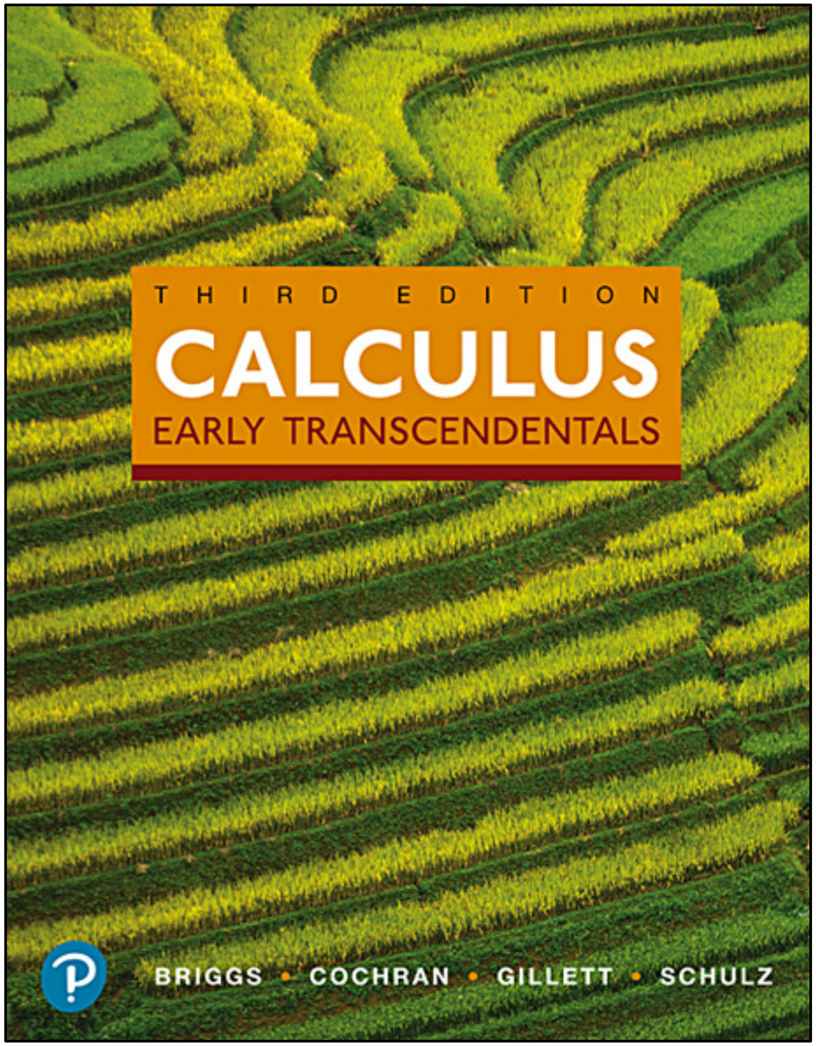
Calculus Early Transcendentals

Third Edition



Chapter 14 $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

Chapter 15 $f(x,y)$, $f(x,y,z)$

Chapter 15

Functions of Several Variables

Lesson	Section	
9	§15.1	Graph and Level Curve
10	§15.2	Limits and Continuity
11	§15.3	Partial Derivative
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Section 15.1 Graphs and Level Curves

Definition Function, Domain, and Range with Two Independent Variables

A **function** $z = f(x, y)$ assigns to each point (*x*, *y*) in a set *D* in $\mathbb{R}^2$ a unique real number *z* in a subset of $\mathbb{R}$.

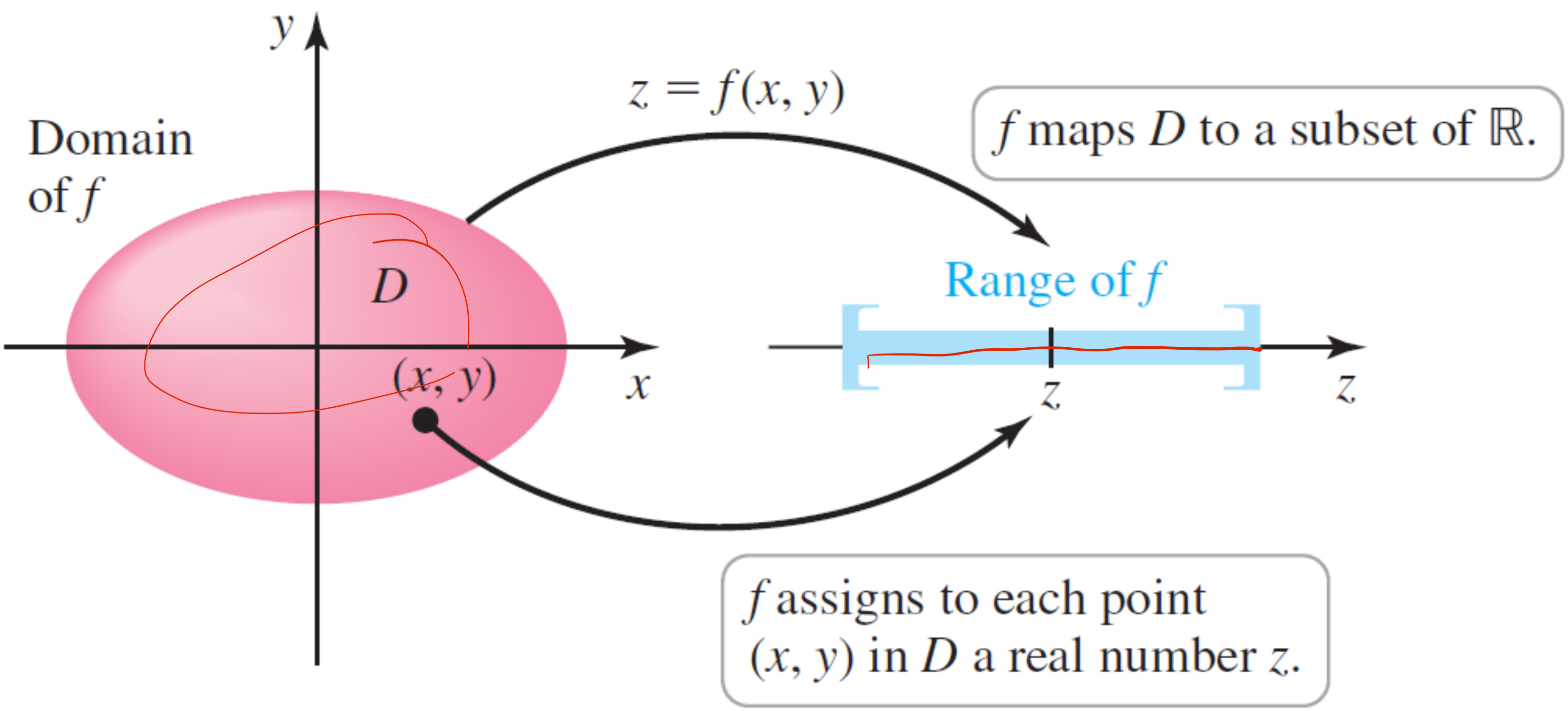
The set *D* is the **domain** of *f*. The range of *f* is the set of real numbers *z* that are assumed as the points (*x*, *y*) vary over the domain (Figure 15.1).

x, y — indep. variable

z — dep. var.

- function of two variables $z = f(x, y)$
- domain $\{(x, y) \in R^2 \mid f(x, y) \text{ is meaningful}\} = \text{dom}$
- range $\{f(x, y) \in R \mid (x, y) \in \text{Dom}\}$

Figure 15.1



y
Domain
of f
D
x
(x, y)
z = f(x, y)
f maps D to a subset of ℝ.
Range of f
z
z
f assigns to each point
(x, y) in D a real number z.

Finding domains

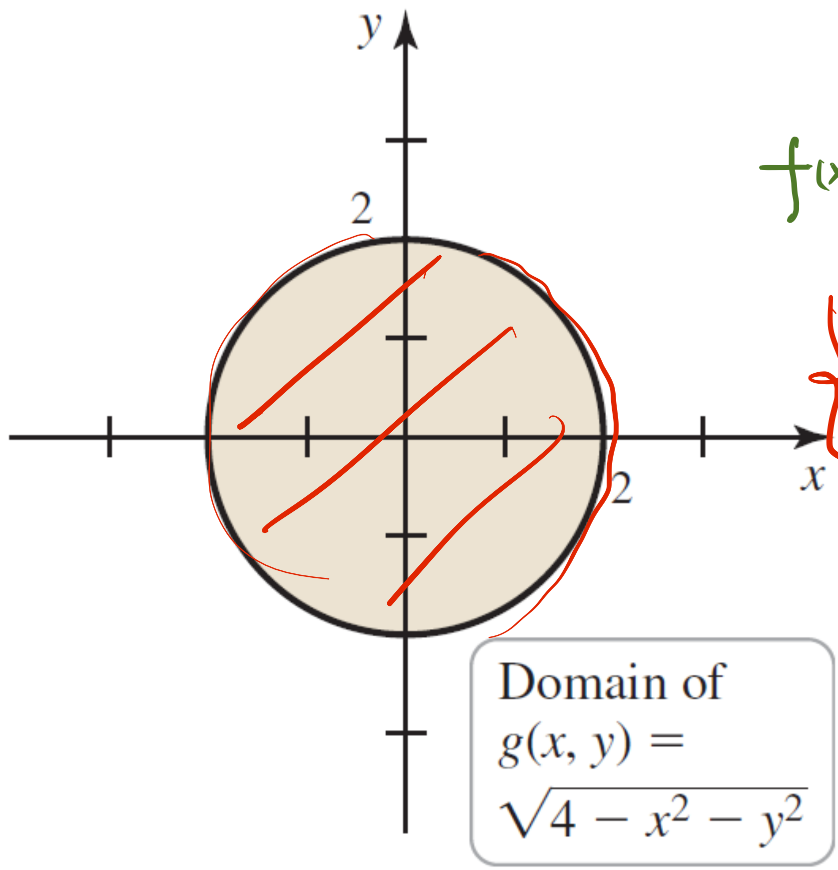
Figure 15.2

Example 1 $g(x,y)=\sqrt{4-x^2-y^2}$

$4-x^2-y^2 \geq 0$

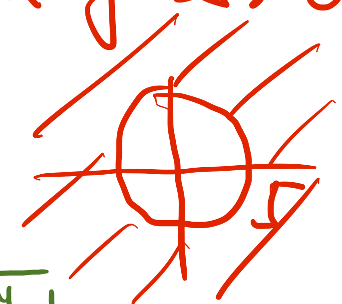
$x^2+y^2 \leq 4 = 2^2$

$x^2+y^2=2^2$



#18 $f(x,y)=\dfrac{1}{\sqrt{x^2+y^2-25}}$

$\begin{cases} x^2+y^2-25 \geq 0 \\ \sqrt{x^2+y^2-25} \neq 0 \end{cases} \Rightarrow x^2+y^2-25>0$



$f(x,y)=\dfrac{\sqrt{x+y-1}}{x-1}$

$\begin{cases} x+y-1 \geq 0 \\ x-1 \neq 0 \end{cases}$

• Graph of $z = f(x, y)$

$y = f(x)$

$(x, f(x))$

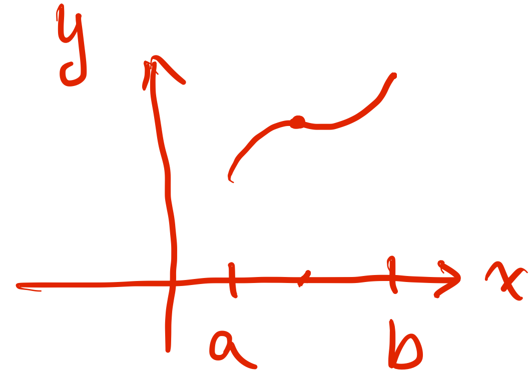
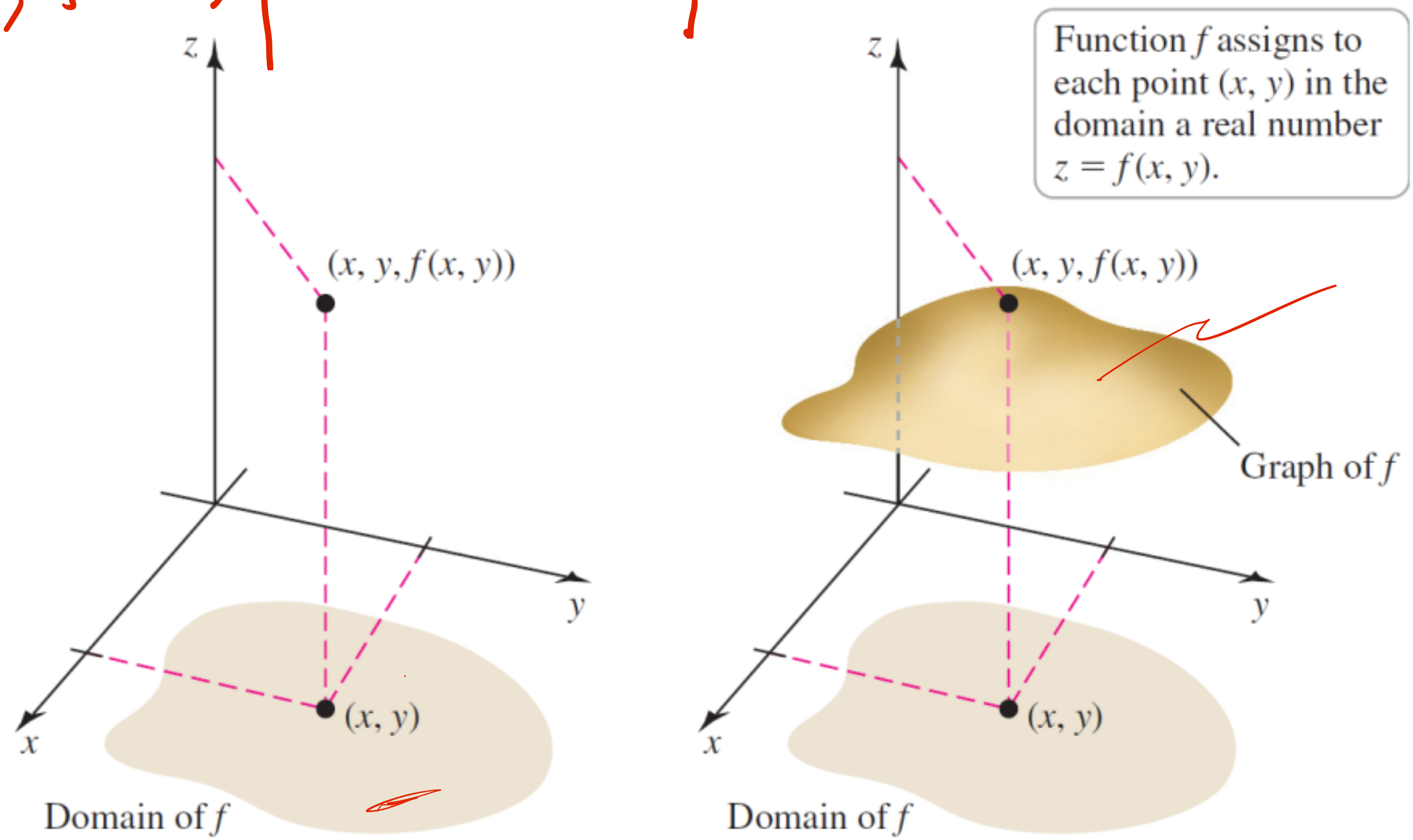


