Chapter 15 Functions of Several Variables

§15.1 Graphs and Level Curves

- functions of two variables

$z = f(x,y)$, $\forall (x,y)$, $f(x,y)$ has a single value

domain $D = \{(x,y) \in R^2 \mid f(x,y) \text{ is well defined}\}$

range $R = \{ f(x,y) \in R \mid (x,y) \in D \}$

example find the domain of the following functions

#18. $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2-25}}$; #20. $f(x,y) = \dfrac{12}{y^2-x^2}$

- graph of $z = f(x,y)$

$$\{(x, y, f(x,y)) \mid (x,y) \in D\}$$

example sketch the graph

$z = f(x,y) = 6-3x-2y$; #29. $z = f(x,y) = -\sqrt{9-x^2-y^2}$

- <u>level curves</u> $\{(x,y) \mid f(x,y)=k\}$

<u>examples</u> sketch some level curves

$$z=f(x,y)=6-3x-2y\,; \quad z=h(x,y)=4x^2+y^2+1$$

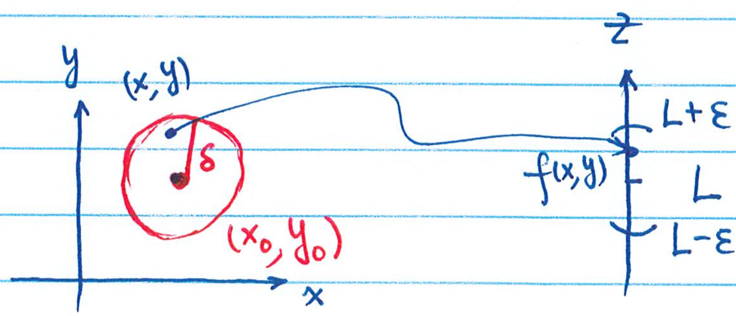
- functions of three variables
  - domain
  - range
  - level surface

<u>example</u> $w=f(x,y,z)=\sqrt{1-x^2-y^2-z^2}$.

§15.2 Limits and Continuity

- <u>Limits</u>

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L \iff \begin{array}{l}\forall \varepsilon>0,\ \exists \delta>0, \text{ s.t.} \\ \text{"}0<|(x,y)-(x_0,y_0)|<\delta \Rightarrow |f(x,y)-L|<\varepsilon\text{"}.\end{array}$$



<u>Limits of Constant and Linear Functions</u> $a, b, c \in \mathbb{R}$

(1) $\lim\limits_{(x,y)\to(a,b)} c = c$; (2) $\lim\limits_{(x,y)\to(a,b)} x = a$; (3) $\lim\limits_{(x,y)\to(a,b)} y = b$.

<u>Limit Laws</u> $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$

$$\lim_{(x,y)\to(a,b)} \left[f(x,y) \pm g(x,y)\right] = L \pm M$$

$$\lim_{(x,y)\to(a,b)} \left[c\, f(x,y)\right] = cL$$

$$\lim_{(x,y)\to(a,b)} f(x,y)\, g(x,y) = LM$$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \text{ provided } M \neq 0.$$

$$\lim_{(x,y)\to(a,b)} \left[f(x,y)\right]^n = L^n ; \quad \lim_{(x,y)\to(a,b)} \left[f(x,y)\right]^{\frac{1}{n}} = L^{\frac{1}{n}}, \quad L > 0 \text{ if } n \text{ is even}$$

<u>examples</u>

<u>#19</u> $\lim\limits_{(x,y)\to(2,0)} \dfrac{x^2 - 3xy^2}{x+y}$, <u>#22</u> $\lim\limits_{(x,y)\to(1,-2)} \dfrac{y^2 + 2xy}{y + 2x}$

<u>#24</u> $\lim\limits_{(x,y)\to(-1,1)} \dfrac{2x^2 - xy - 3y^2}{x+y}$, <u>#27</u> $\lim\limits_{(x,y)\to(1,2)} \dfrac{\sqrt{y} - \sqrt{x+1}}{y - x - 1}$

- $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ DNE $\iff \lim\limits_{\substack{(x,y)\to(a,b)\\ \text{along path } C_1}} f(x,y) = L_1 \neq L_2 = \lim\limits_{\substack{(x,y)\to(a,b)\\ \text{along path } C_2}} f(x,y)$

examples #30 $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy}{3x^2+y^2}$; #33 $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^3+x^3}{xy^2}$

