5 \cos t, 5 \sin t, 5 \sin 2t \rangle$
$0 \le t \le 2\pi$
z
$\mathbf{u}(t)$
y
x
$|\mathbf{u}(t)| = 1$,
for $0 \le t \le 2\pi$
Tennis ball curve

- Two-dimensional motion in a gravitational field

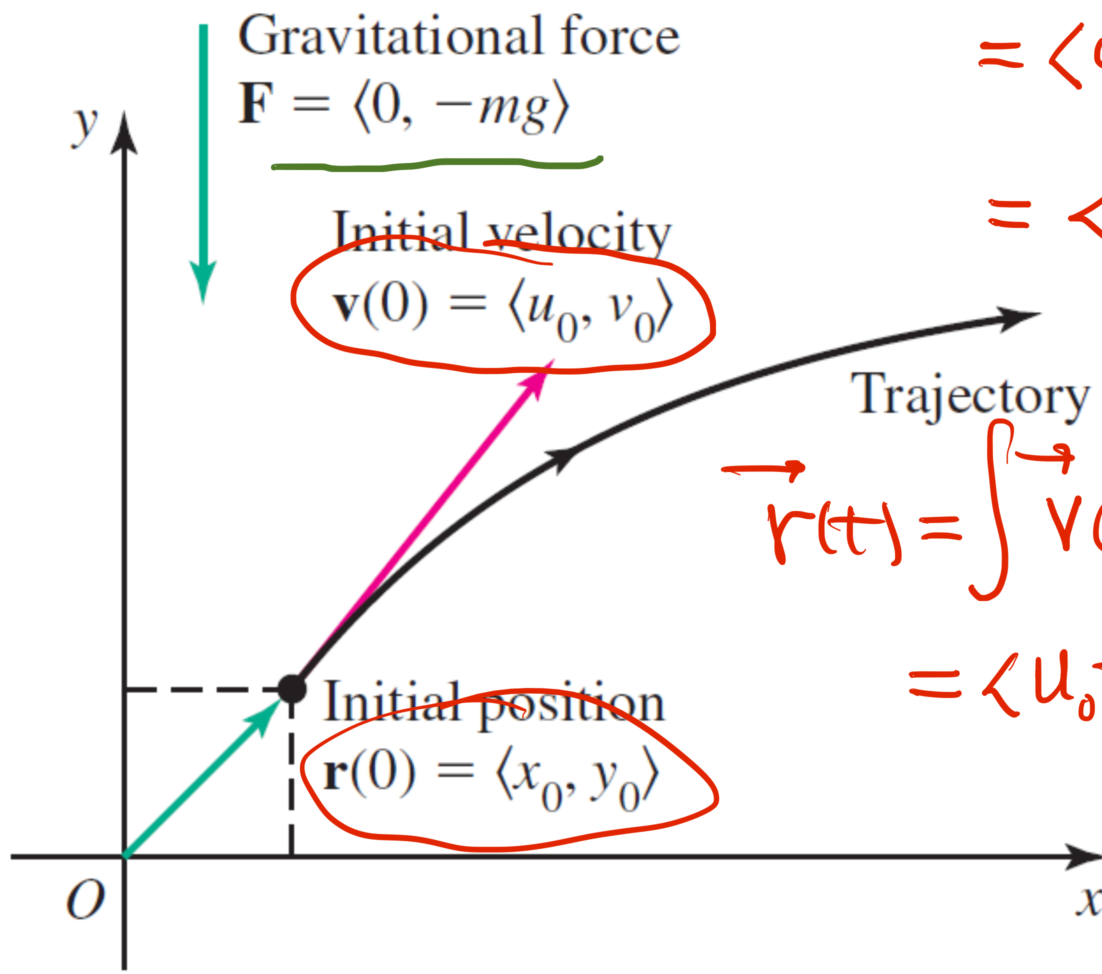
Newton's second law $\boxed{m\vec{a} = \vec{F} = \langle 0, -mg \rangle} = m\langle 0, -g \rangle$

Figure 14.20

$$\Rightarrow \vec{a} = \langle 0, -g \rangle, \quad \vec{v}(t) = \int \vec{a} + \vec{v}(0)$$

$$= \langle 0, -gt \rangle + \vec{v}(0)$$

$$= \langle u_0, -gt + v_0 \rangle$$



$$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{r}(0)$$

$$= \langle u_0 t + x_0, \; -\frac{1}{2}gt^2 + v_0 t + y_0 \rangle$$

Summary Two-Dimensional Motion in a Gravitational Field

Consider an object moving in a plane with a horizontal *x*-axis and a vertical *y*-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and the initial position $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, the velocity of the object, for $t \geq 0$, is

$$\vec{a} = \langle 0, -g \rangle \Rightarrow \mathbf{v}(t) = \langle x'(t),\ y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

and the position is

$$\Rightarrow \mathbf{r}(t) = \langle x(t),\ y(t) \rangle = \left\langle u_0 t + x_0, -\frac{1}{2} g t^2 + v_0 t + y_0 \right\rangle.$$

Example 4 A baseball is hit from 3 ft above home plate with an initial velocity in ft/s of $\vec{v}(0)=\langle u_0, v_0\rangle=\langle 80,80\rangle$. Neglect all forces other than gravity.

(a) Find the position and velocity of the ball between the time it is hit and the time it first hits the ground.

(b) Show that trajectory of the ball is a segment of a parabola.

(c) How far does the ball travel horizontally? (d) What is the maximum height of the ball?

(e) Does the ball clear a 20-ft fence that is 380 ft from home plate?

Figure 14.21

Given $\vec{r}(0)=\langle 0,3\rangle$, $\vec{v}(0)=\langle 80,80\rangle$

(a) $\vec{r}(T)=\langle 80T, -\frac{1}{2}gT^2+80T+3\rangle$

$\Rightarrow 0=-16T^2+80T+3$

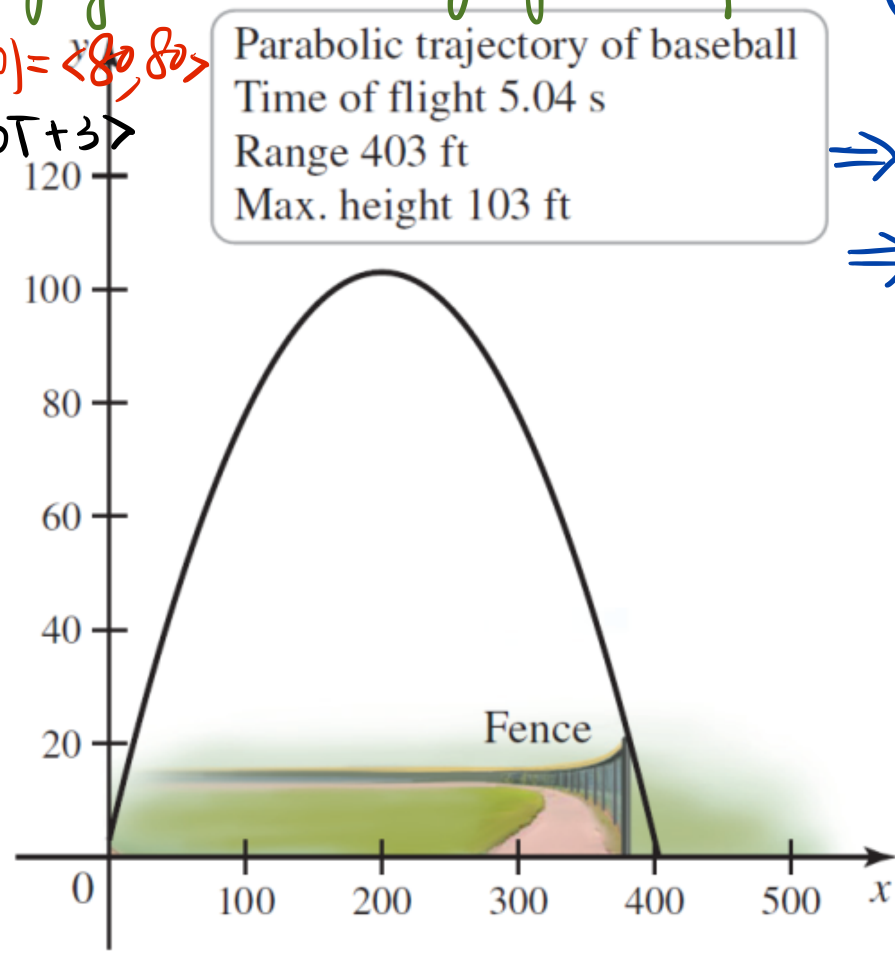
$=-16(T^2-5T)+3$

$=-16\left(T-\frac{5}{2}\right)^2+103$

$\Rightarrow T=\frac{5}{2}+\sqrt{\frac{103}{16}}$

$\vec{r}(T)=$

$\vec{v}(T)=$



(b) $\begin{cases} x=80t \\ y=-16t^2+80t+3 \end{cases}$

$\Rightarrow t=\frac{x}{80}$

$\Rightarrow y=-16\frac{x^2}{80^2}+x-$

Range, Time of Flight, Maximum Height

$\vec{r}_0 = \langle 0, 0 \rangle$

Assume that the motion of an object begins at the origin, and that it is launched at an angle of $\alpha \in [0, \frac{\pi}{2}]$ above the horizontal with an initial speed $|\vec{v}_0|$