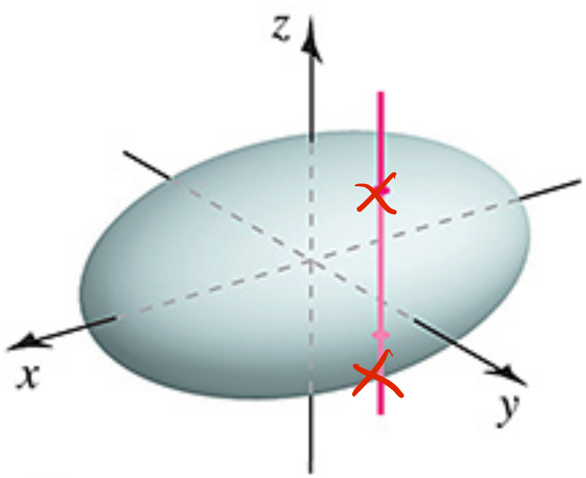
Figure 15.3

$\{(x, y, f(x, y)) \mid (x, y) \in D\}$

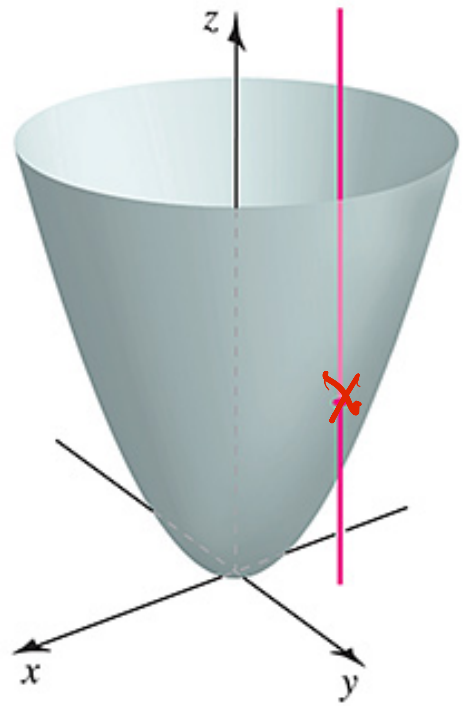


• vertical line test

Figure 15.4



An ellipsoid does not pass the vertical line test: not the graph of a function.

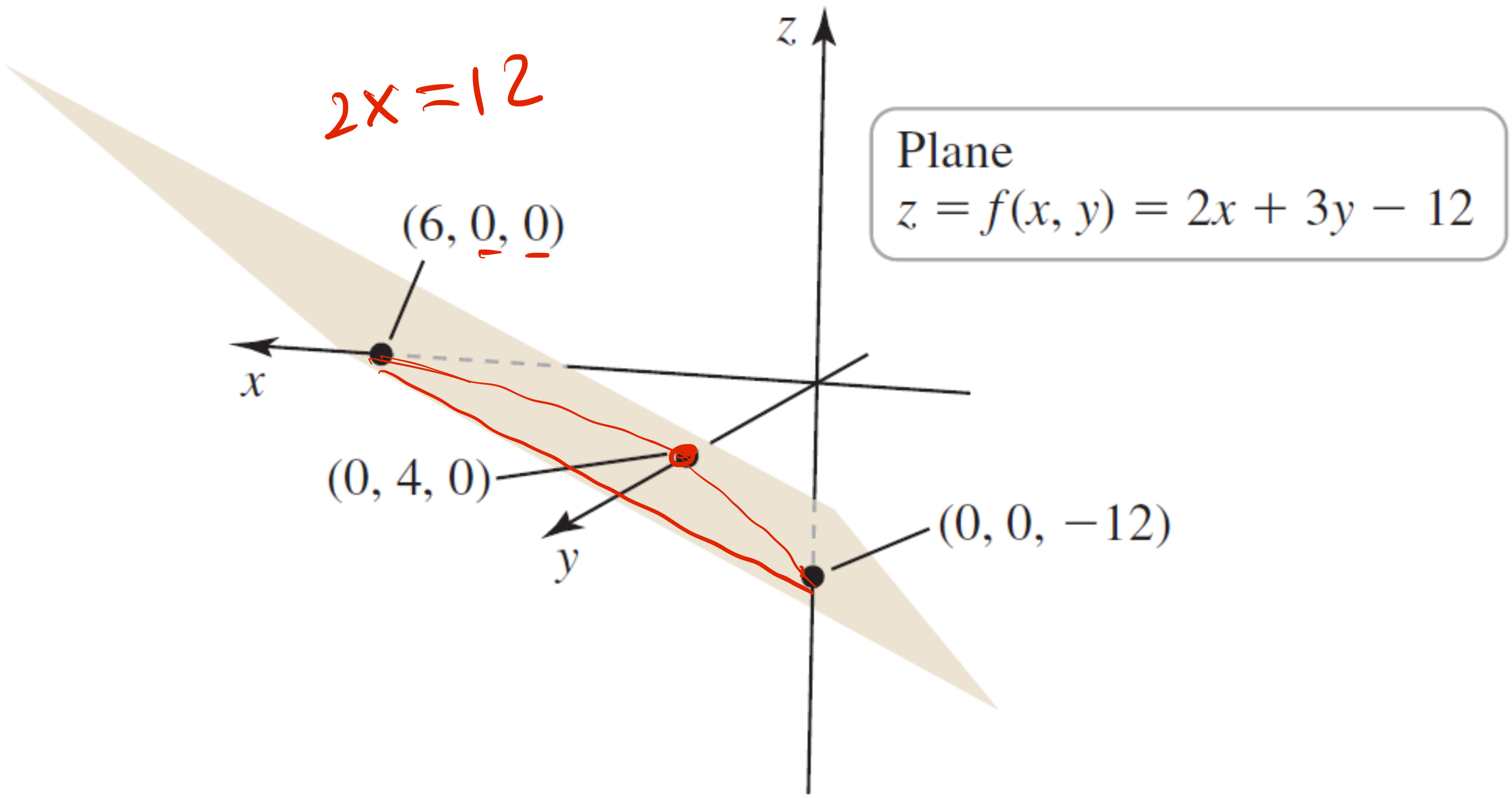


This elliptic paraboloid passes the vertical line test: graph of a function.

• Sketch graphs (a) $z = f(x,y) = 2x + 3y - 12$

$2x + 3y - z = 12$

Figure 15.5



(b) $z = f(x, y) = x^2 + y^2$

Figure 15.6

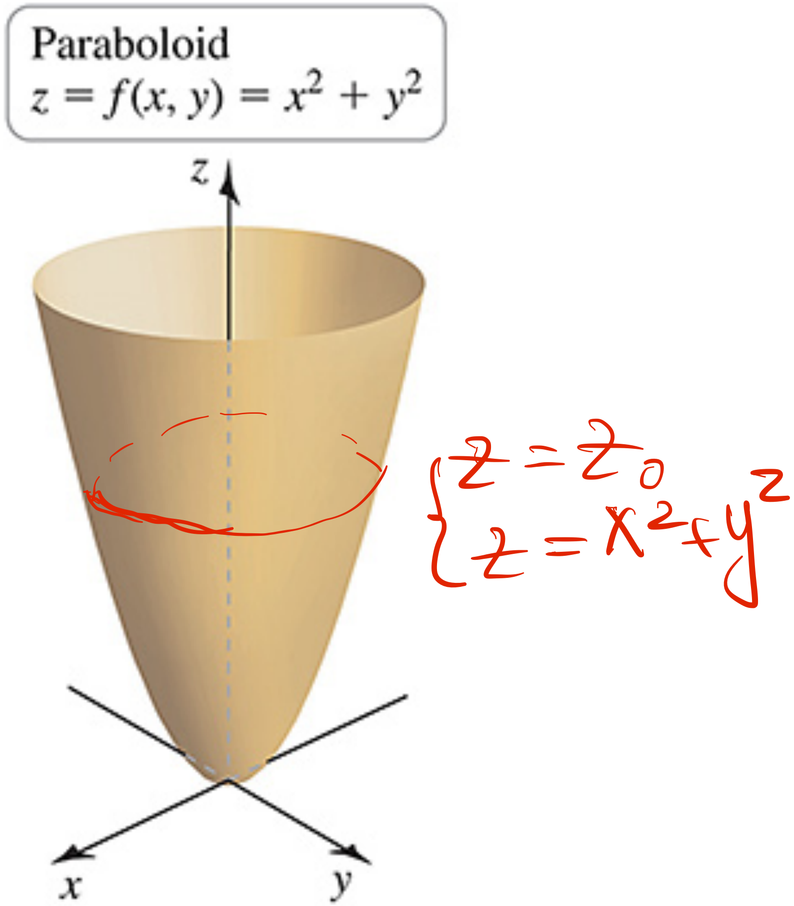
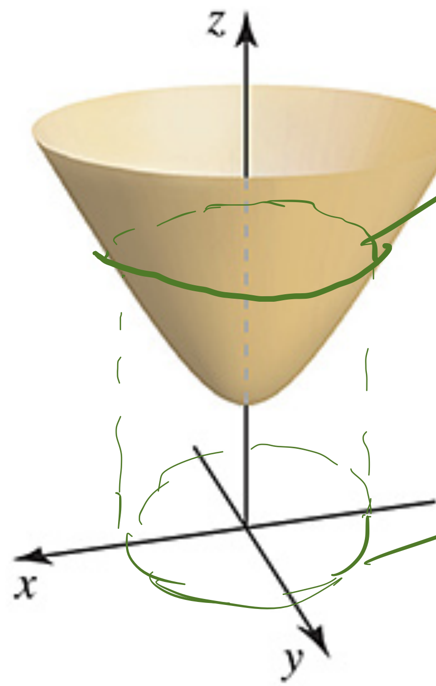


Figure 15.7

(c) $z = h(x,y) = \sqrt{1+x^2+y^2}$

$z^2 = 1+x^2+y^2 \Rightarrow z^2 - x^2 - y^2 = 1$

Upper sheet of hyperboloid of two sheets
$z = \sqrt{1 + x^2 + y^2}$



contours

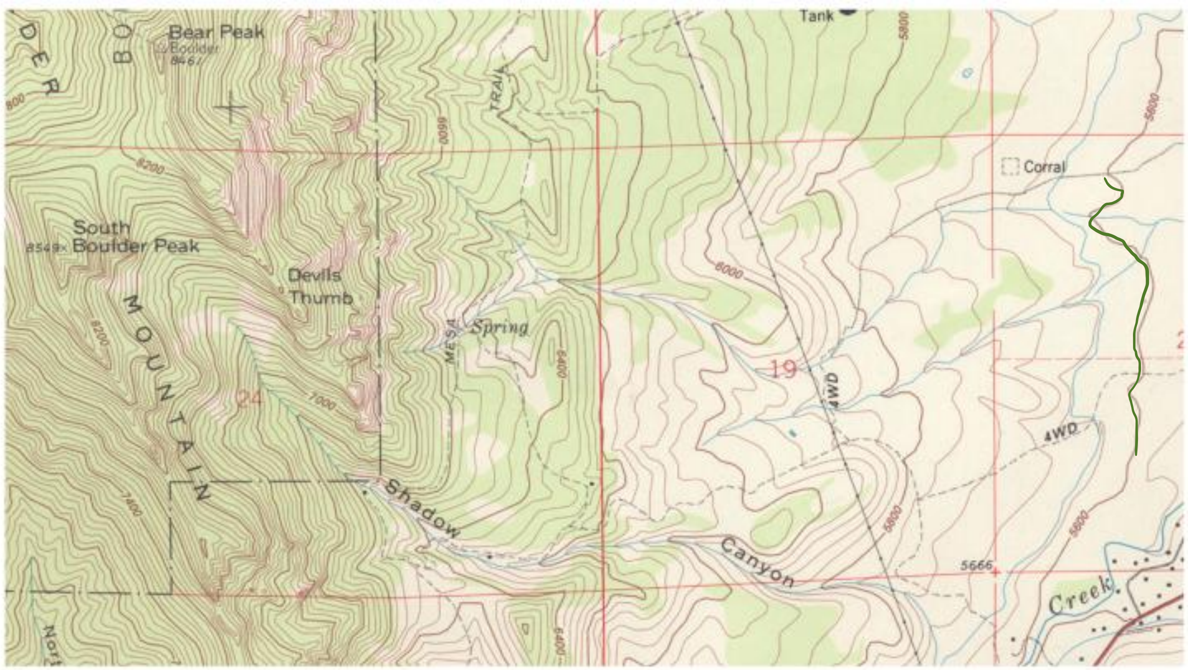
trace
$\begin{cases} z = z_0 \\ z = \sqrt{1+x^2+y^2} \end{cases}$

level curve
$z_0 = \sqrt{1+x^2+y^2}$

• Level Curve

Figure 15.8

Closely spaced contours: rapid changes in elevation



Widely spaced contours: slow changes in elevation

Figure 15.9

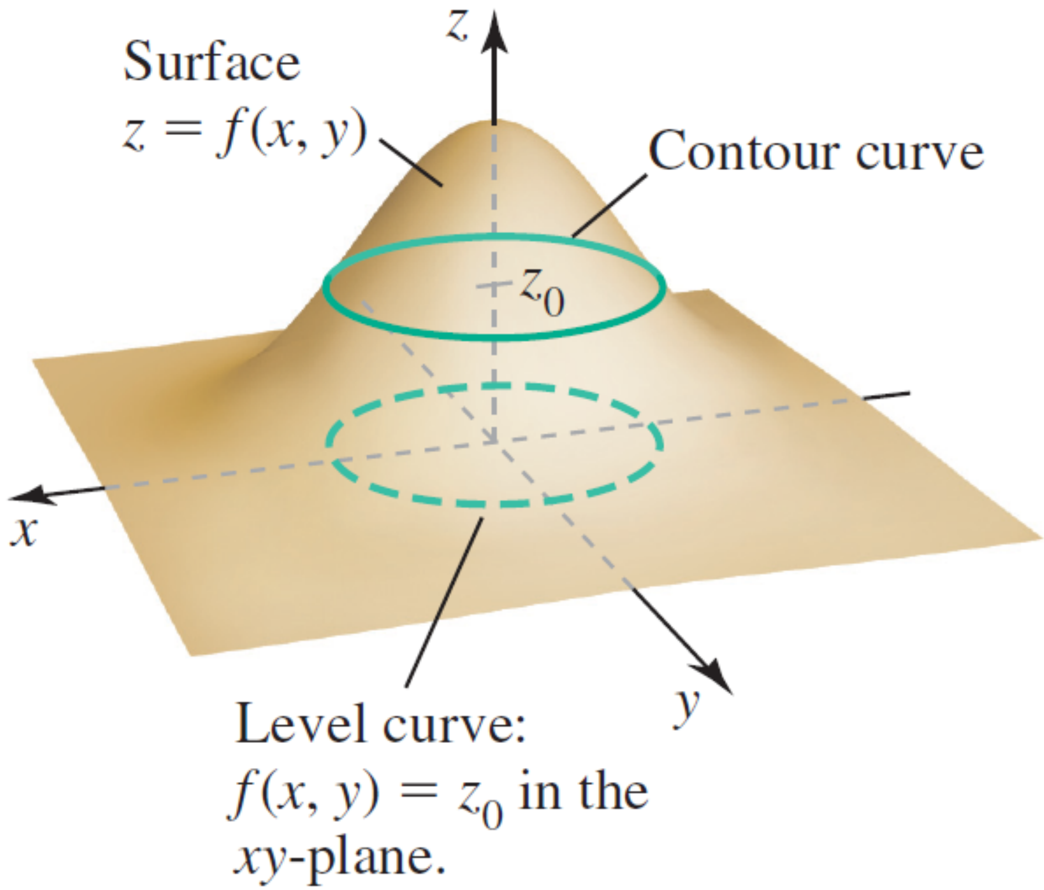

Surface
z = f(x, y)
z
Contour curve
z0
x
y
Level curve:
f(x, y) = z0 in the
xy-plane.