- continuity

$f(x,y)$ is cont. at $(a,b) \iff \lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

examples

#42 $f(x,y) = \begin{cases} \dfrac{y^4-2x^2}{y^4+x^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

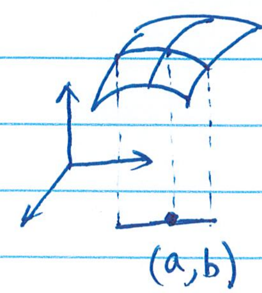
#54 $f(x,y) = \begin{cases} \dfrac{1-\cos(x^2+y^2)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

§15.3 Partial Derivatives

$z = f(x,y)$

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$



examples

#16, 22, 29, 31

- higher-order derivative

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \cdots$$

examples #38, 39

- partial derivative and continuity

$$f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

- $\lim_{(x,y)\to(0,0)} f(x,y) = ?$
- f is not cont. at $(0,0)$
- $\frac{\partial f}{\partial x}(0,0) = ?$ $\frac{\partial f}{\partial y}(0,0) = ?$

Clairaut's Thrm f is defined on a disk $D \ni (a,b)$

f_{xy} and f_{yx} are cont. on $D \Longrightarrow f_{xy}(a,b) = f_{yx}(a,b)$.

#59, 91

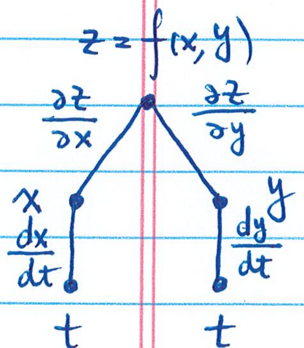
§15.4 The Chain Rule

- function of one variable

$$\begin{matrix} y = f(x) \\ x = g(t) \end{matrix} \Longrightarrow y = f(g(t)) : \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(g(t))\,g'(t)$$

- function of two variables

one indep. variable $z = f(x,y)$ $\begin{cases} x = g(t) \\ y = h(t) \end{cases} \Longrightarrow z = f\big(g(t), h(t)\big)$

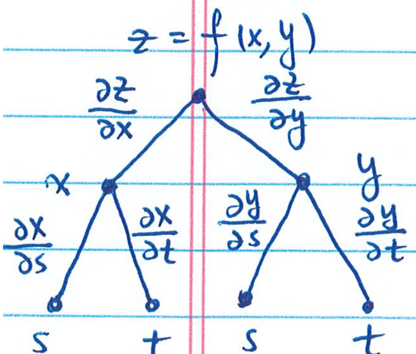


$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= f_x\big(g(t), h(t)\big)\, g'(t) + f_y\big(g(t), h(t)\big)\, h'(t)$$

#9, 13

two indep. variable $z = f(x,y)$ $\begin{cases} x = g(t,s) \\ y = h(t,s) \end{cases} \Longrightarrow z = f\big(g(t,s), h(t,s)\big)$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

#19, 22

implicit differentiation

$$F(x,y)=0 \implies y=f(x)$$

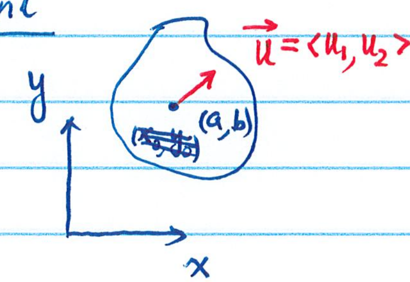
$$\implies F(x,f(x))=0 \implies \frac{d}{dx}F(x,f(x))=0$$

$$0=\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx} \implies \frac{dy}{dx}=-\frac{F_x}{F_y}.$$

#35, 38

§15.5 Directional Derivative and the Gradient

$$z=f(x,y),\quad (x,y)\in D\subset R^2$$

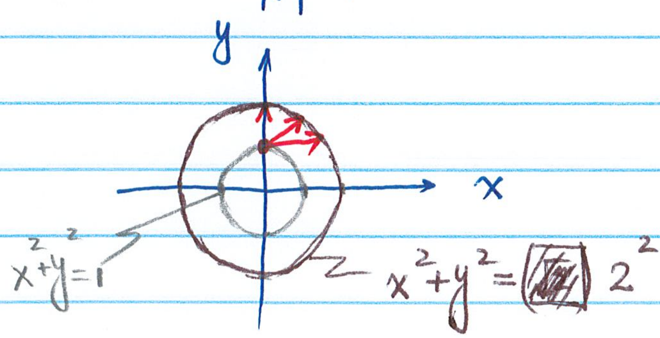


Question In which direction at (a,b) does the value of $f(x,y)$ increase most rapidly?