Figure 14.22

- Time of Flight

$$\boxed{y(T) = 0}$$

$$= -\frac{1}{2} g T^2 + |\vec{v}_0| \sin\alpha \, T + 0$$

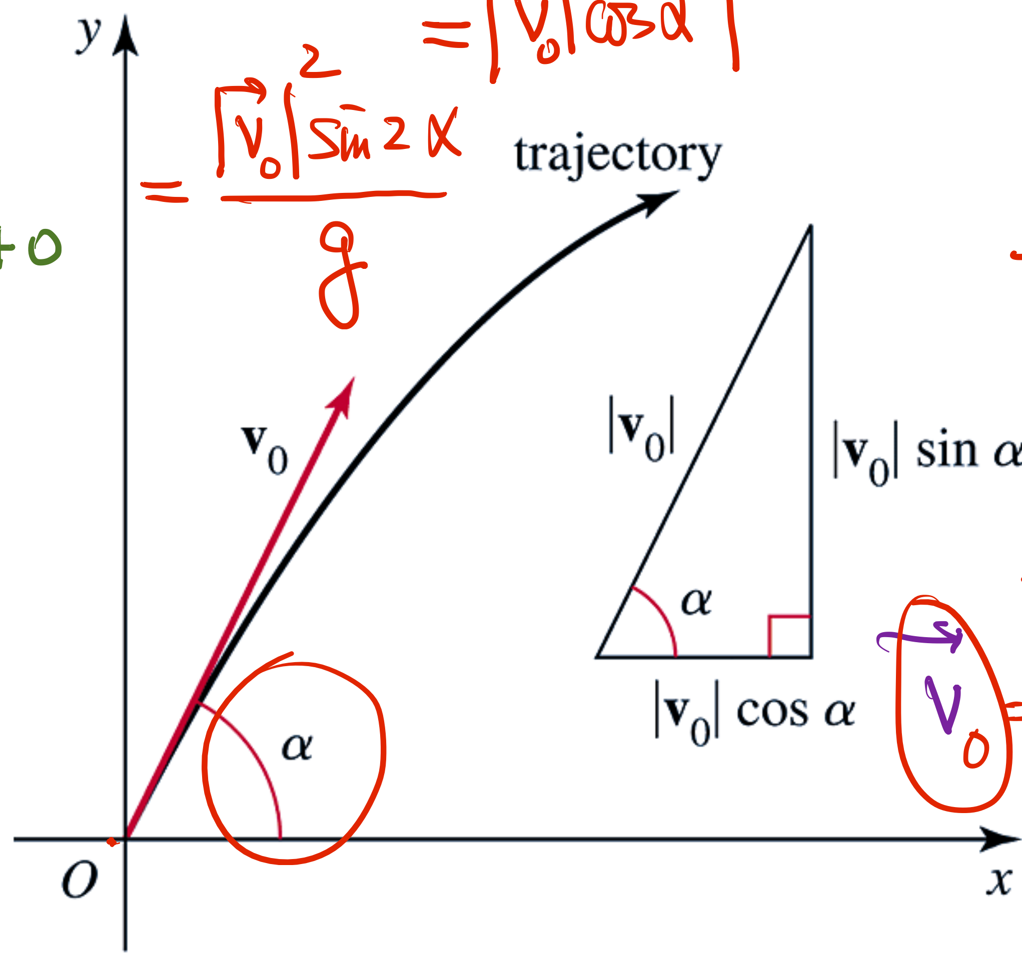
$$\Rightarrow T = \frac{2|\vec{v}_0| \sin\alpha}{g}$$

- Range

$$\boxed{\text{Range} = x(T)}$$

$$= |\vec{v}_0| \cos\alpha \, T$$

$$= \frac{|\vec{v}_0|^2 \sin 2\alpha}{g}$$



- Maximum Height

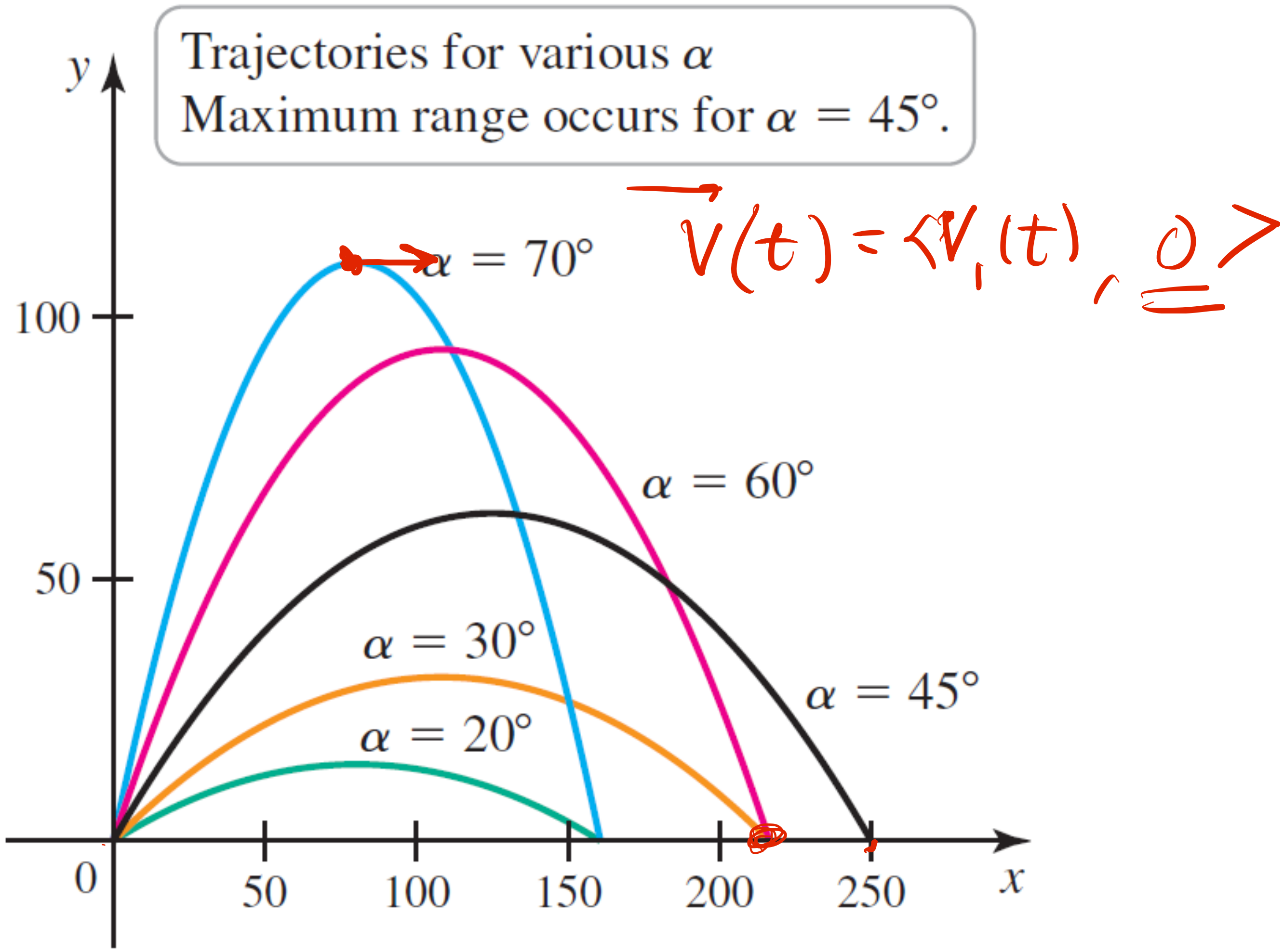
$$-gt + |\vec{v}_0| \sin\alpha = 0$$

$$t = \frac{|\vec{v}_0| \sin\alpha}{g} = \frac{T}{2}$$

$$y\left(\frac{T}{2}\right) = -g\frac{T}{2} + |\vec{v}_0| \sin\alpha$$

$$\vec{v}_0 = \langle |\vec{v}_0| \cos\alpha, |\vec{v}_0| \sin\alpha \rangle$$

Figure 14.23



Trajectories for various α
Maximum range occurs for $\alpha = 45°$.
$\vec{V}(t) = \langle V_1(t), 0 \rangle$
$\alpha = 70°$
$\alpha = 60°$
$\alpha = 45°$
$\alpha = 30°$
$\alpha = 20°$
y
x
100
50
0
50
100
150
200
250

Summary Two-Dimensional Motion

Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ and initial velocity $\langle u_0,\ v_0 \rangle = \langle |\mathbf{v}_0| \cos a, |\mathbf{v}_0| \sin a \rangle$. The trajectory, which is a segment of a parabola, has the following properties.

$$\text{time of flight } = T = \frac{2|\mathbf{v}_0|\sin\alpha}{g}$$

$$\text{range } = \frac{|\mathbf{v}_0|^2 \sin 2\alpha}{g}$$

$$\text{maximum height } = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0|\sin\alpha)^2}{g}$$

motion in gravitational field

$\vec{r}(t)$, $\vec{v}(t) = \vec{r}\,'(t)$, $\vec{a} = \vec{v}\,'(t)$

Figure 14.24

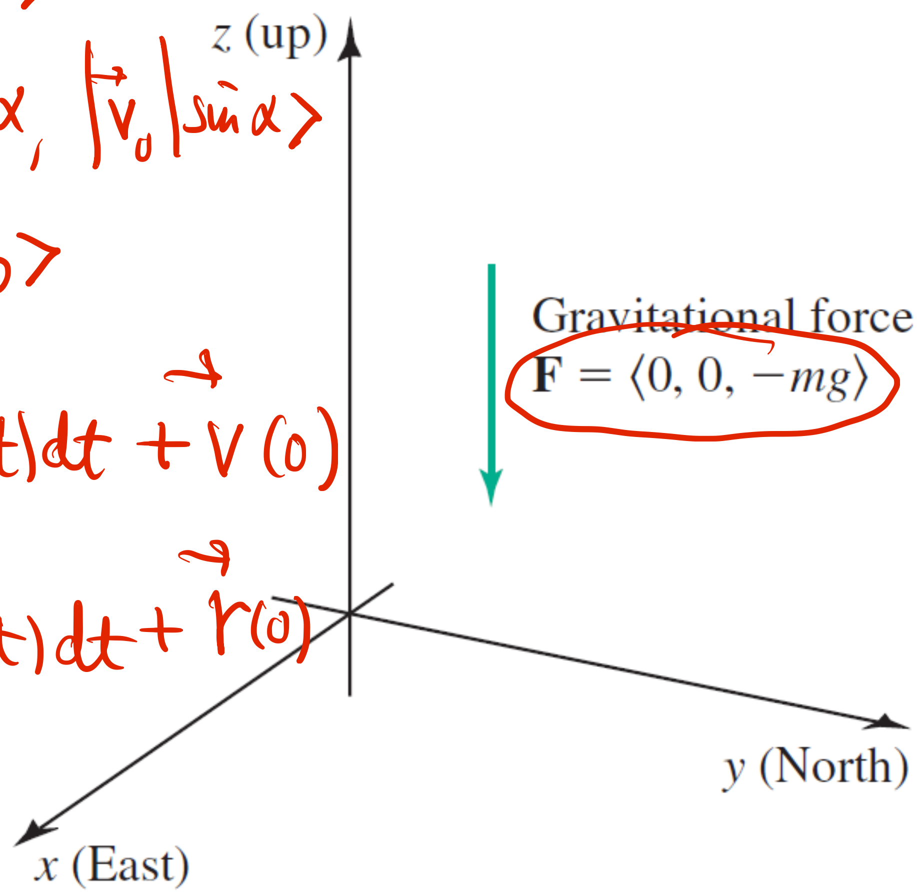
$$\vec{a}(t) = \langle 0, -g \rangle$$

$$\vec{v}(0) = \langle |\vec{v}_0| \cos\alpha, |\vec{v}_0| \sin\alpha \rangle$$

$$\vec{r}(0) = \langle x_0, y_0 \rangle$$

$$\Rightarrow \vec{v}(t) = \int \vec{a}(t)\,dt + \vec{v}(0)$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{r}(0)$$



Example 6 A small projectile is fired to the east over horizontal ground with an initial speed of $|\vec{v}_0| = 300$ m/s at an angle of $\alpha = 30°$ above the horizontal. A crosswind blows from south to north, producing an acceleration of the projectile of 0.36 m/s² to the north.