Figure 15.10

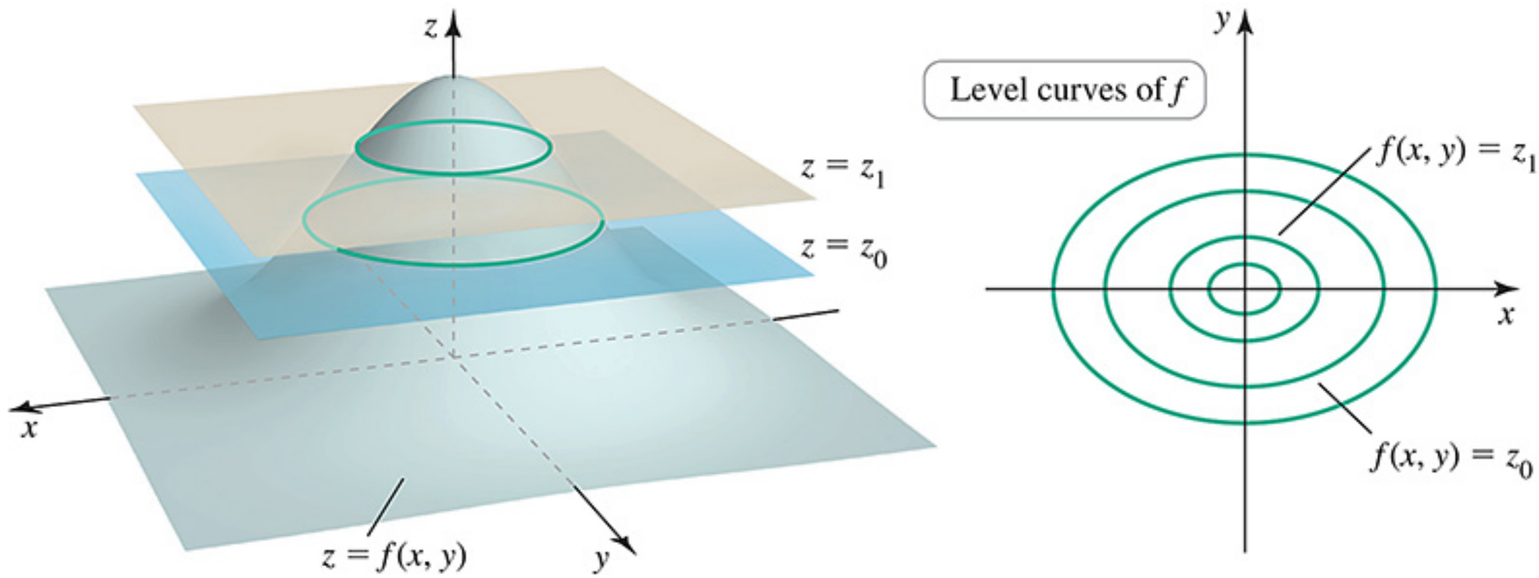
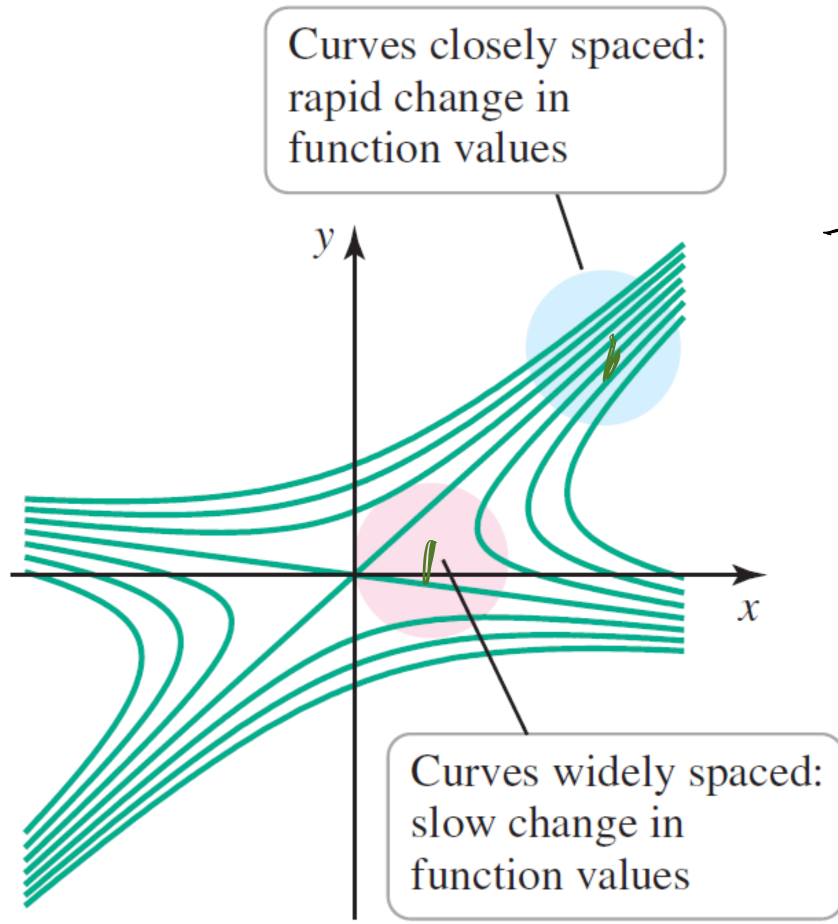


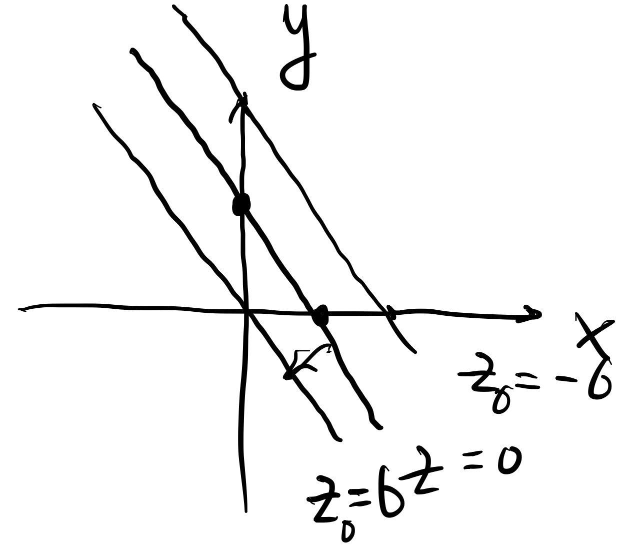
Figure 15.11



- Sketch level curve

$z = f(x,y) = 6 - 3x - 2y$

$3x + 2y = 6 - z_0$



$z_0 = 0$

$z_0 = -6$ $3x + 2y = 12$

Example 3 Sketch level curves (a) $z = f(x, y) = y - x^2 - 1$

$$z_0 = y - x^2 - 1 \Rightarrow y = x^2 + (1 + z_0)$$

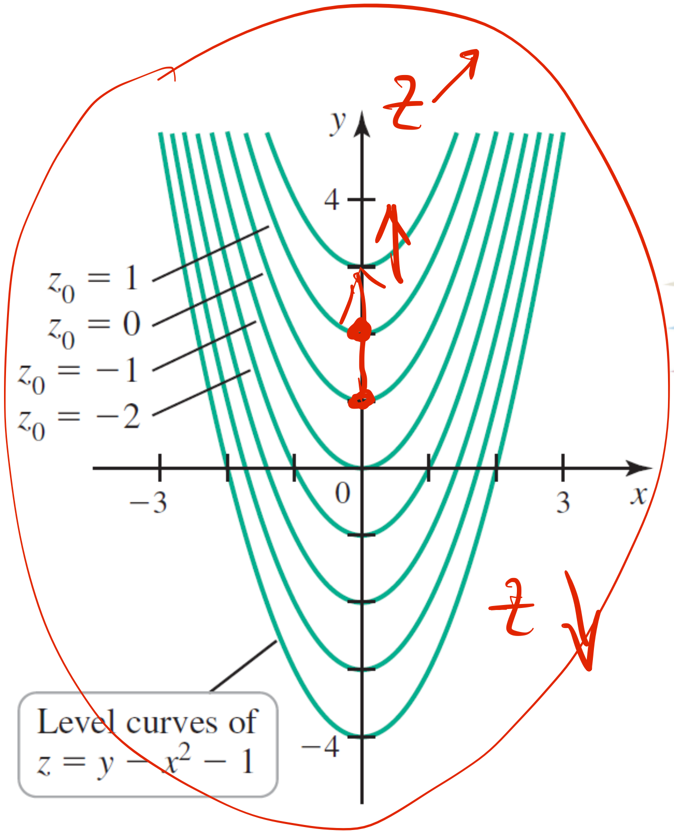
$z_0 = 0$: $y = x^2 + 1$

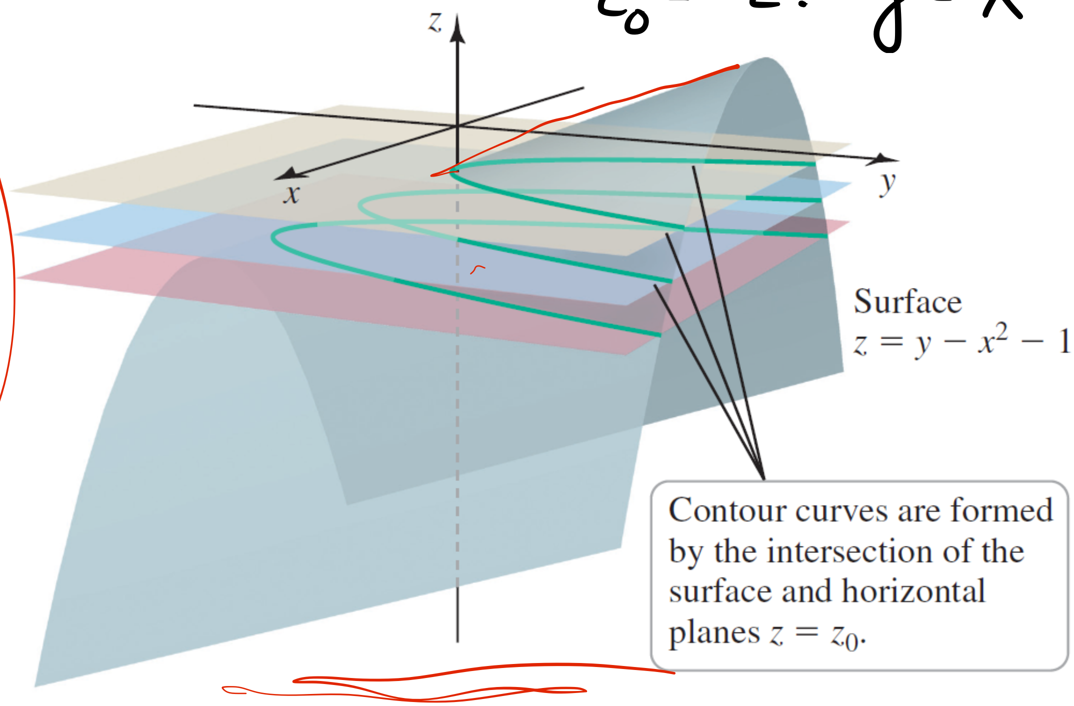
$z_0 = 1$: $y = x^2 + 2$

$z_0 = -1$: $y = x^2$

$z_0 = -2$: $y = x^2 - 1$

Figure 15.12 (a & b)