$\iff$ Find $\vec{u}=\langle u_1,u_2\rangle$ s.t.

$$\frac{d}{dt}f\left((a,b)+t\vec{u}\right)\Big|_{t=0}=\max_{\substack{\vec{v}\in R^2\\|\vec{v}|=1}}\frac{d}{dt}f\left((a,b)+t\vec{v}\right)\Big|_{t=0}$$

example $z=f(x,y)=x^2+y^2$



$$\frac{d}{dt} f\left((a,b)+t\vec{u}\right)\Big|_{t=0} = \max_{\vec{v}\in\mathbb{R}^2, |\vec{v}|=1} \frac{d}{dt} f\left((a,b)+t\vec{v}\right)\Big|_{t=0}$$

$(a,b)+t\vec{v}$
$=(a+tv_1, a+tv_2)$

$$= \max_{|\vec{v}|=1} \left\{ \frac{\partial f}{\partial x}(a,b)\, v_1 + \frac{\partial f}{\partial y}(a,b)\, v_2 \right\}$$

$$= \max_{|\vec{v}|=1} \nabla f(a,b)\cdot \vec{v}$$

$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$

$$= \max_{|\vec{v}|=1} \left|\nabla f(a,b)\right| \cos\theta$$

θ is the angle between $\vec{v}$ and ∇f

$$= \left|\nabla f(a,b)\right| \quad \text{if } \theta = 0 \iff \vec{u} \parallel \nabla f(a,b)$$

- directional derivative along $\vec{u}$ ~ unit vector

$$D_{\vec{u}} f(a,b) = \nabla f(a,b)\cdot\vec{u}$$

$$= \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

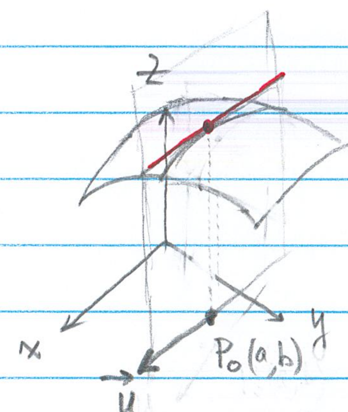


Fig 15.45

- gradient

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix}$$

Thrm 15.11 (Directions of Change) f is diff. at (a,b) with $\nabla f(a,b) \neq \vec{0}$

(1) f has its max rate of increase at (a,b) in the direction of the gradient $\nabla f(a,b)$;

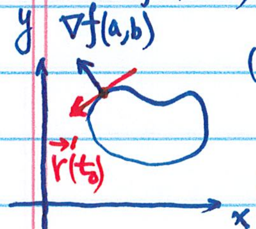
(2) f .. .. decrease .. .. $-\nabla f(a,b)$;

(3) the directional derivative is zero in any direction orthogonal to $\nabla f(a,b)$.

examples 1-5.

- <u>the gradient and level curves</u> f is diff at (a,b) with $\nabla f(a,b) \neq \vec{0}$

$\nabla f(a,b)$ is orthogonal to the level curve $f(x,y) = z_0$ at (a,b)