$\vec{a}(t) = \langle 0, 0.36, -9.8 \rangle$

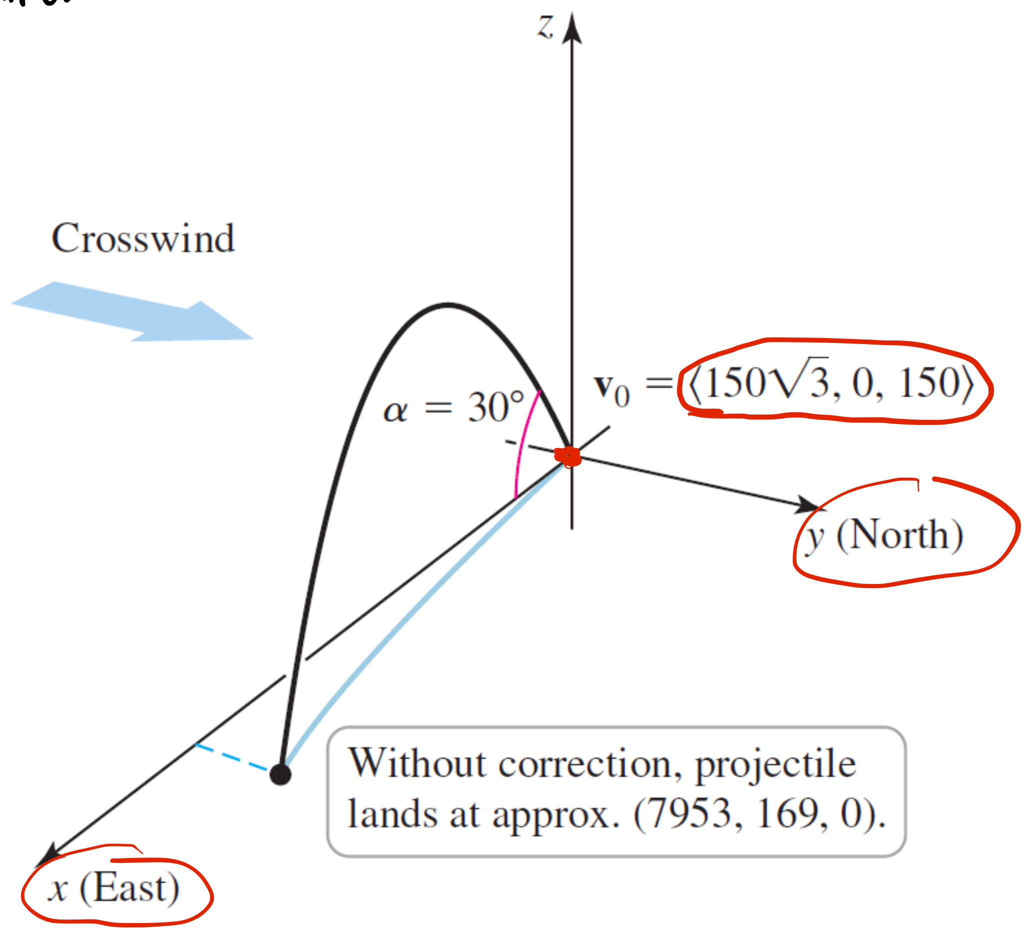
$\vec{v}(0) = \langle |\vec{v}_0| \cos\frac{\pi}{6}, 0, |\vec{v}_0| \sin\frac{\pi}{6} \rangle$

$\vec{r}(0) = \langle 0, 0, 0 \rangle$

Figure 14.25 (a)

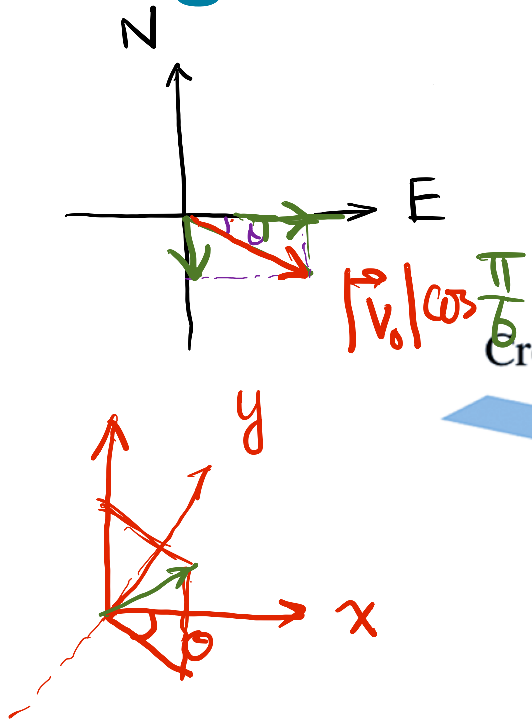
(a) Where does the projectile land?
How far does it land from its lauch site?

$\vec{r}(t) =$



(b) In order to correct for the crosswind and make the projectile land due east of the lauch site, at what angle from due east must it be fired?

Figure 14.25 (b)



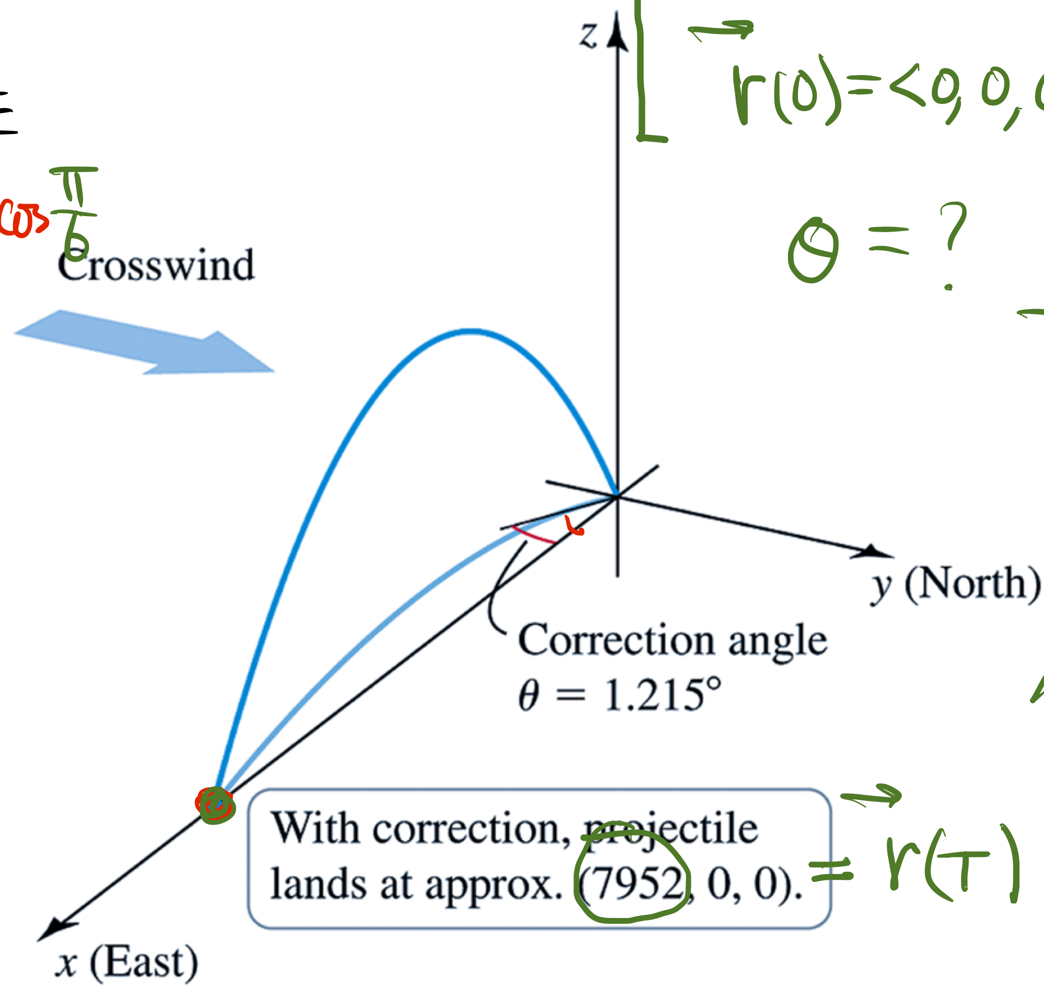
$\vec{a}(t) = \langle 0, 0.36, -9.8 \rangle$

$\vec{v}(0) = \langle |\vec{v}_0|\cos\frac{\pi}{6}\cos\theta, |\vec{v}_0|\cos\frac{\pi}{6}\sin\theta, |\vec{v}_0|\sin\frac{\pi}{6} \rangle$