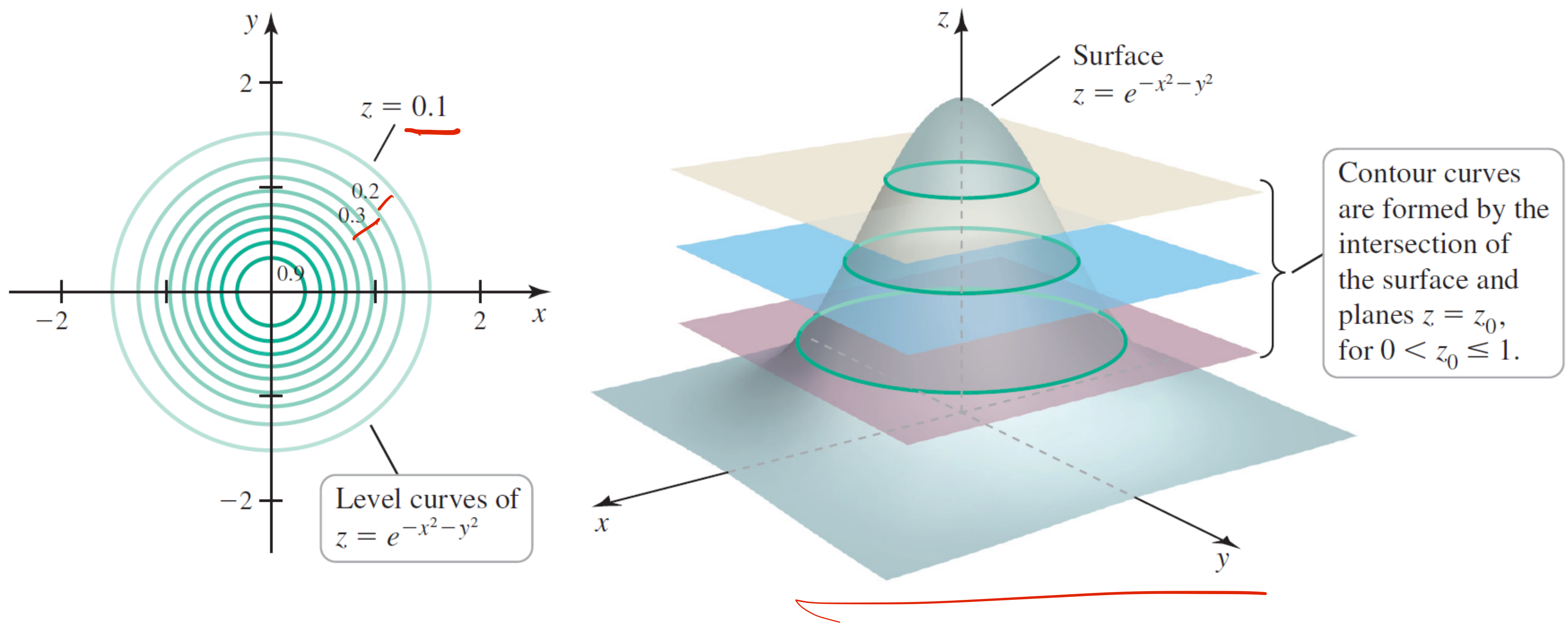




(b) $z = f(x, y) = e^{-x^2-y^2}$

$z_0 = e^{-(x^2+y^2)}$ $\ln z_0 = -(x^2+y^2)$, $x^2+y^2 = \ln \frac{1}{z_0}$

Figure 15.13 (a & b)



Definition Function, Domain, and Range with *n* Independent Variables

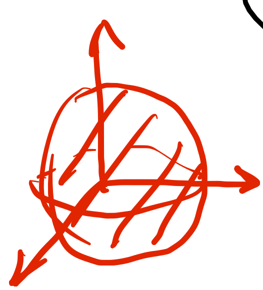
The **function** $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number x_{n+1} to each point $(x_1, x_2, \ldots, x_n)$ in a set *D* in $\mathbb{R}^n$. The set *D* is the **domain** of *f*. The **range** is the set of real numbers x_{n+1} that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

Example 7 Finding domains

(a) $g(x,y,z) = \sqrt{16 - x^2 - y^2 - z^2}$

$16 - x^2 - y^2 - z^2 \geq 0$

$x^2 + y^2 + z^2 \leq 4^2$

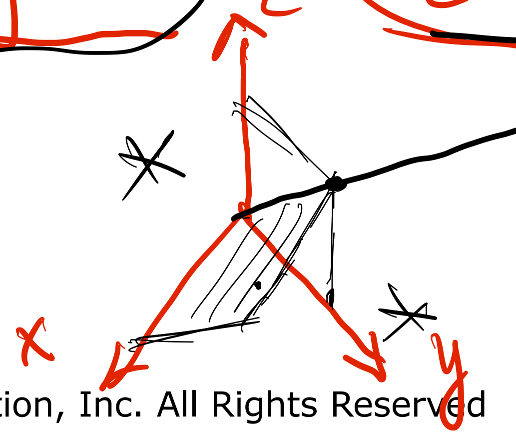


(b) $h(x,y,z) = \dfrac{12y^2}{z-y}$

$\text{Dom} = \{(x,y,z) \mid z - y \neq 0\}$

$z - y \neq 0$

$z = y$



$z = y$

• Graphs of Functions of More than Two Variables

$w = f(x, y, z)$

$\{(x, y, z, f(x,y,z)) \mid (x, y, z) \in D\}$

Figure 15.17

$z = f(x, y)$ graph

level curve

$z_0 = f(x, y)$

$w = f(x, y, z)$

$w_0 = f(x, y, z)$

level surface

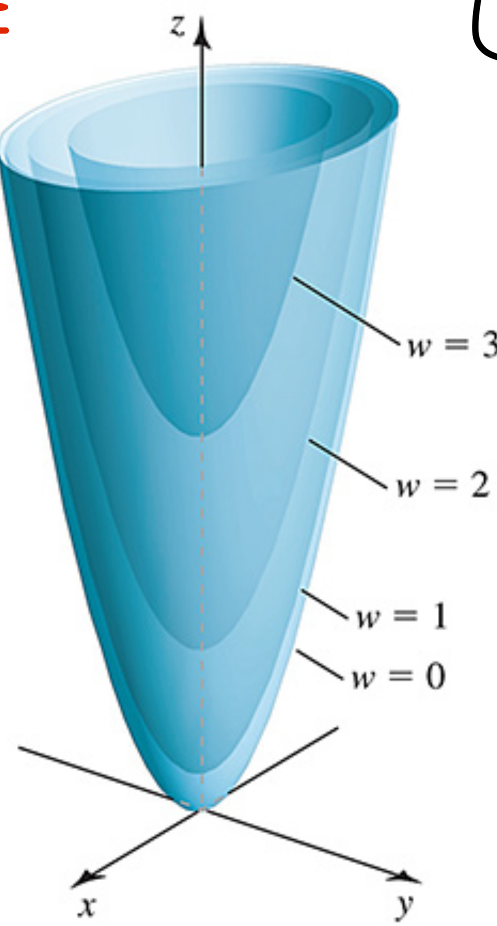
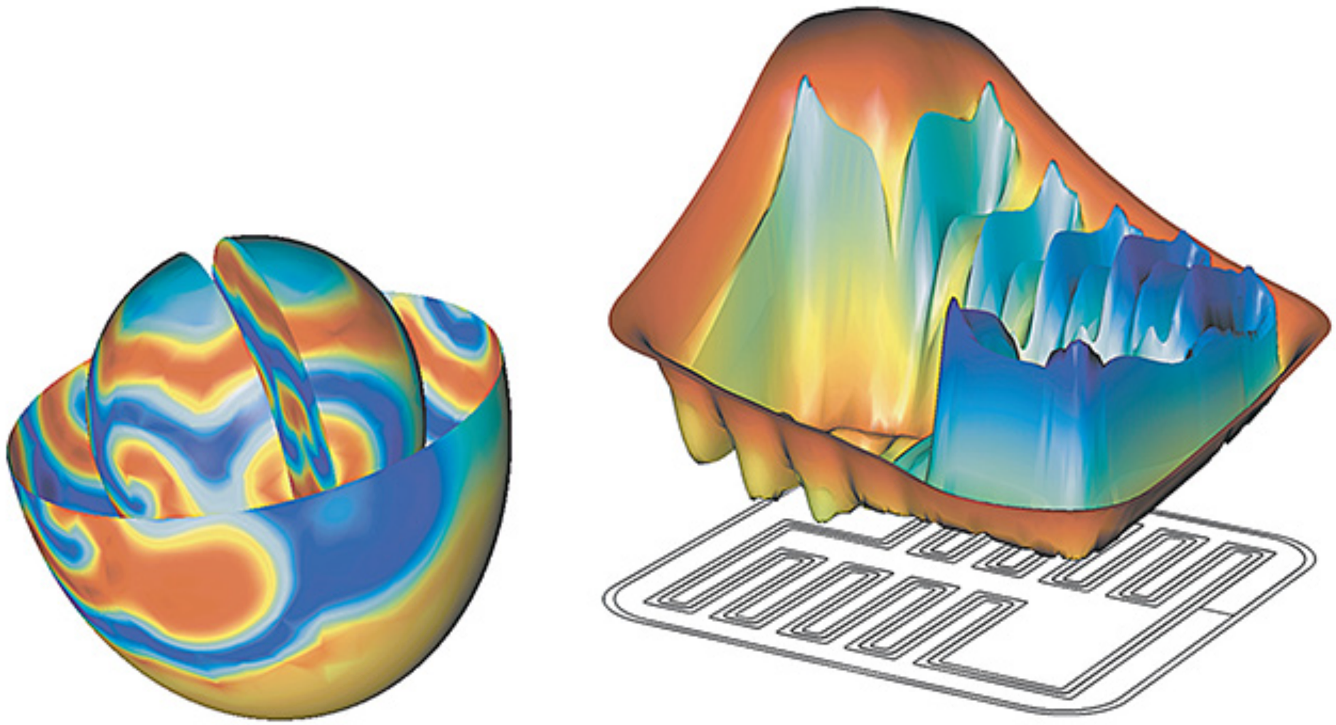


Figure 15.18 (a & b)



Section 15.2 Limits and Continuity

$$\lim_{x \to a} f(x) = L$$

$\forall \varepsilon > 0$ $\exists \delta_\varepsilon > 0$ $|x-a| < \delta_\varepsilon \Longrightarrow |f(x) - L| < \varepsilon$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

$\forall \varepsilon > 0$ $\exists \delta_\varepsilon > 0$ s.t. $|(x,y)-(a,b)| < \delta \Longrightarrow |f(x,y) - L| < \varepsilon$

$|(x,y)-(a,b)| = \sqrt{(x-a)^2 + (y-b)^2}$

$(x-a)^2+(y-b)^2 \leq \delta^2$

Figure 15.19 (a & b)

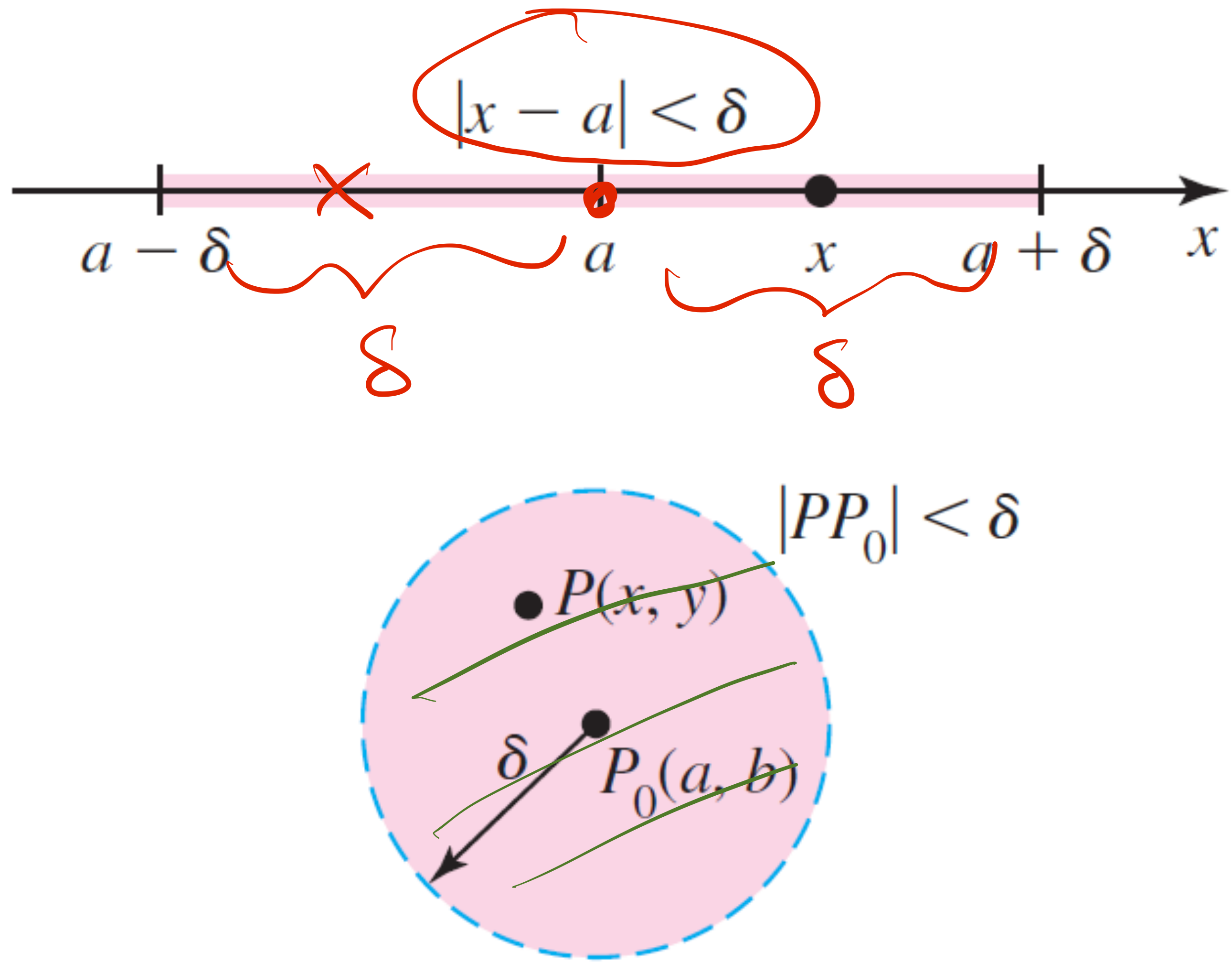
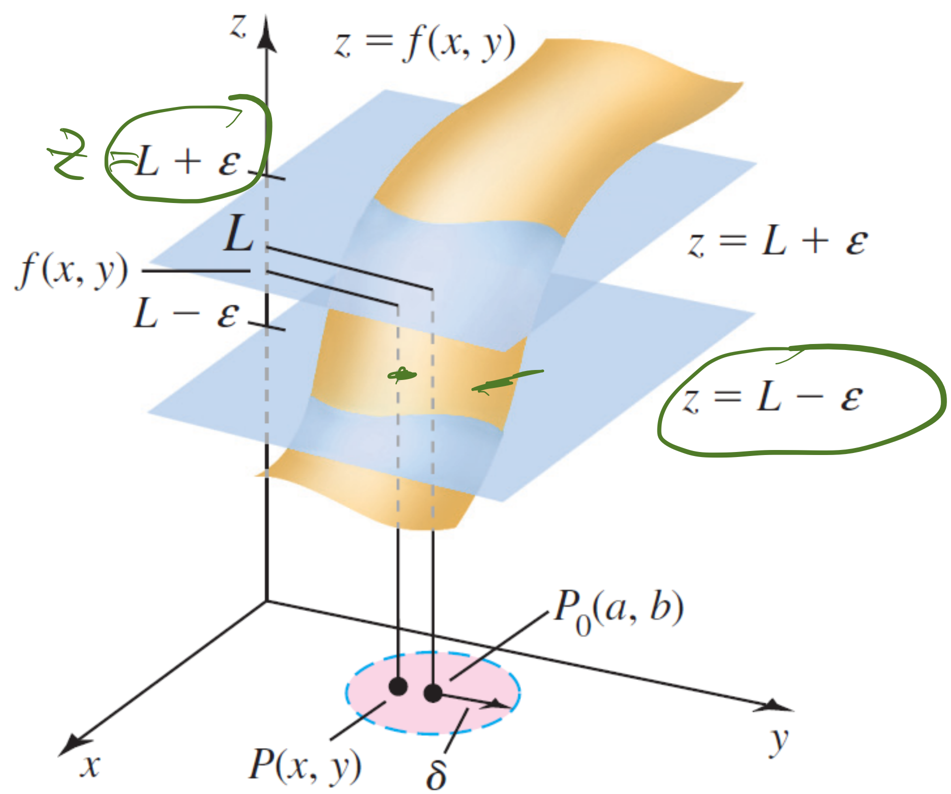


Figure 15.20



$f(x, y)$ is between $L - \varepsilon$ and $L + \varepsilon$ whenever $P(x, y)$ is within δ of P_0.