$(a,b) \sim \vec{r}(t_0)$

level curve: $f(x,y) = z_0$

parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$

<u>Proof</u> $z_0 = f(x,y) = f(\vec{r}(t))$

$$\Rightarrow 0 = \frac{d}{dt} f(\vec{r}(t)) \Big|_{t=t_0}$$

$$= \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \Big|_{t=t_0}$$

$$\Rightarrow \nabla f(a,b) \perp \vec{r}\,'(t_0)$$

#

<u>examples 6-7</u>

- <u>three dimensions</u>

§15.6 Tangent Planes and Linear Approximation

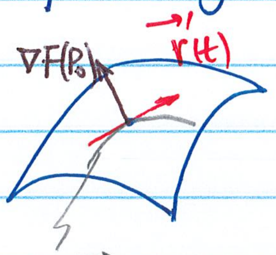
- surface in R^3

graph $\quad z = f(x, y)$

level surface $\quad F(x,y,z) = 0$

- tangent planes for level surface at $P_0(a,b,c)$

$\nabla F(P_0)$ is orthogonal to the tangent plane at P_0



C: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
on the level surface

$$0 = F(x(t), y(t), z(t))$$

$$0 = \frac{d}{dt} F(x(t), y(t), z(t)) \Big|_{t=0}$$

$$= \nabla F(P_0) \cdot \vec{r}\,'(0)$$

$$\Rightarrow 0 = \nabla F(P_0) \cdot \langle x-a, y-b, z-c \rangle$$

$$= F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c)$$

#15, ~~24~~ 25

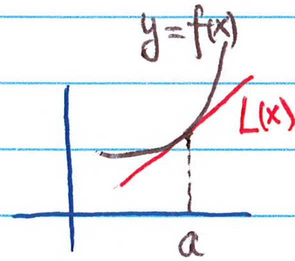
- tangent planes for graph at $(a, b, f(a,b))$

$$z = f(x,y) \Longrightarrow F(x,y,z) \equiv z - f(x,y) = 0$$

$$\Longrightarrow \nabla F(a,b,f(a,b)) = \langle -f_x(a,b), -f_y(a,b), 1 \rangle$$

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

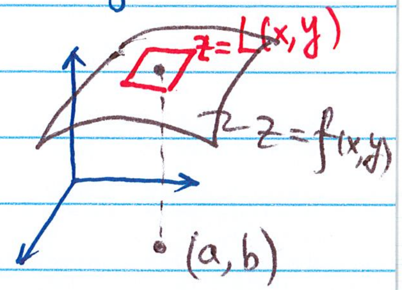
#19, 26



- <u>Linear Approximation</u>

<u>$y = f(x)$</u> $\qquad f(x) \approx L(x) = f(a) + f'(a)(x-a)$

<u>$y = f(x,y)$</u> $\qquad f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$



#34

- <u>differentials and change</u>

$$\Delta z \equiv f(x,y) - f(a,b)$$

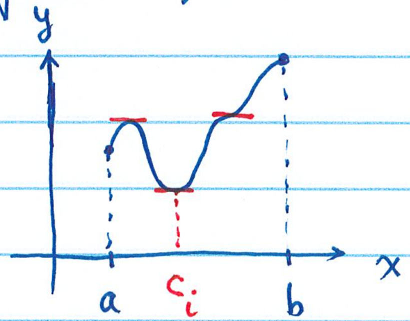
$$\approx L(x,y) - f(a,b) = f_x(a,b)\underbrace{(x-a)}_{dx} + f_y(a,b)\underbrace{(y-b)}_{dy} \cancel{\equiv dz}$$

$$= f_x(a,b)\,dx + f_y(a,b)\,dy \equiv dz$$

<u>Ex. 4, 6</u>

§15.7 Maximum / Minimum Problems

- functions of one variable



$f(c_i)=0$

$$\max_{x\in[a,b]}/\min f(x) = \max/\min \{f(c_i), f(a), f(b)\}$$

a, b — boundary pts

c_i — critical pts

- functions of two variables

Definition (Local Max/Min Values)

(1) $f(a,b)$ is a local max value of $f \iff f(a,b) \geq f(x,y),\ \forall (x,y)\in D_r(a,b)$

(2) .. .. min .. .. .. $\iff f(a,b) \leq f(x,y),\ \forall (x,y)\in D_r(a,b)$

Theorem (First Derivative Test)

(1) f has a local max/min at an interior pt (a,b)

(2) $f_x(a,b)$ and $f_y(a,b)$ exist

$\Bigg\} \Longrightarrow \begin{array}{l} f_x(a,b)=0 \\ f_y(a,b)=0 \end{array}$

Proof auxiliary function $g(t) = f\left(a+h_1 t,\ b+h_2 t\right)$ has a local max/min at $t=0$

$$\Rightarrow 0 = g'(t)\Big|_{t=0} = \frac{d}{dt} f(a+h_1 t, b+h_2 t)\Big|_{t=0} = \nabla f(a,b)\cdot\langle h_1, h_2\rangle \quad \forall (h_1,h_2)\in\mathbb{R}^2$$

$$\Rightarrow \nabla f(a,b) = \langle 0, 0\rangle. \quad \#$$

Definition (Critical Points)

$$(a,b) \text{ is a critical pt of } f \iff \begin{cases} (1)\ f_x(a,b) = f_y(a,b) = 0 \\ (2)\ \text{either } f_x(a,b) \text{ or } f_y(a,b) \text{ DNE.} \end{cases}$$

Ex. 1

Definition (Saddle Point)

(a,b) is a saddle pt of $f \iff (a,b)$ is a critical pt, but $f(a,b)$ is not a local max/min.