$\vec{r}(0) = \langle 0, 0, 0 \rangle$

$\theta = ?$

$\vec{r}(t) =$

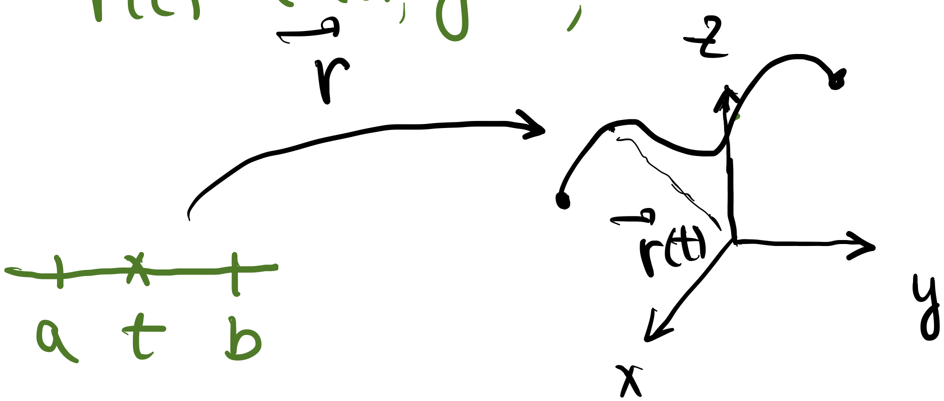


$= \vec{r}(T)$

Section 14.4 Length of Curves

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad t \in [a, b]$

$L = \int_a^b \left| \vec{r}\,'(t) \right| dt$

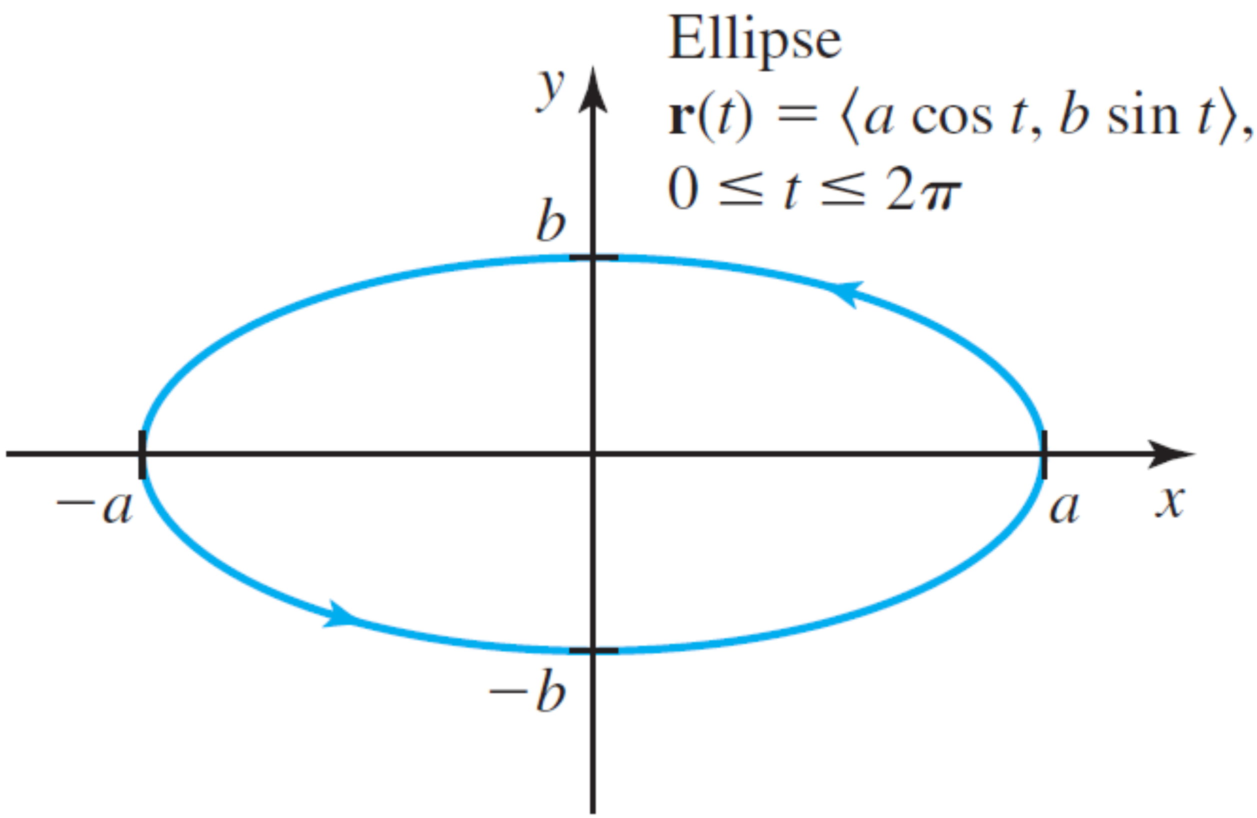


Definition Arc Length for Vector Functions

Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \le t \le b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Figure 14.26



Ellipse
$\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle,$
$0 \le t \le 2\pi$
y
b
−a
a
x
−b

Table 14.1

Planet	Semimajor axis, *a* (AU)	Semiminor axis, *b* (AU)	$\alpha = b / a$	Orbit length (AU)
Mercury	0.387	0.397	0.979	2.407
Venus	0.723	0.723	1.000	4.543
Earth	1.000	0.999	0.999	6.280
Mars	1.524	1.517	0.995	9.554
Jupiter	5.203	5.179	0.995	32.616
Saturn	9.539	9.524	0.998	59.888
Uranus	19.182	19.161	0.999	120.458
Neptune	30.058	30.057	1.000	188.857

$\vec{r}\,'(t) = \langle -250\sin t,\ 250\cos t,\ 100 \rangle$

Figure 14.27

Example 2 An eagle rises at a rate of 100 vertical ft/min on a helical path given by

$\vec{r}(t) = \langle 250\cos t, 250\sin t, 100t \rangle$

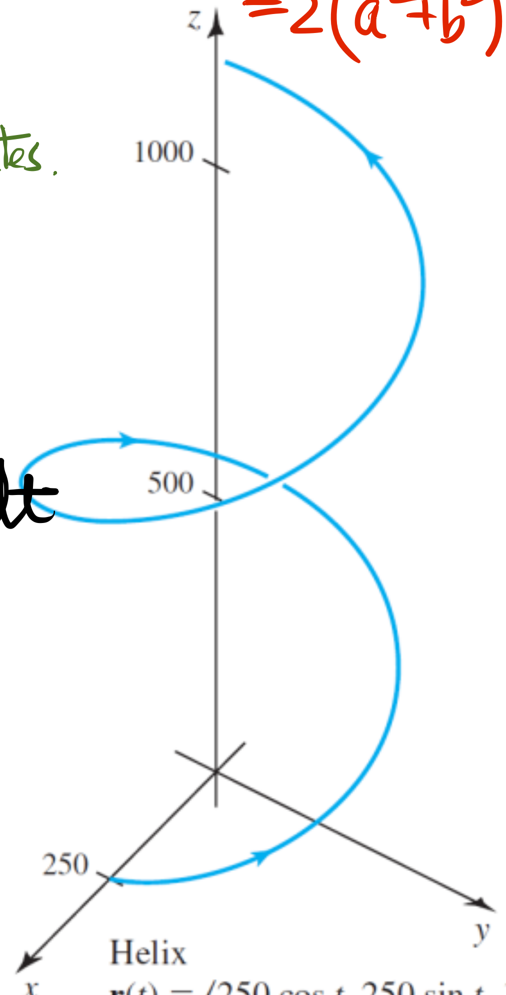
where $\vec{r}$ is measured in feet and t in minutes. How far does it travel in 10 min?

$$L = \int_0^{10} |\vec{r}\,'(t)|\,dt$$

$$= \int_0^{10} \sqrt{250^2(\sin^2 t + \cos^2 t) + 100^2}\,dt$$

$$= \sqrt{250^2 + 100^2} \int_0^{10} dt$$

$$= 10\sqrt{250^2 + 100^2}$$



Helix
$\mathbf{r}(t) = \langle 250\cos t, 250\sin t, 100t \rangle$

$$(a-b)^2 + (a+b)^2 = a^2 - 2ab + b^2$$
$$+ a^2 + 2ab + b^2$$
$$2a^2 + 2b^2$$
$$= 2(a^2 + b^2)$$

#14 $\vec{r} = \langle \cos t + \sin t, \cos t - \sin t \rangle$

$\vec{r}\,' = \langle -\sin t + \cos t, -\sin t - \cos t \rangle$

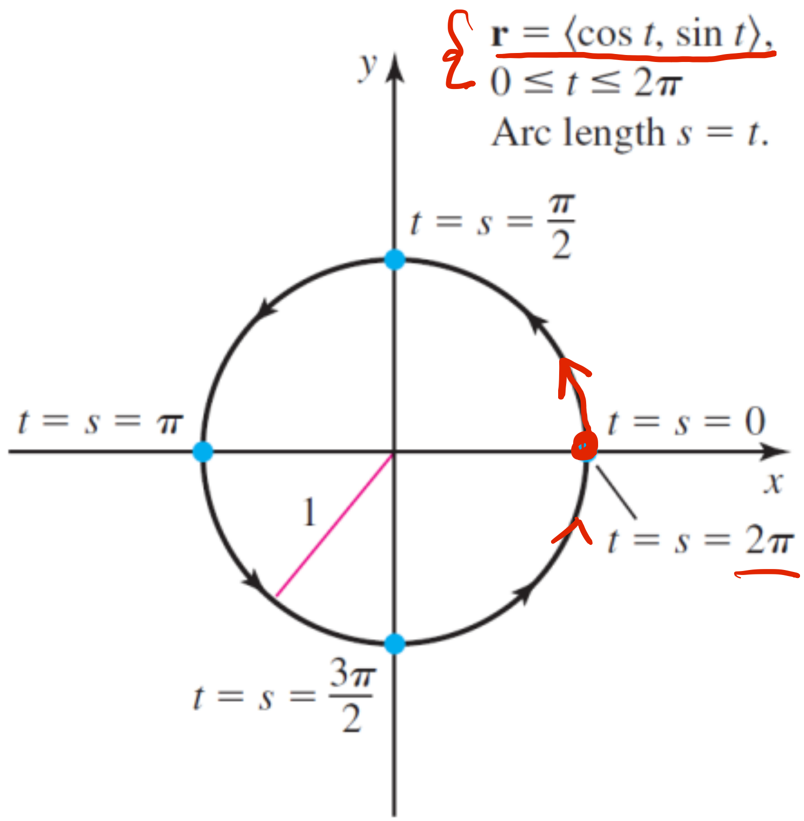
$$L = \int_c^d \sqrt{(-\sin t + \cos t)^2 + (\sin t + \cos t)^2}\,dt$$

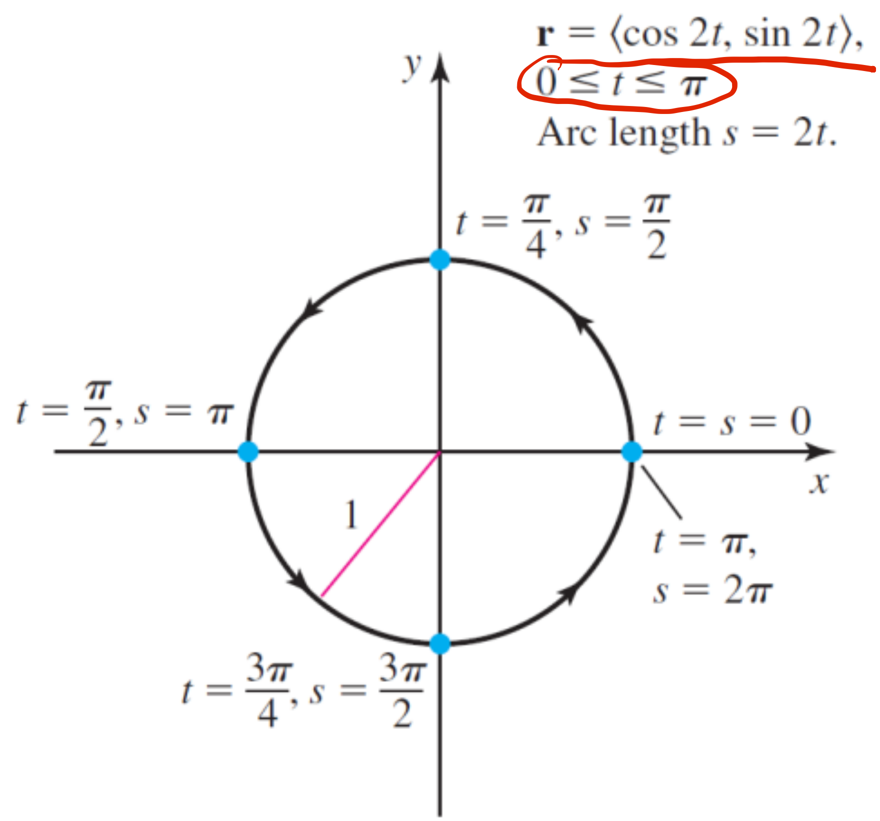
$$= \int_c^d \sqrt{2(\cos^2 t + \sin^2 t)}\,dt$$

$$= \sqrt{2}\,(d-c)$$

• parametrization of the unit circle $C: x^2+y^2=1$

Figure 14.28 (a & b)





Theorem 14.3 Arc Length as a Function of a Parameter

Let $\mathbf{r}(t)$ describe a smooth curve, for $t \geq a$. The arc length is given by

$\frac{d}{dt}\int_{a(t)}^{b(t)} f(u)\,du$

$= f(b(t))\cdot b'(t) - f(a(t))\cdot a'(t)$

$$s(t) = \int_a^t |\mathbf{v}(u)|\,du,$$

$\frac{d}{dt}s(t) = |\vec{v}(t)|\cdot 1 - |\vec{v}(a)|\cdot 0$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\frac{ds}{dt} = |\mathbf{v}(t)|$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

Example 3 (arc length parametrization)

$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4t \rangle, \ t \geq 0$

(a) $s(t) = \int_0^t |\vec{r}\,'(t)|\,dt$

$= \int_0^t \sqrt{4\sin^2 t + 4\cos^2 t + 4^2}\,dt$

$= \int_0^t \sqrt{20}\,dt = 2\sqrt{5}\,t$

$s = 2\sqrt{5}\,t$

$\Downarrow$

$t = \frac{1}{2\sqrt{5}}s$

(b) Use arc length to parametrize the same curve.