Definition Limit of a Function of Two Variables

The function *f* has the **limit** *L* as $P(x, y)$ approaches $P_0(a,\ b)$, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{P\to P_0} f(x,y) = L,$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon$$

whenever (*x*, *y*) is in the domain of *f* and

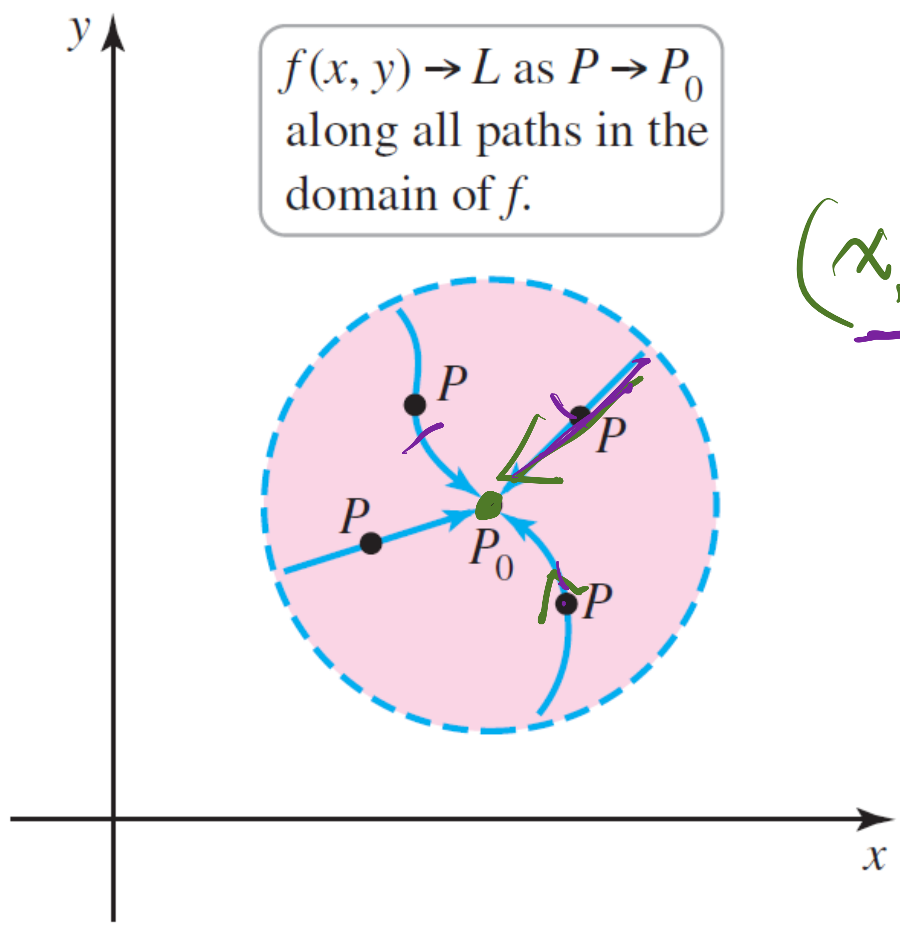
$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Figure 15.21

$\lim_{x \to a} f(x) = L$

$a-\delta \quad a \quad a+\delta$

$(x, y) \to (a, b)$



Theorem 15.1 Limits of Constant and Linear Functions

Let *a*, *b*, and *c* be real numbers.

1. Constant function $f(x,y) = c: \lim_{(x,y)\to(a,b)} c = c$
2. Linear function $f(x,y) = x: \lim_{(x,y)\to(a,b)} x = a$
3. Linear function $f(x,y) = y: \lim_{(x,y)\to(a,b)} y = b$

Theorem 15.2 Limit Laws for Functions of Two Variables

Let *L* and *M* be real numbers and suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M.$

Assume *c* is a constant, and *n* > 0 is an integer.

1. **Sum** $\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M.$
2. **Difference** $\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M.$
3. **Constant multiple** $\lim_{(x,y)\to(a,b)} cf(x,y) = cL$
4. **Product** $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM$
5. **Quotient** $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, provided $M \neq 0$
6. **Power** $\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n.$
7. **Root** $\lim_{(x,y)\to(a,b)} (f(x,y))^{1/n} = L^{1/n},$ where we assume *L* > 0 if *n* is even.

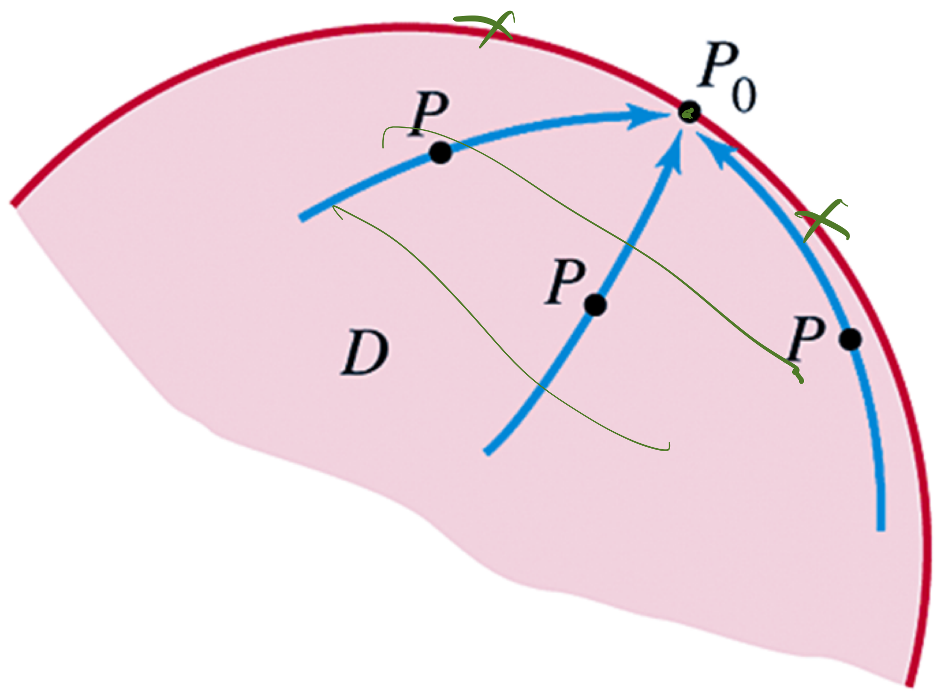
• Limits at Boundary Points

$\lim_{x \to a} f(x) = L$

a

$x \neq a$

Figure 15.23



P must approach P_0 along all paths in the domain *D* of *f*.

$\frac{0}{0}$ $\quad a^2-b^2=(a+b)(a-b)$

$$\lim_{(x,y)\to(4,4)} \frac{x^2-y^2}{x-y} = \lim \frac{(x+y)(x-y)}{x-y} = \lim (x+y) = 4+4=8$$

$x-y \neq 0$

Figure 15.24

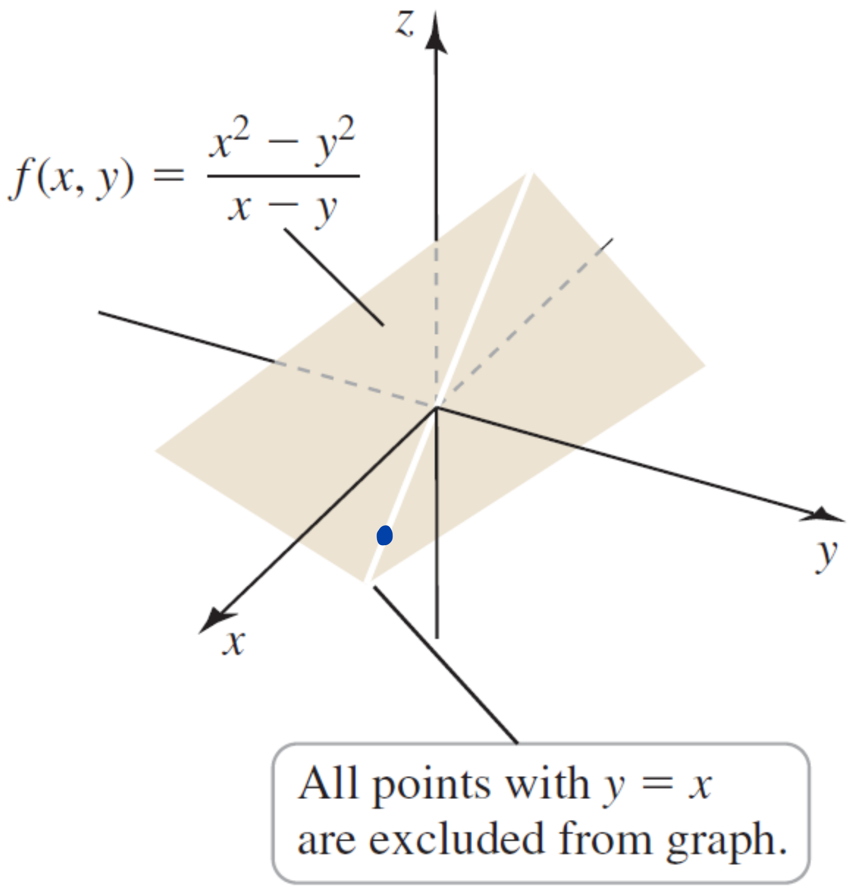


Figure 15.25

$\frac{0}{0}$ $\quad a^2 - b^2 = (a+b)(a-b)$

$$\lim_{(x,y)\to(4,1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}} \cdot \frac{\sqrt{x} + 2\sqrt{y}}{\sqrt{x} + 2\sqrt{y}}$$

$xy - 4y^2 = y(x - 4y)$

$$= \lim \frac{y(x-4y)(\sqrt{x} + 2\sqrt{y})}{x - 4y} = \lim y(\sqrt{x} + 2\sqrt{y})$$