Theorem (Second Derivative Test) $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$.

$$\begin{matrix} (1)\ f_{xx}, f_{xy}, f_{yy} \in C^0(D_r(a,b)) \\ (2)\ f_x(a,b) = f_y(a,b) = 0 \end{matrix} \Rightarrow \begin{cases} (1)\ D>0 \begin{cases} f_{xx}(a,b)>0 \Rightarrow \text{a local max} \\ f_{xx}(a,b)<0 \Rightarrow \text{a local min} \end{cases} \\ (2)\ D<0 \Rightarrow (a,b) \text{ is a saddle pt} \\ (3)\ D=0 \Rightarrow \text{it is inconclusive.} \end{cases}$$

Ex. 2, 3, 4

Definition (Absolute Maximum/Minimum Values) f is defined on a set $R \subset \mathbb{R}^2$ containing (a,b)

(1) $f(a,b)$ is an absolute max value of $f \iff f(a,b) \geq f(x,y),\ \forall (x,y) \in R$

(2) .. .. min .. .. .. $\iff f(a,b) \leq f(x,y),\ \forall (x,y) \in R$.

Procedure (Finding abs. max./min. values on closed bounded sets)

R is a closed bounded set in $\mathbb{R}^2$ and $f \in C^0(R)$.

(1) calculate values of f at all critical pts

(2) find the max and min values of f on the boundary of R

(3) $\begin{cases} \text{abs. max. value} = \text{the largest values in (1) and (2).} \\ \text{abs. min. value} = \text{the smallest values in (1) and (2).} \end{cases}$

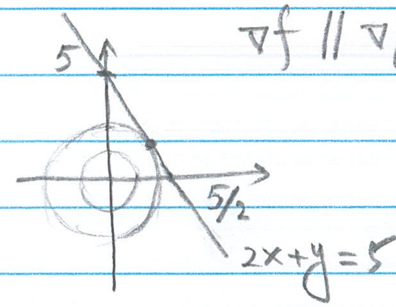
Ex. 5, 6, 7, 8, 9

§15.8 Lagrange Multipliers

constrained maximum/minimum problems

$$\max_{g(x,y,z)=0}/\min \; f(x,y,z)$$

$\nabla f \parallel \nabla g$



Ex. 1 $$\min_{\underset{g(x,y)}{\underset{\parallel}{2x+y}}=5} \left(\underset{f(x,y)}{\underset{\parallel}{x^2+y^2}} \right)$$

Ex. 2 Find (x,y,z) on $g(x,y,z)=x^2-z^2-1=0$ that are closest to $(0,0,0)$

~~Solution~~ $$\min_{g(x,y,z)=0} \sqrt{x^2+y^2+z^2} \quad \text{or} \quad \min_{g(x,y,z)=0} \left(\underbrace{x^2+y^2+z^2}_{f(x,y,z)} \right)$$

Solution 1 $g(x,y,z)=0 \Longrightarrow z^2 = x^2-1$ or $x^2=z^2+1$

- $h(x,y) = f(x,y,z)\Big|_{z^2=x^2-1} = 2x^2+y^2-1$

  $\min h(x,y) \Rightarrow$ critical pt $(x,y)=(0,0) \Rightarrow z^2=-1$ wrong

- $k(y,z) = f(x,y,z)\Big|_{x^2=z^2+1} = y^2+2z^2+1$

  $\min k(y,z) \Rightarrow$ critical pt $(y,z)=(0,0) \Rightarrow x^2=1 \Rightarrow x=\pm 1$

Solution 2

$\nabla f \parallel \nabla g$

- <u>the method of Lagrange multipliers</u>

Assume that $\begin{cases} (1)\ f(x,y,z) \text{ and } g(x,y,z) \text{ are differentiable} \\ (2)\ \nabla g \neq 0 \text{ where } g = 0 \end{cases}$

- critical pts of $\max\limits_{g(x,y,z)=0}/\min f(x,y,z)$ satisfy

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

<u>Ex. 3</u> $f(x,y) = xy$ $\qquad \max\limits_{g=1}/\min f$

$g(x,y) = \dfrac{x^2}{8} + \dfrac{y^2}{2}$

<u>Ex. 4</u> $f(x,y) = 3x + 4y$

$g(x,y) = x^2 + y^2$

<u>#24</u> $f(x,y,z) = xy - z$

$g(x,y,z) = x^2 + y^2 + z^2 - xy$