$\vec{R}(s) = \vec{r}(t) = \vec{r}\left(\frac{1}{2\sqrt{5}}s\right)$

$= \langle 2\cos\frac{s}{2\sqrt{5}}, 2\sin\frac{s}{2\sqrt{5}}, \frac{2s}{\sqrt{5}} \rangle$

Section 14.5 Curvature and ~~Normal Vectors~~

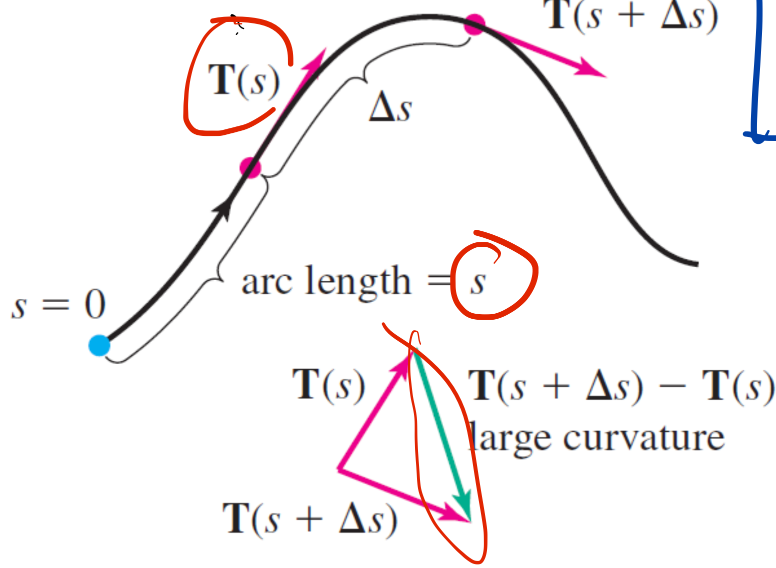
$\vec{r}(t) \longrightarrow \vec{R}(s) \qquad \frac{\vec{R}'(s)}{|\vec{R}'(s)|}$

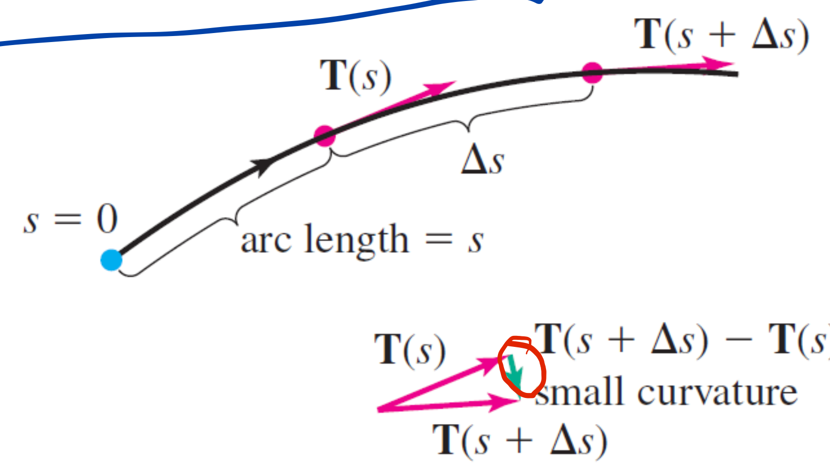
Figure 14.29

$$\kappa = \left|\frac{d\vec{T}(t)}{ds}\right| = \left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right|$$

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\vec{r}'(t)|}$$

$$\boxed{\kappa = \frac{1}{|\vec{r}'(t)|}\left|\frac{d\vec{T}}{dt}\right|}$$





Definition Curvature

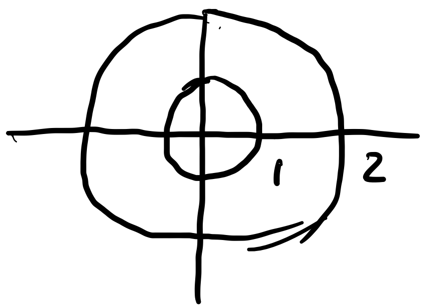
Let $\boldsymbol{r}$ describe a smooth parameterized curve. If s denotes arc length and

$$\boldsymbol{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$$

is the unit tangent vector, the **curvature** is $\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|.$

Theorem 14.4 Curvature Formula

Let $\mathbf{r}(t)$ describe a smooth parameterized curve, where *t* is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and **T** is the unit tangent vector, then the curvature is



$$\kappa(t) = \frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right| = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|}.$$

#14 $\vec{r}(t) = \langle \cos t^2, \sin t^2 \rangle$, $x^2 + y^2 = 1$

$\vec{r}\,' = \langle -2t \sin t^2, 2t \cos t^2 \rangle$

$|\vec{r}\,'| = \sqrt{4t^2} = 2t$, $\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \langle -\sin t^2, \cos t^2 \rangle$

$\vec{T}\,' = -2t \langle \cos t^2, \sin t^2 \rangle$

$|\vec{T}\,'| = 2t \Rightarrow \kappa = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{2t}{2t} = 1$

Compute $\vec{T}$ and κ

#12 $\vec{r}(t) = \langle 2\cos t, -2\sin t \rangle$

$x^2 + y^2 = (2\cos t)^2 + (-2\sin t)^2 = 2^2$

$\vec{r}\,' = 2\langle -\sin t, -\cos t \rangle$

$|\vec{r}\,'| = 2$, $\vec{T}(t) = \frac{-2\langle \sin t, \cos t \rangle}{2}$

$= -\langle \sin t, \cos t \rangle$

$\vec{T}\,'(t) = -\langle \cos t, -\sin t \rangle$

$|\vec{T}\,'(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\kappa = \frac{1}{2}$

Theorem 14.5 Alternative Curvature Formula

Let **r** be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

#22 $\vec{r}(t) = \langle 4t, 3\sin t, 3\cos t \rangle$

$\vec{v}(t) = \langle 4, 3\cos t, -3\sin t \rangle$

$\vec{a}(t) = \langle 0, -3\sin t, -3\cos t \rangle$

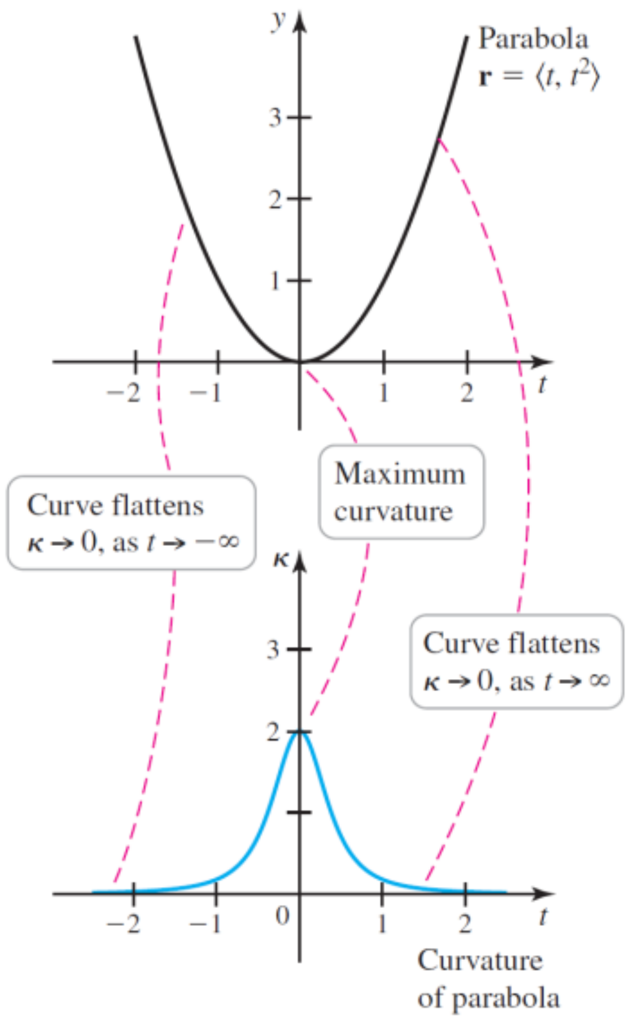
$\vec{v}(t) \times \vec{a}(t) = \langle -9, 12\cos t, -12\sin t \rangle$

$|\vec{v} \times \vec{a}| = \sqrt{81 + 12^2}$

$|\vec{v}| = \sqrt{4^2 + 3^2} = 5$

$K = \frac{\sqrt{81+12^2}}{5^3} = \frac{15}{5^3} = \frac{3}{25}$

Figure 14.30



Parabola
$\mathbf{r} = \langle t, t^2 \rangle$
Maximum curvature
Curve flattens
$\kappa \to 0$, as $t \to -\infty$
Curve flattens
$\kappa \to 0$, as $t \to \infty$
Curvature of parabola

Definition Principal Unit Normal Vector

Let ***r*** describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point *P* on the curve at which $k \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

For other parameters, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of *t* corresponding to *P*.

Theorem 14.6 Properties of the Principal Unit Normal Vector

Let **r** describe a smooth parameterized with unit tangent vector **T** and principal unit normal vector **N**.

1. **T** and **N** are orthogonal at all points of the curve; that is, $\mathbf{T} \cdot \mathbf{N} = 0$

   at all points where **N** is defined.