$$= 1 \cdot (\sqrt{4} + 2\sqrt{1})$$

$$= 4$$

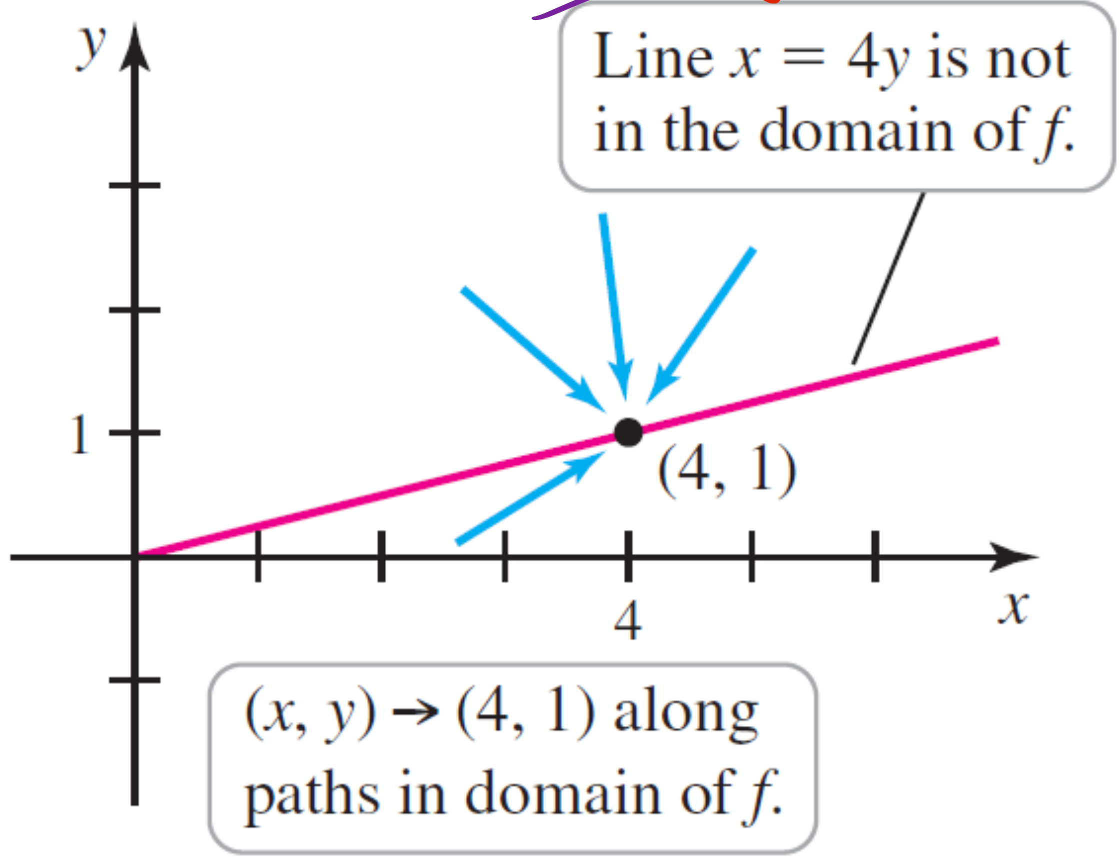
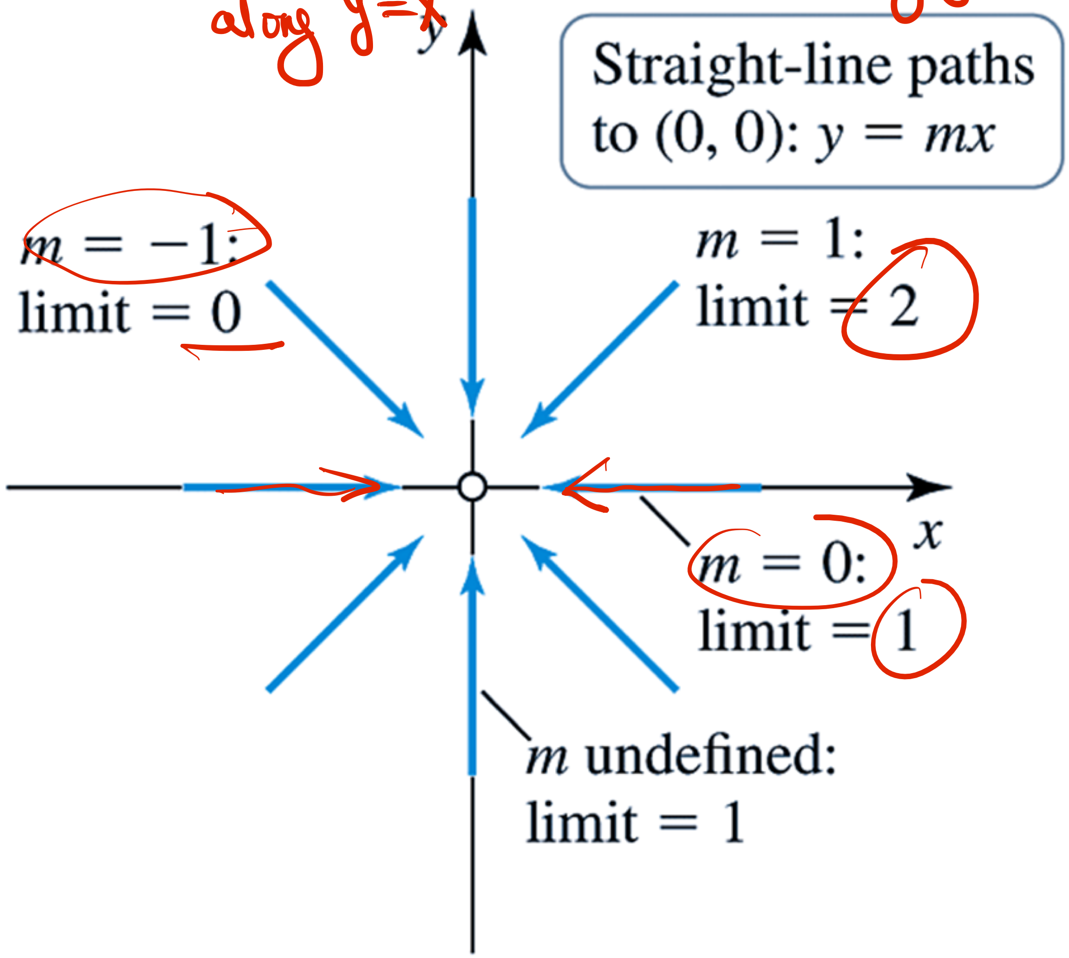
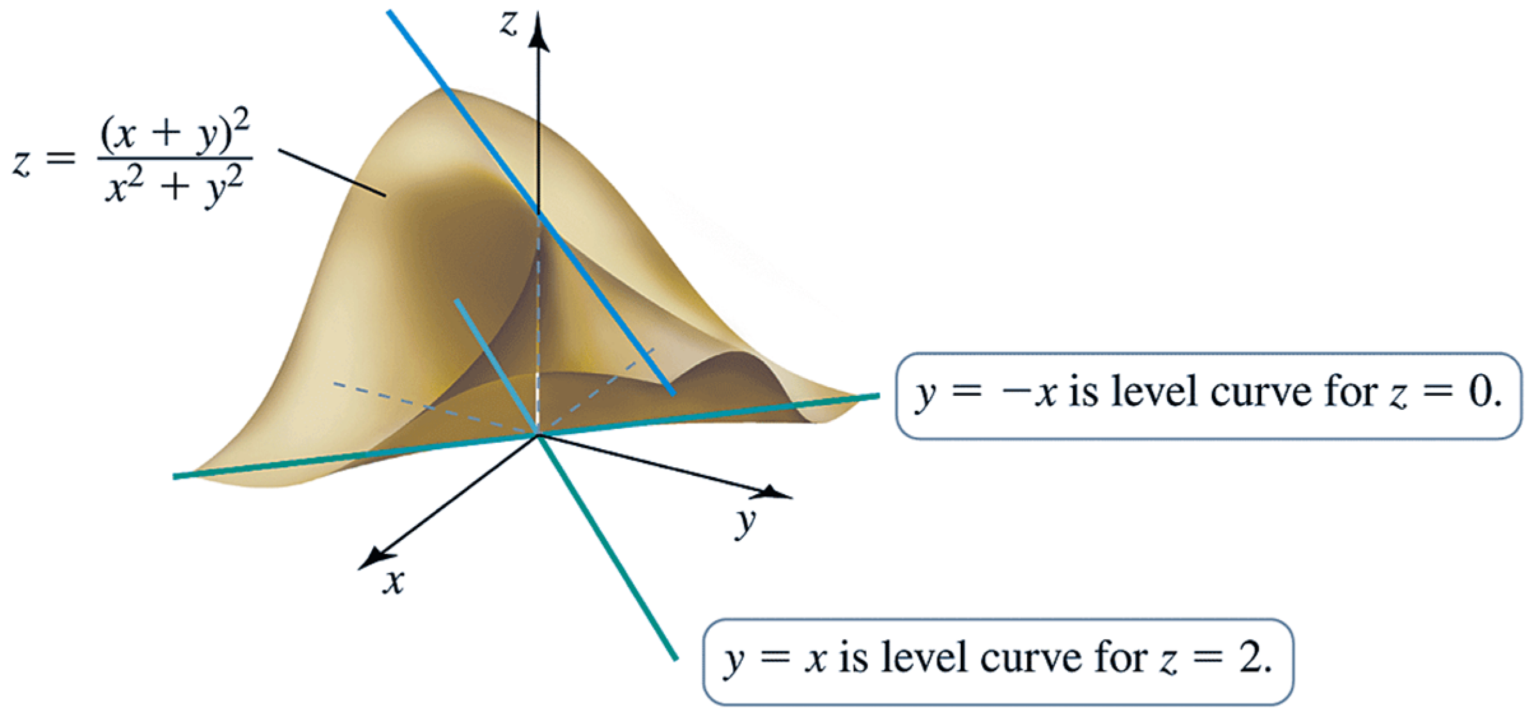


Figure 15.26



Straight-line paths to (0, 0): y = mx
m = −1:
limit = 0
m = 1:
limit = 2
m = 0:
limit = 1
m undefined:
limit = 1
x
y

Figure 15.27


$z = \frac{(x + y)^2}{x^2 + y^2}$
z
y
x
$y = -x$ is level curve for $z = 0$.
$y = x$ is level curve for $z = 2$.

Procedure Two-Path Test for Nonexistence of Limits

If $f(x, y)$ approaches two different values as (*x*, *y*) approaches (*a*, *b*) along two different paths in the domain of *f*, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

§15.2 Examples

#19

$$\lim_{(x,y)\to(2,0)} \frac{x^2-3xy^2}{x+y} = \frac{\lim (x^2-3xy^2)}{\lim (x+y)} = \frac{2^2-0}{2+0} = 2 \qquad \neq \frac{0}{0}$$

#22

$$\lim_{(x,y)\to(1,-2)} \frac{y^2+2xy}{y+2x} = \frac{y(y+2x)}{y+2x} = \lim y = -2$$

#24 $\frac{0}{0}$

$$\lim_{(x,y)\to(-1,1)} \frac{2x^2-xy-3y^2}{x+y} =$$

$$2(x^2-y^2) - (xy+y^2)$$

$$2(x+y)(x-y) - (x+y)y$$

$$= (x+y)[2(x-y)-y]$$

$$= \lim (2x-3y) = -5$$

#27

$$\lim_{(x,y)\to(1,2)} \frac{\sqrt{y}-\sqrt{x+1}}{y-x-1} \cdot \frac{\sqrt{y}+\sqrt{x+1}}{\sqrt{y}+\sqrt{x+1}} = \lim \frac{1}{\sqrt{y}+\sqrt{x+1}} = \frac{1}{2\sqrt{2}}$$

#30 $\frac{0}{0}$

$$\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2} = \qquad y = mx$$

$$\frac{4x\cdot mx}{3x^2+(mx)^2} = \frac{4mx^2}{(3+m^2)x^2}$$

$$= \frac{4m}{3+m^2} = \begin{cases} 0, & m=0 \\ 1, & m=1 \end{cases} \quad \text{DNE}$$

#33

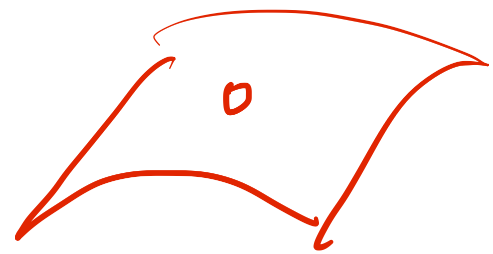
$$\lim_{(x,y)\to(0,0)} \frac{y^3+x^3}{xy^2} \qquad y = mx$$

$$\frac{m^3x^3+x^3}{x\cdot m^2x^2} = \frac{m^3+1}{m^2} = \begin{cases} 2, & m=1 \\ 0, & m=-1 \end{cases}$$

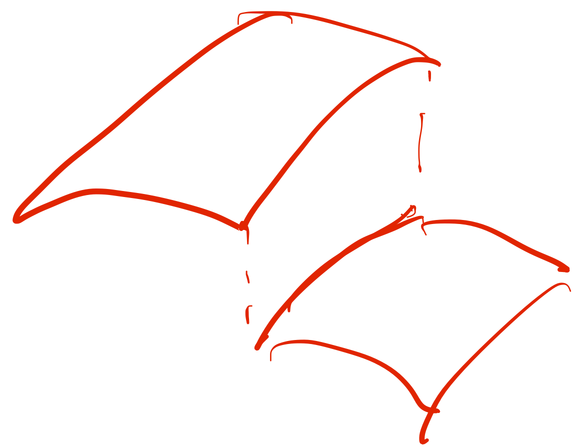
DNE

Definition Continuity

The function *f* is continuous at the point (*a*, *b*) provided



1. *f* is defined at (*a*, *b*).
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$



Example 4

$$f(x,y)=\begin{cases}\dfrac{3xy^2}{x^2+y^4}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Determine the pts at which $f(x,y)$ is continuous.

Figure 15.28

$$\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^4} \stackrel{?}{=} 0$$

$y = mx$: $\dfrac{3m^2x^3}{x^2+m^4x^4} = \dfrac{3m^2x}{1+m^4x^2} \to 0$

$x = my^2$: $\dfrac{3\cdot my^2\cdot y^2}{m^2y^4+y^4} = \dfrac{3m}{m^2+1} = \begin{cases}0, & m=0\\ \frac{3}{2}, & m=1\end{cases}$

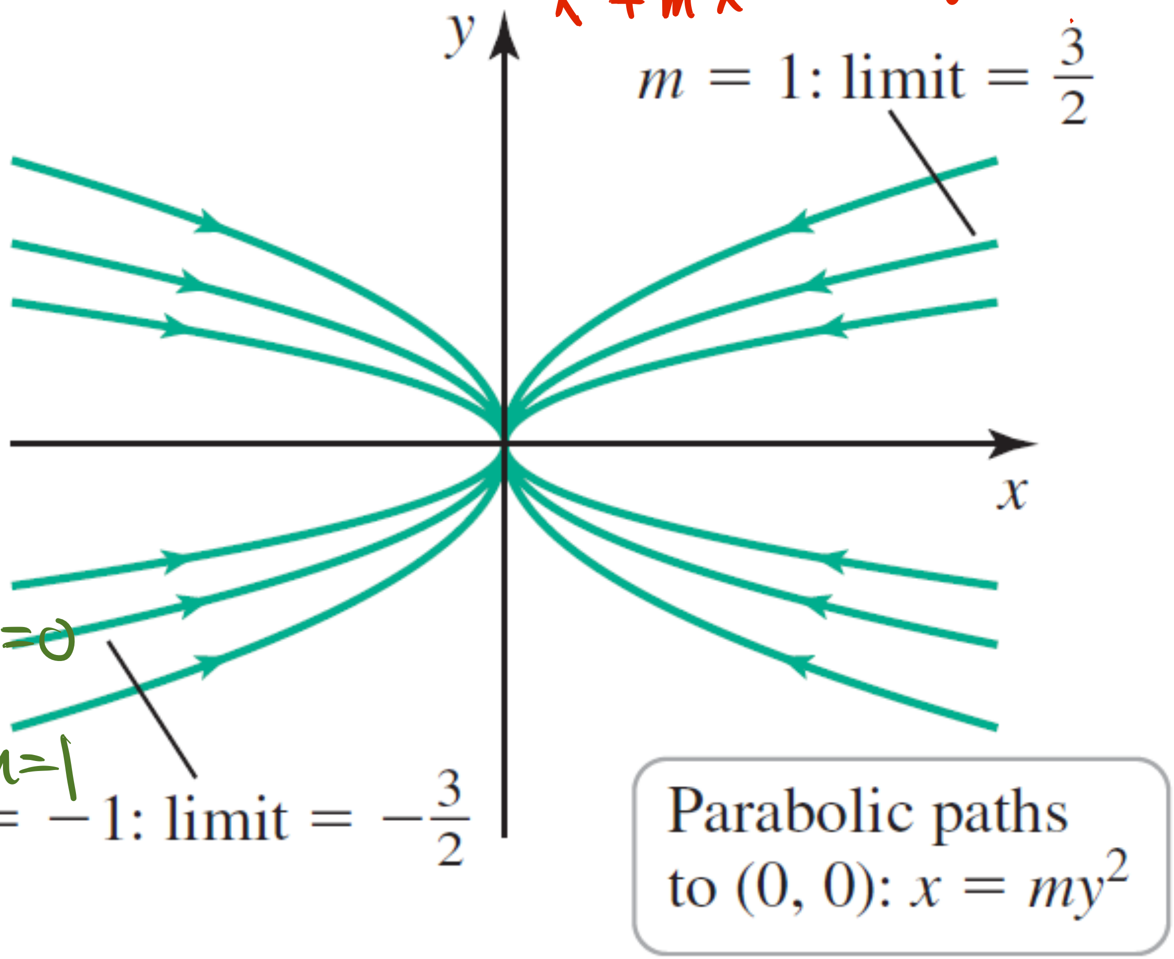
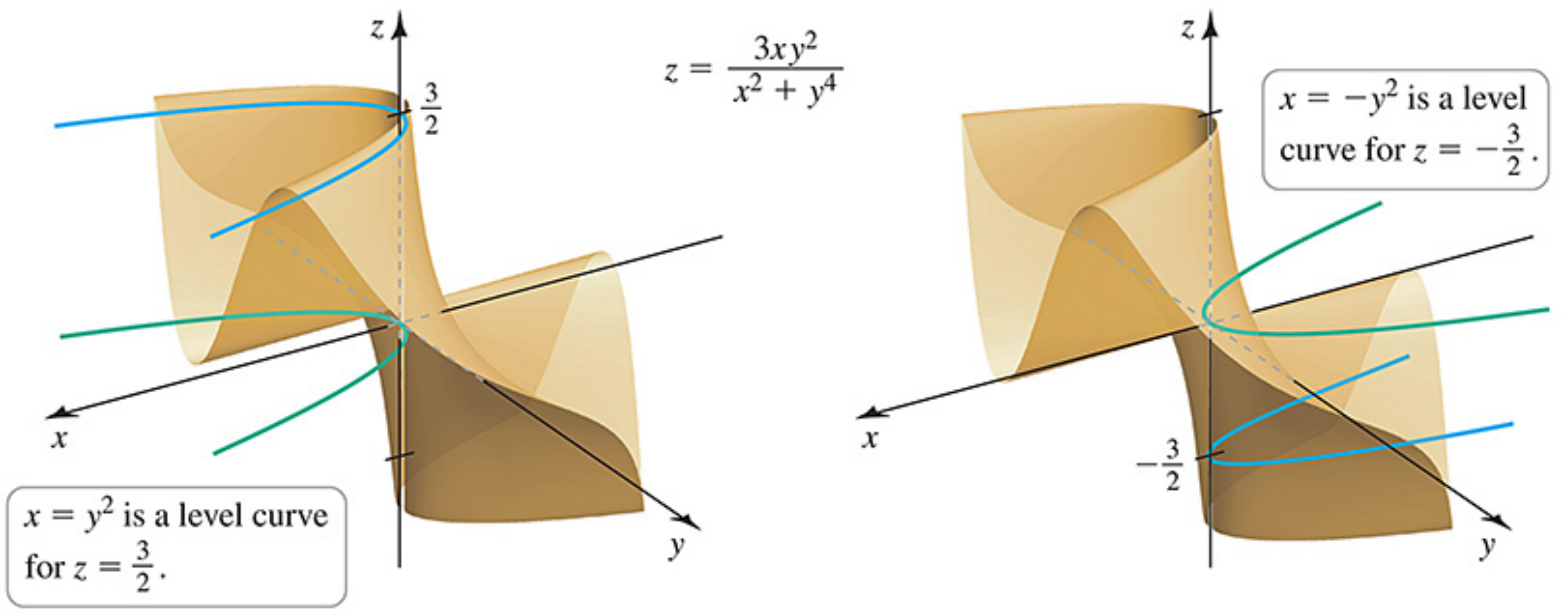