2. The principal unit normal vector points to the inside of the curve–in the direction that the curve is turning.

Figure 14.31

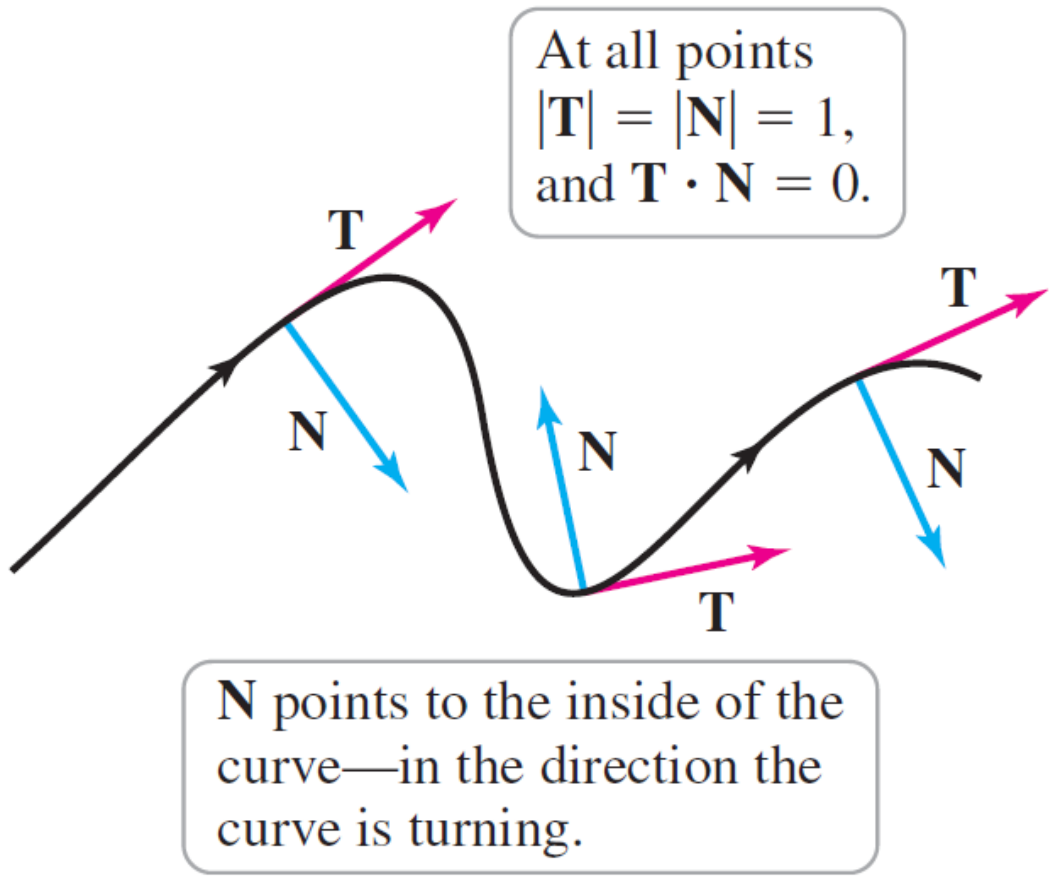

At all points
$|\mathbf{T}| = |\mathbf{N}| = 1$,
and $\mathbf{T} \cdot \mathbf{N} = 0$.
T
N
N
T
T
N
N points to the inside of the curve—in the direction the curve is turning.

Figure 14.32

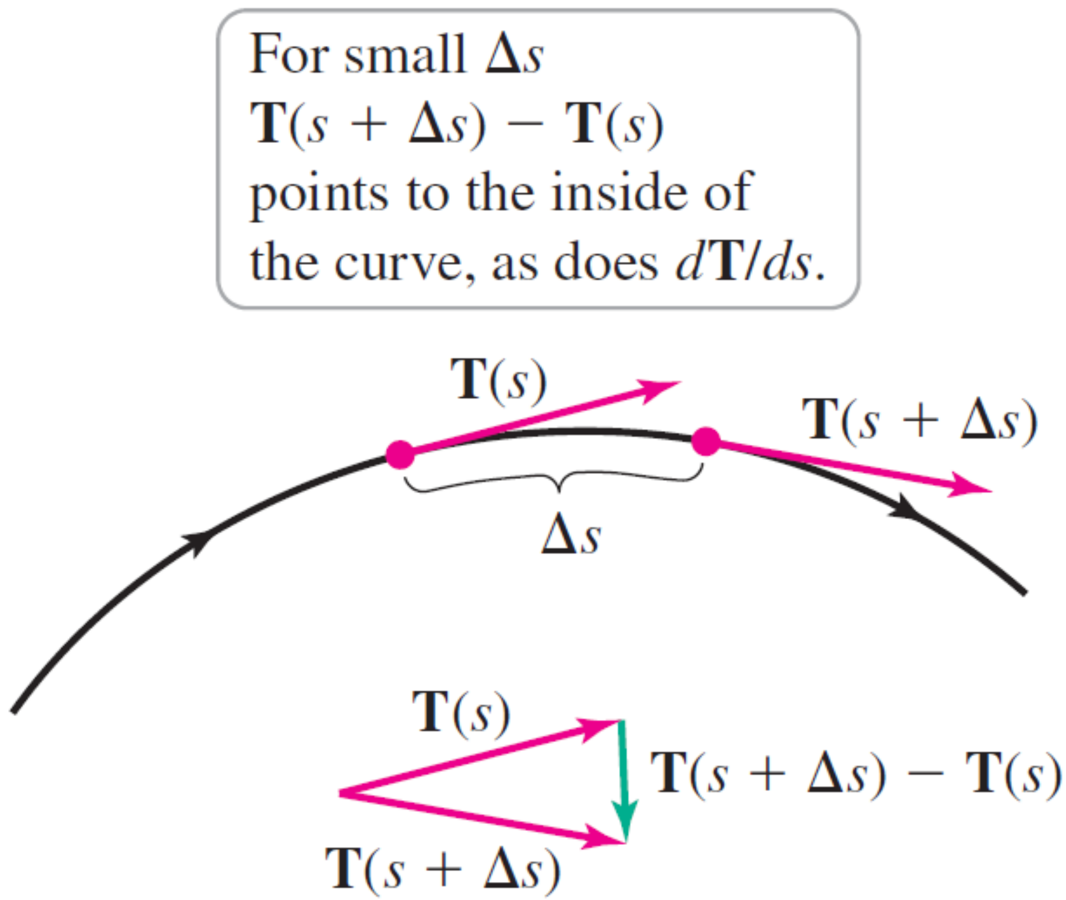
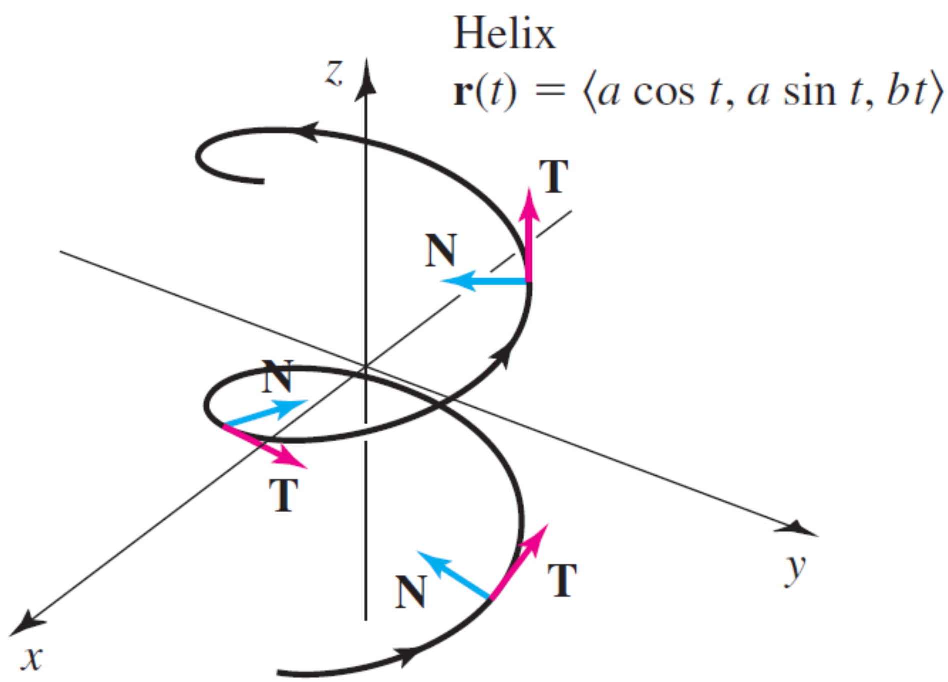

For small Δs
$\mathbf{T}(s + \Delta s) - \mathbf{T}(s)$
points to the inside of
the curve, as does $d\mathbf{T}/ds$.
$\mathbf{T}(s)$
$\mathbf{T}(s + \Delta s)$
Δs
$\mathbf{T}(s)$
$\mathbf{T}(s + \Delta s) - \mathbf{T}(s)$
$\mathbf{T}(s + \Delta s)$

Figure 14.33



$\mathbf{T} \cdot \mathbf{N} = 0$ at all points of the curve.
$\mathbf{T}$ points in the direction of the curve.
$\mathbf{N}$ points to the inside of the curve.

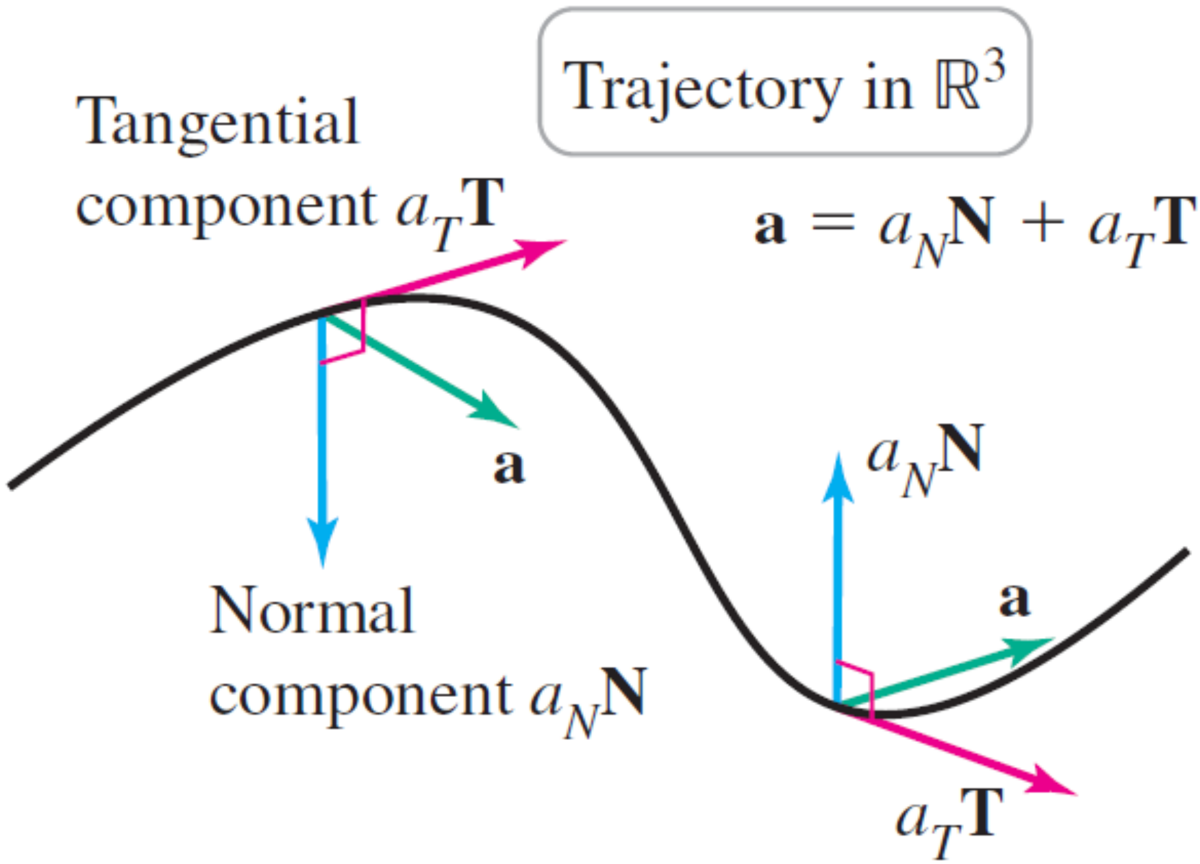
Theorem 14.7 Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of **T**) and its **normal component** a_N (in the direction of **N**):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T},$$