Figure 15.29



z
$\frac{3}{2}$
$z = \frac{3xy^2}{x^2 + y^4}$
x
y
$x = y^2$ is a level curve for $z = \frac{3}{2}$.
z
$x = -y^2$ is a level curve for $z = -\frac{3}{2}$.
x
$-\frac{3}{2}$
y

At what points of R^2 are the following functions continuous?

#42 $$f(x,y)=\begin{cases}\dfrac{y^4-2x^2}{y^4+x^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\lim_{\substack{\text{along}\\ x=my^2}} \frac{y^4-2x^2}{y^4+x^2} = \lim \frac{y^4-2m^2y^4}{y^4+m^2y^4} = \frac{1-2m^2}{1+m^2} = \begin{cases}1, & m=0\\ 0, & m=1\end{cases}$$ DNE

f is cont. for all $(x,y)\neq(0,0)$

#54 $$f(x,y)=\begin{cases}\dfrac{1-\cos(x^2+y^2)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\lim \frac{1-\cos(x^2+y^2)}{x^2+y^2} \overset{?}{=} 0$$

$x=r\cos\theta$

$y=r\sin\theta$

$$= \lim_{r\to0} \frac{1-\cos r^2}{r^2} = \lim_{r\to0} \frac{2r\sin r^2}{2r} = 0$$

f is cont. everywhere.

Theorem 15.3 Continuity of Composite Functions

If $u = g(x, y)$ is continuous at (*a*, *b*) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (*a*, *b*).

L'hopital's Rule

$\frac{0}{0}$: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

#49 $f(x,y) = \ln(x^2+y^2)$

$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)}$ $\frac{0}{0}$

- $a^2-b^2=(a+b)(a-b)$
- $= \lim_{u\to 0} \frac{\hat{f}(u)}{\hat{g}(u)}$
- DNE $\lim_{\text{along a path}_1} f \neq \lim_{\text{along a path}_2} f$

$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2} = \lim_{u\to 0} \frac{1-\cos u}{u} = \lim_{u\to 0} \frac{\sin u}{1} = 0$

$x^2+y^2 = u \to 0$

C: $y = mx$

$y = mx^2$

Section 15.3 Partial Derivatives

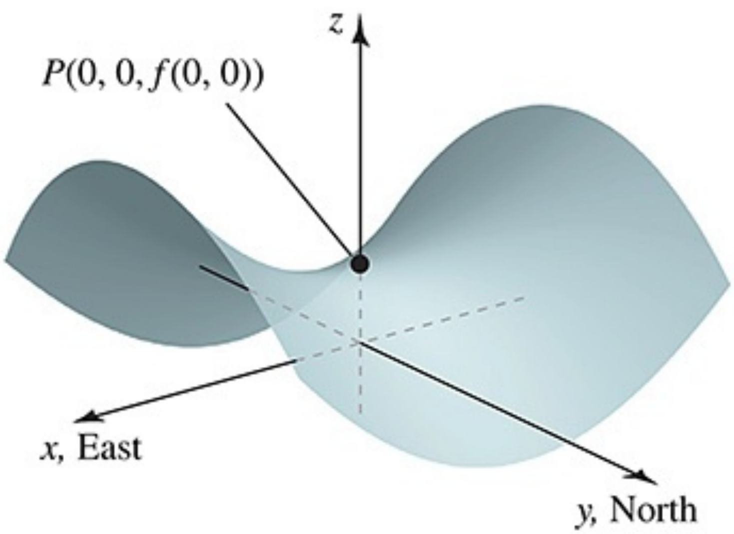
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$f(x, y)$

$$\lim_{\|(h_1, h_2)\| \to 0} \frac{f(x+h_1, y+h_2) - f(x, y)}{\|(x+h_1, y+h_2) - (x, y)\|}$$

$\rightarrow (h_1, h_2)$

Figure 15.30 (a & b)



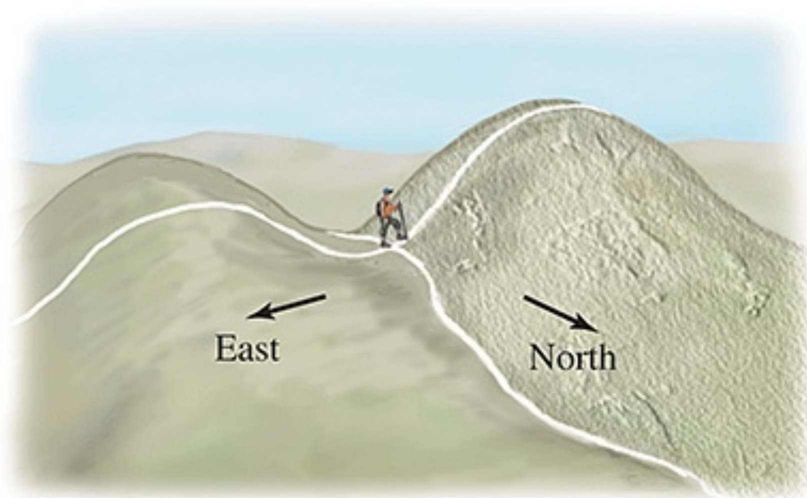


Figure 15.31

$$\begin{cases} z = f(x, y) \\ y = b \end{cases}$$

$\frac{\partial f}{\partial x}(a, b) =$

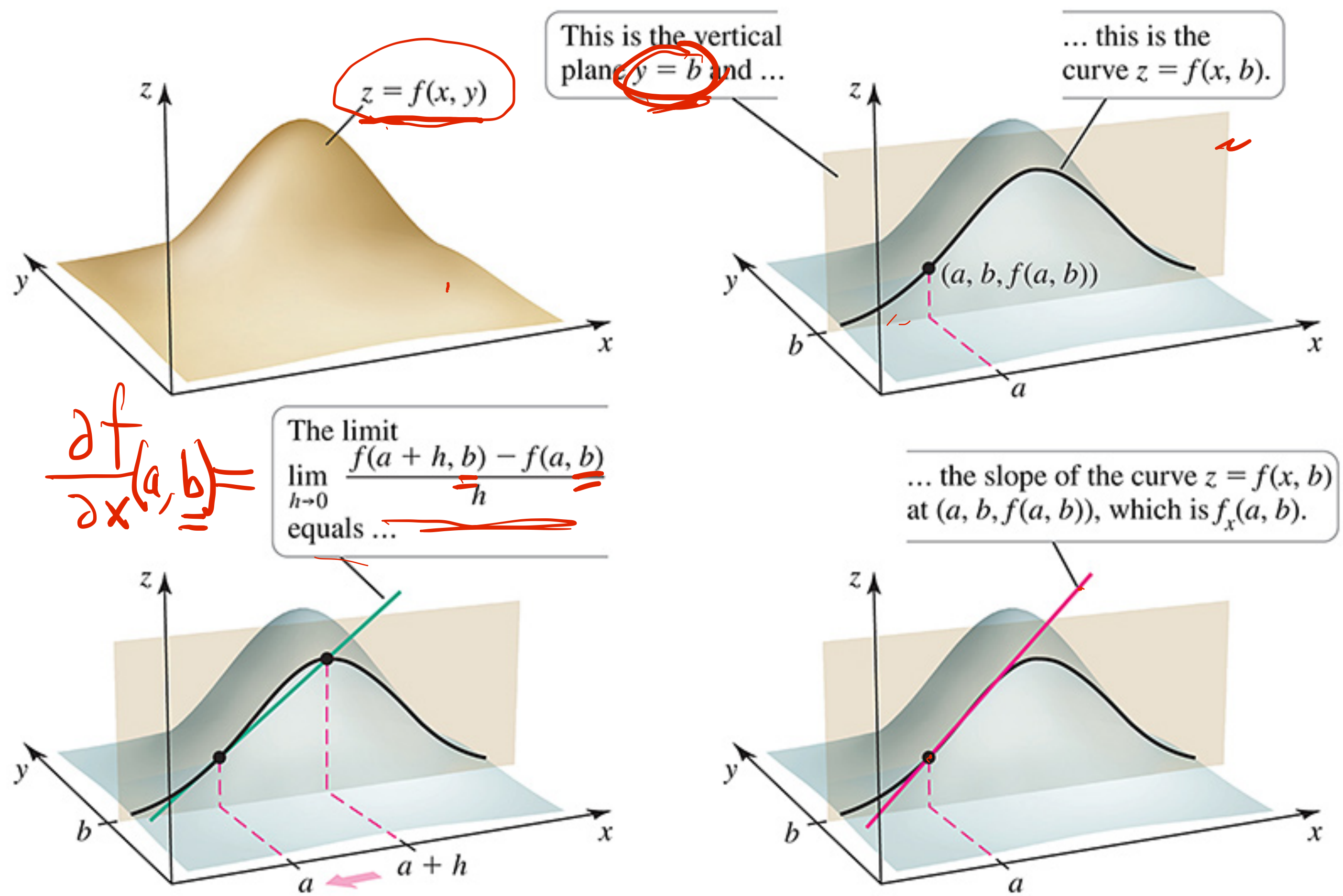
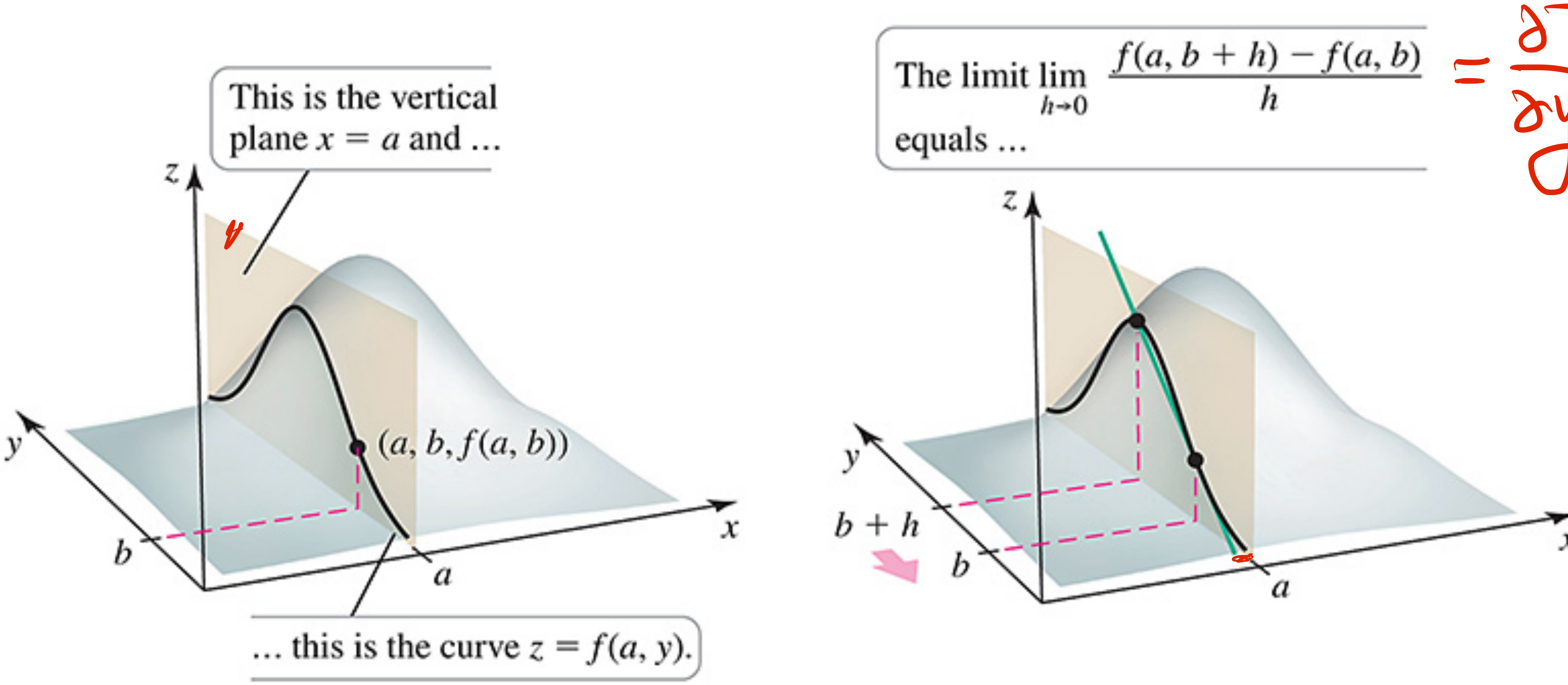