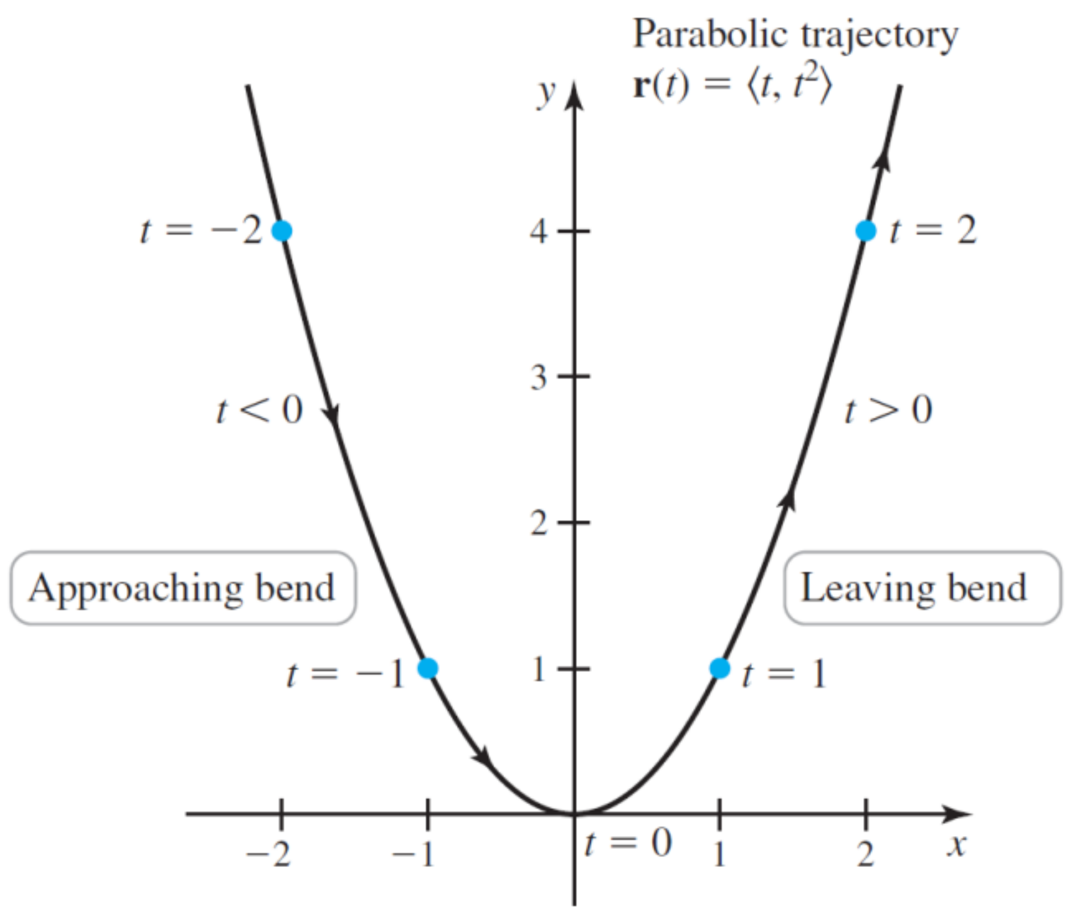
where $a_N = k\,|\mathbf{v}|^2 = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2s}{dt^2}.$

Figure 14.34



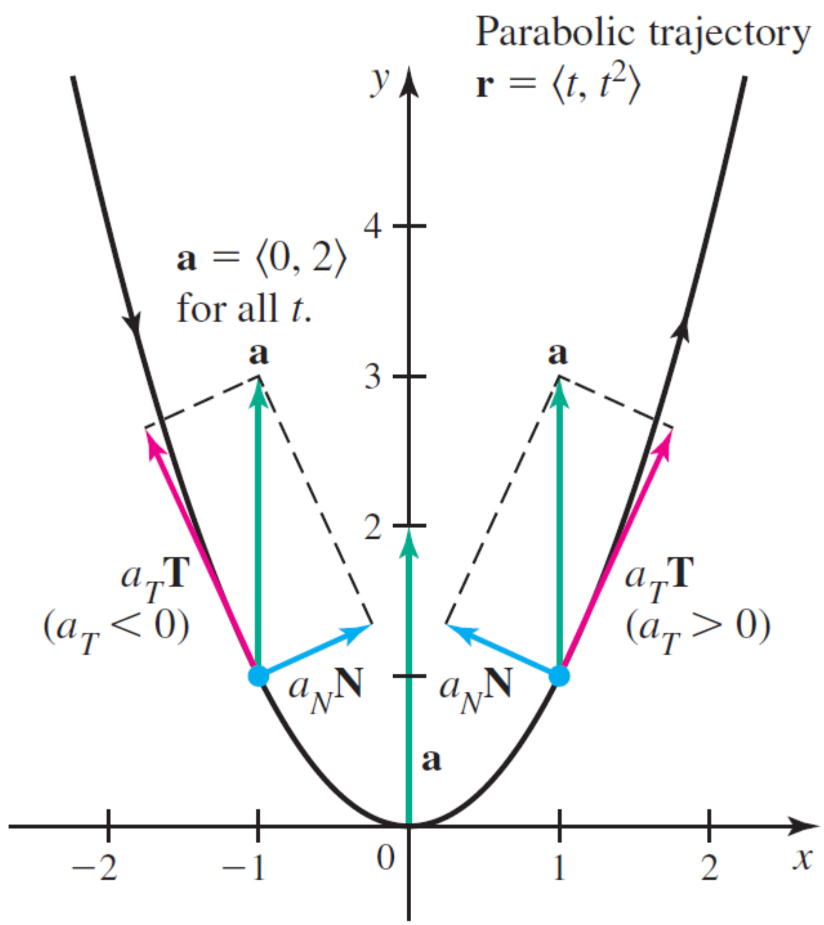
Trajectory in $\mathbb{R}^3$
Tangential component $a_T\mathbf{T}$
$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$
$\mathbf{a}$
$a_N\mathbf{N}$
$\mathbf{a}$
Normal component $a_N\mathbf{N}$
$a_T\mathbf{T}$

Figure 14.35



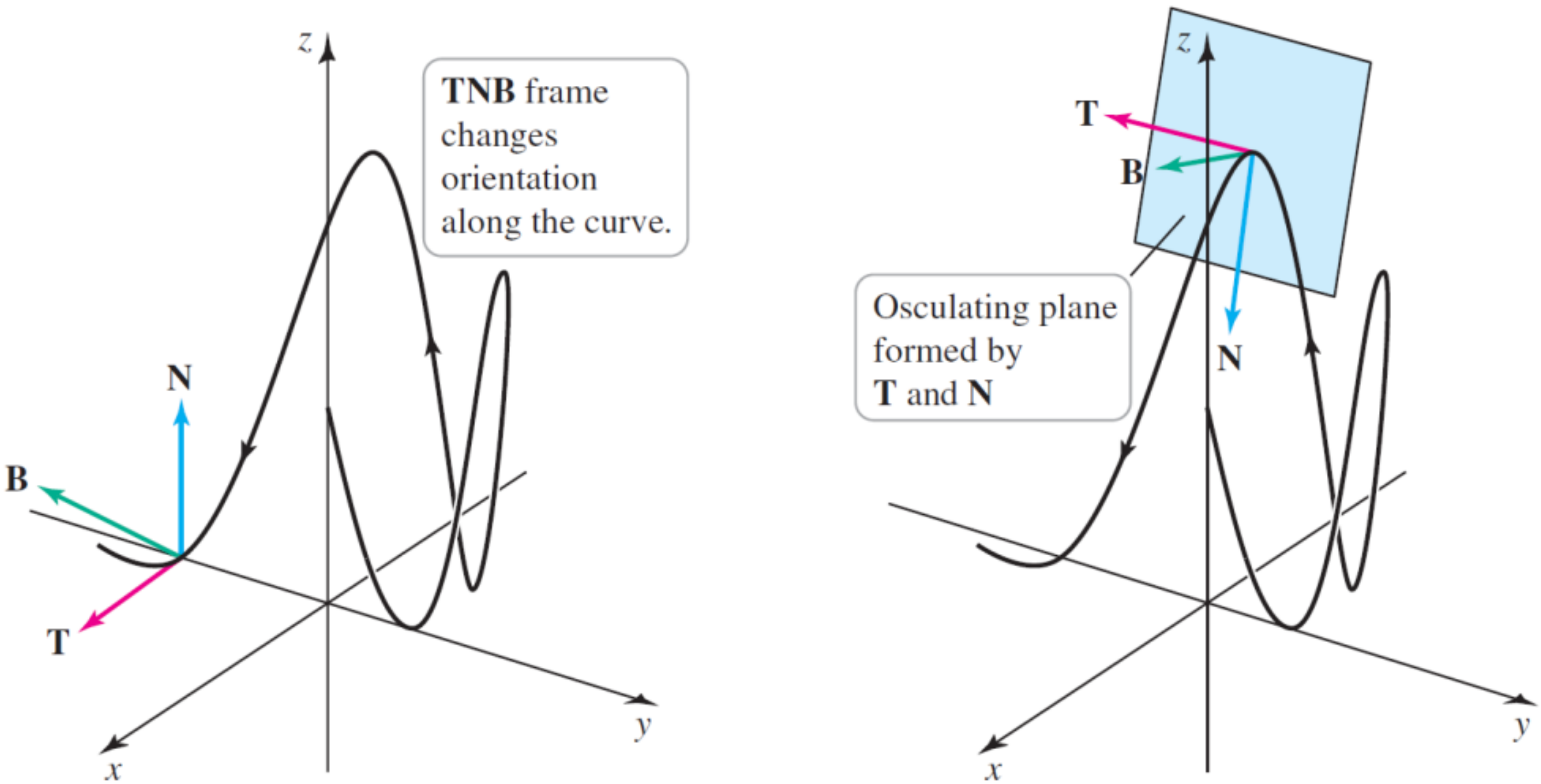
Parabolic trajectory
$\mathbf{r}(t) = \langle t, t^2 \rangle$
y
$t = -2$
$t = 2$
$t < 0$
$t > 0$
Approaching bend
Leaving bend
$t = -1$
$t = 1$
$t = 0$
4
3
2
1
−2
−1
1
2
x

Figure 14.36



Parabolic trajectory
$\mathbf{r} = \langle t, t^2 \rangle$
$\mathbf{a} = \langle 0, 2 \rangle$
for all t.
$\mathbf{a}$
$\mathbf{a}$
$a_T\mathbf{T}$
$(a_T < 0)$
$a_T\mathbf{T}$
$(a_T > 0)$
$a_N\mathbf{N}$
$a_N\mathbf{N}$
$\mathbf{a}$
x
y
−2
−1
0
1
2
2
3
4

Figure 14.37 (a & b)



Definition Unit Binormal Vector and Torsion

Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors **T** and **N**, respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binormal vector** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$

Figure 14.38

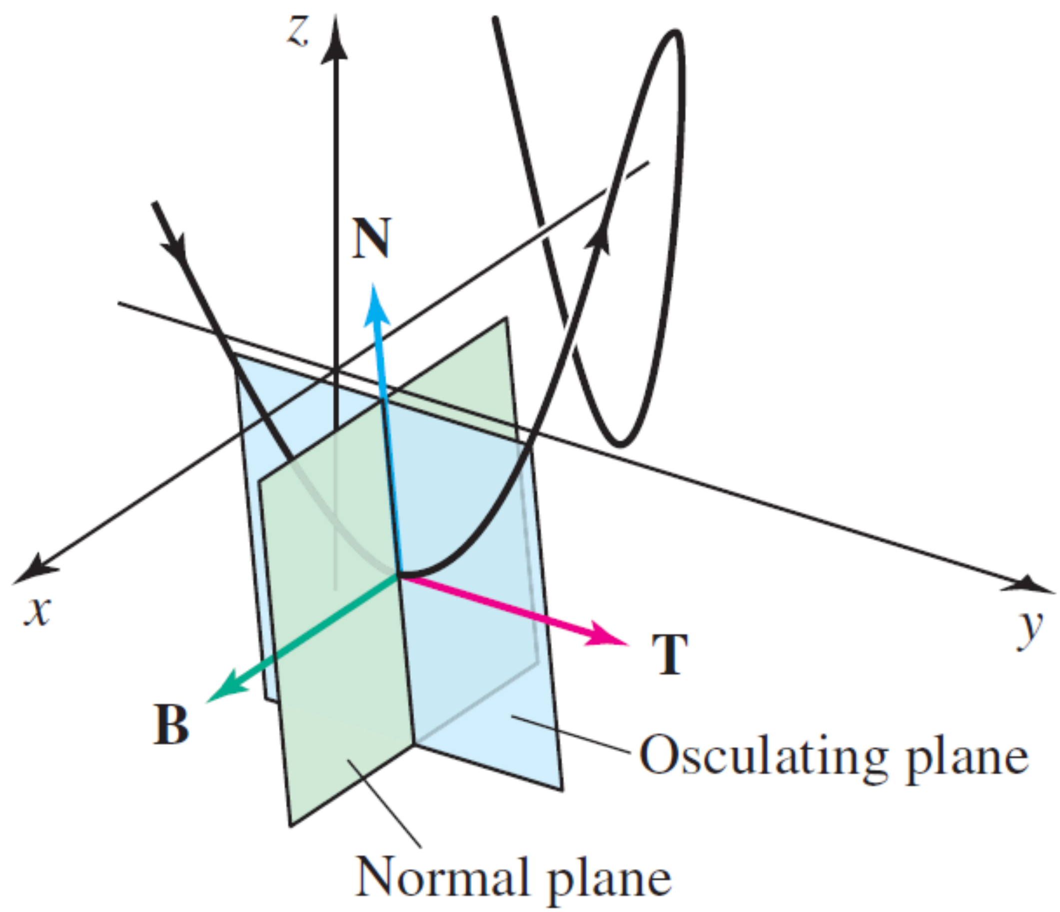
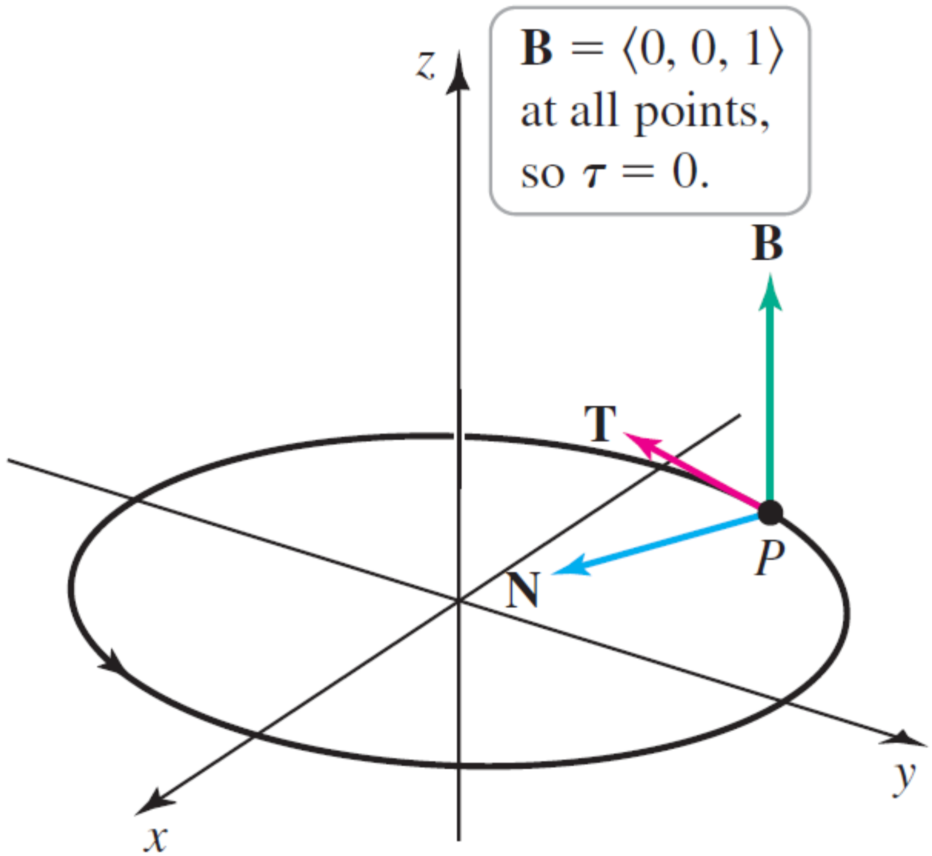
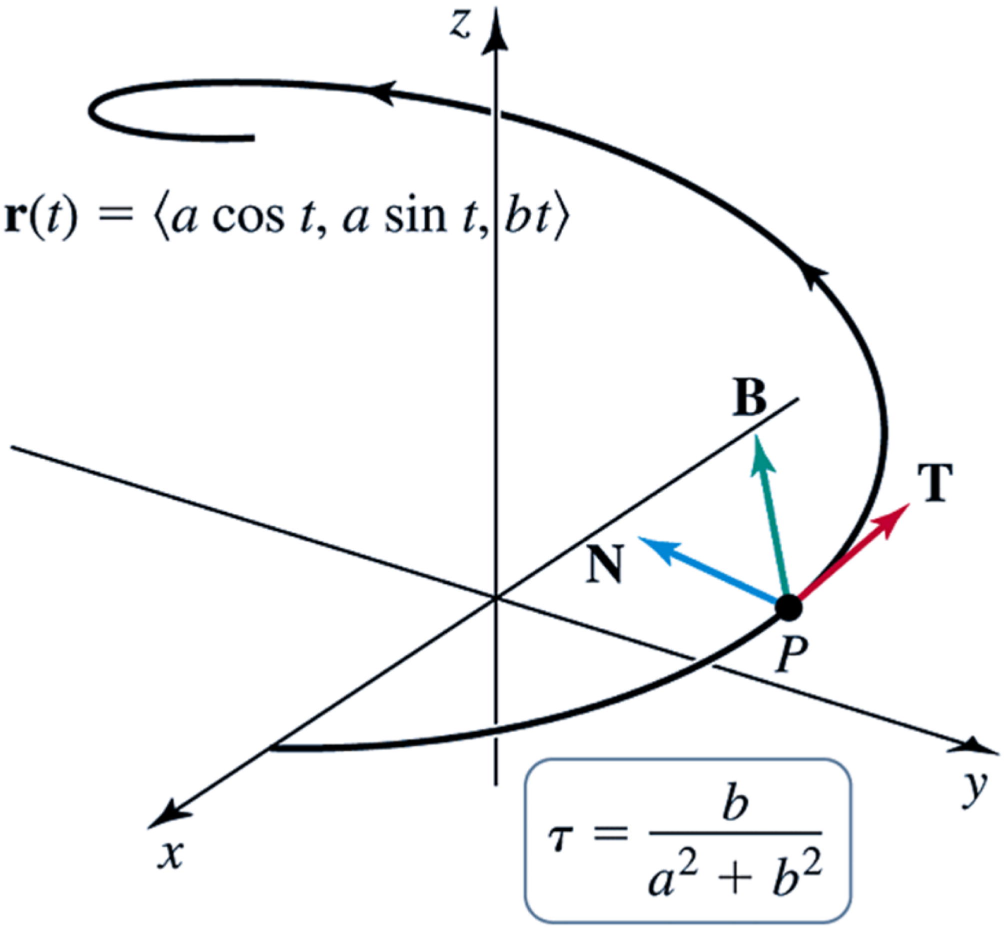


Figure 14.39



C: $\mathbf{r}(t) = \langle R \cos t, R \sin t \rangle$

Figure 14.40


z
$\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle$
B
T
N
P
y
x
$\tau = \dfrac{b}{a^2 + b^2}$

Summary Formulas for Curves in Space (1 of 2)

Position function: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Velocity: $\mathbf{v} = \mathbf{r}'$

Acceleration: $\mathbf{a} = \mathbf{v}'$

Unit tangent vector: $\boldsymbol{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq 0$)

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Summary Formulas for Curves in Space (2 of 2)

Components of acceleration: $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where $a_N = \kappa\,|\mathbf{v}|^2 = \dfrac{|\mathbf{v}\times a|}{|\mathbf{v}|}$

and $a_T = \dfrac{d^2s}{dt^2} = \dfrac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|}$

Unit binormal vector: $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \dfrac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$

Torsion:

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2}$$