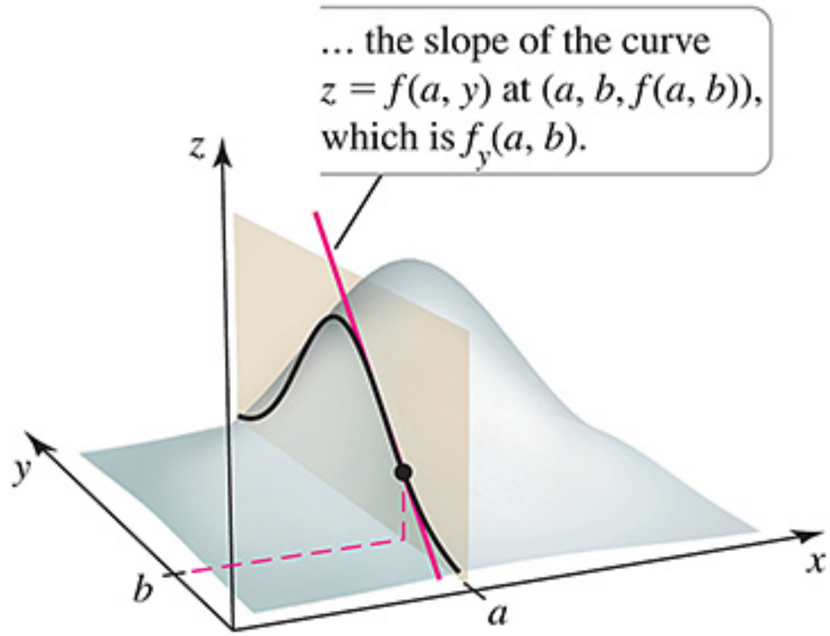
Figure 15.32 (1 of 2)

$$\begin{cases} z = f(x, y) \\ x = a \end{cases}$$



$$= \frac{\partial f}{\partial y}(a, b)$$

Figure 15.32 (2 of 2)



Definition Partial Derivatives

The **partial derivative of *f* with respect to *x* at the point (*a*, *b*)** is

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

The **partial derivative of *f* with respect to *y* at the point (*a*, *b*)** is

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

provided these limits exist.

#16 $f(x,y) = 4x^3y^2 + 3x^2y^3 + 10$

$$\frac{\partial f}{\partial x} = 12x^2y^2 + 6xy^3$$

$$\frac{\partial f}{\partial y} = 8x^3y + 9x^2y^2$$

$$(y^{-2})' = -2y^{-3}$$

$$(\tan^{-1}x)' = \frac{1}{1+x^2}$$

#22 $f(x,y) = \tan^{-1}\left(\frac{x^2}{y^2}\right)$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x^2}{y^2}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{x^2}{y^2}\right)$$

$$= \frac{y^4}{y^4+x^4} \cdot \frac{2x}{y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x^2}{y^2}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{x^2}{y^2}\right)$$

$$= \frac{y^4}{y^4+x^4}\, x^2 \cdot (-2y^{-3})$$

$$f(x) = \int_{h(x)}^{g(x)} l(t)\,dt$$

$$f'(x) = l(g(x)) \cdot g'(x) - l(h(x)) \cdot h'(x)$$

#31 $f(x,y) = \int_{x}^{y^3} e^{t^2}\,dt$

$$\frac{\partial f}{\partial x} = e^{(y^3)^2} \cdot \frac{\partial y^3}{\partial x} - e^{x^2} \cdot \frac{\partial x}{\partial x}$$

$$= 0 - e^{-x^2}$$

$$\frac{\partial f}{\partial y} = e^{(y^3)^2} \cdot \frac{\partial y^3}{\partial y} - e^{x^2} \cdot \frac{\partial x}{\partial y} = 3y\,e^{y^6} - 0$$

Table 15.3

$f(x)$, $f'(x)$, $f''(x) = (f'(x))' = \frac{d^2f}{dx^2}$

$f(x,y)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

Notation 1	Notation 2	What we say …
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	*d* squared *f dx* squared or *f* − *x* − *x*
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	*d* squared *f dy* squared or *f* − *y* − y
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$	$(f_y)_x = f_{yx}$	*f* − *y* − x
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$	$(f_x)_y = f_{xy}$	*f* − *x* − y

mixed 2nd-order der.

Theorem 15.4 (Clairaut) Equality of Mixed Partial Derivatives

Assume *f* is defined on an open set *D* of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout *D*. Then $f_{xy} = f_{yx}$ at all points of *D*.

#38 $f(x,y) = x^2 \sin y$

$\frac{\partial f}{\partial x} = 2x \sin y \rightarrow \frac{\partial^2 f}{\partial x^2} = 2\sin y$

$\frac{\partial^2 f}{\partial y \partial x} = 2x\cos y$

$\frac{\partial f}{\partial y} = x^2 \cos y$

$\frac{\partial^2 f}{\partial y^2} = -x^2 \sin y$

$\frac{\partial^2 f}{\partial x \partial y} = 2x \cos y$

#39 $h(x,y) = x^3 + xy^2 + 1$

$h_x = 3x^2 + y^2 \rightarrow h_{xx} = 6x$ $\qquad h_{xy} = 2y$

$h_y = 2xy \rightarrow h_{yy} = 2x$ $\qquad h_{yx} = 2y$

#92 Show that $u(x,y) = e^{ax} \cos ay$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for any } a \in R$$

$\frac{\partial u}{\partial x} = a e^{ax} \cos ay$

$\frac{\partial^2 u}{\partial x^2} = a^2 e^{ax} \cos ay$

LHS = RHS

$\frac{\partial u}{\partial y} = -a e^{ax} \sin ay$, $\frac{\partial^2 u}{\partial y^2} = -a^2 e^{ax} \cos ay$

$LHS = a^2 e^{ax} \cos ay - a^2 e^{ax} \cos ay = 0$

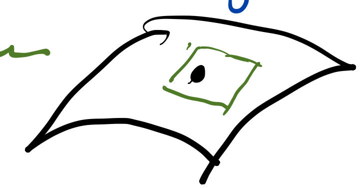
Definition Differentiability

The function $z = f(x, y)$ is **differentiable at (*a*, *b*)** provided

$f_x(a, b)$ and $f_y(a, b)$ exist and the change $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ equals

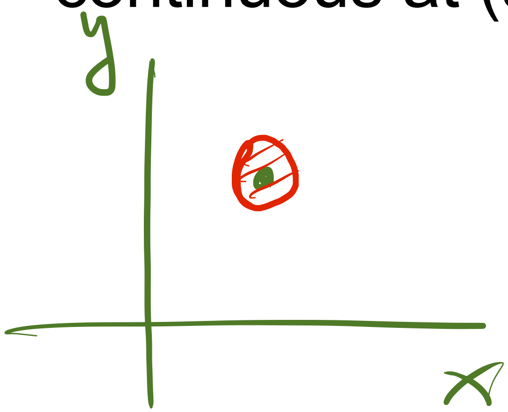
$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where for fixed *a* and *b*, ε_1 and ε_2 are functions that depend only on Δx and Δy, with $(\varepsilon_1, \varepsilon_2) \to (0,0)$ as $(\Delta x, \Delta y) \to (0,0)$. A function is **differentiable** on an open set *R* if it is differentiable at every point of *R*.



Theorem 15.5 Conditions for Differentiability

Suppose the function *f* has partial derivatives f_x and f_y defined on an open set containing (*a*, *b*), with f_x and f_y continuous at (*a*, *b*). Then *f* is differentiable at (*a*, *b*).



Theorem 15.6 Differentiable Implies Continuous

If a function f is differentiable at (a, b), then it is continuous at (a, b).

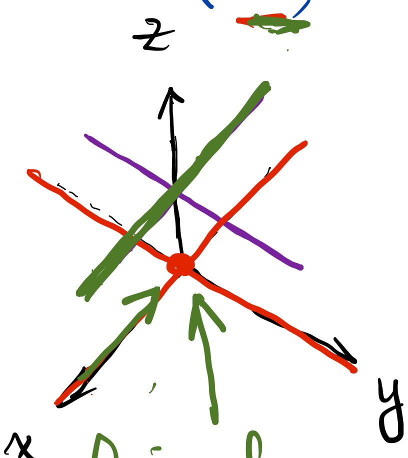
- partial derivatives and continuity ⟨ not cont. / der.

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

$xy = 0 \iff x=0$ or $y=0$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$0 = \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{1-1}{h} = 0$$



$\lim f = 1$ along $y=0$

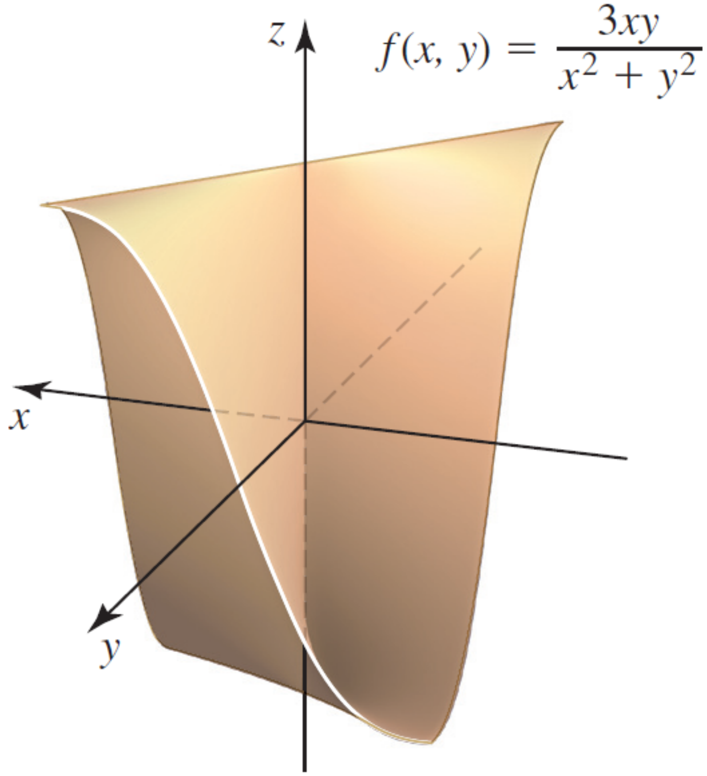
$\lim f = 0$ along $x=y$

DNE

Example 7 Discuss the differentiability and continuity of

$$f(x,y)=\begin{cases} \dfrac{3xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Figure 15.34



f is not continuous at (0, 0), even though $f_x(0, 0) = f_y(0, 0) = 0$.

Section 15.4 The Chain Rule

$y = f(x) = f(g(t))$ $\qquad \frac{dy}{dt} = f'(g(t)) \cdot g'(t)$

$x = g(t)$

$y = f(g(h(t)))$, $\quad \frac{dy}{dt} = f'(\ )\,g'(\ )\,h'(\ ) = \frac{df}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

One independent variable $z = f(x, y) = f(g(t), h(t))$

$$\begin{cases} x = g(t) \\ y = h(t) \end{cases}$$

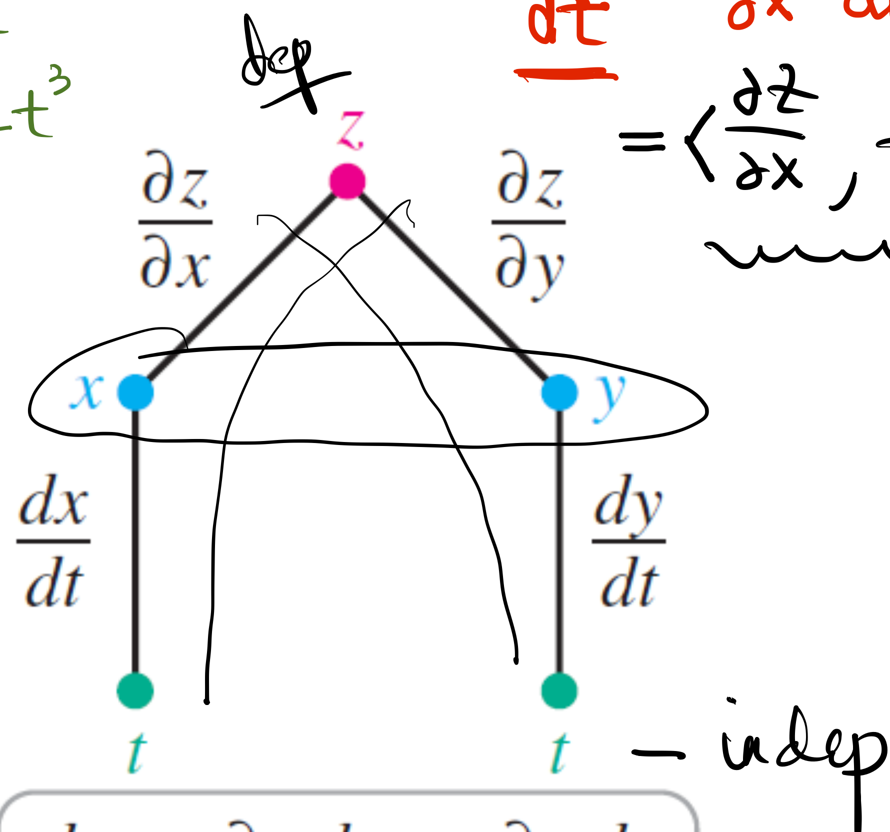
t - indep.
x, y -
z - dep

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

Figure 15.36

#9 $z = x\sin y$ $\quad \begin{cases} x = t^2 \\ y = 4t^3 \end{cases}$

$\frac{dz}{dt} =$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Theorem 15.7 Chain Rule (One Independent Variable)

Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

#13 $z = (x+2y)^{10}$, $\begin{cases} x = \sin^2 t \\ y = (3t+4)^5 \end{cases}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= 10(x+2y)^9 \cdot 1 \cdot 2\sin t \cdot \cos t$$

$$+ 10(x+2y)^9 \cdot 2 \cdot 5(3t+4)^4 \cdot 3$$

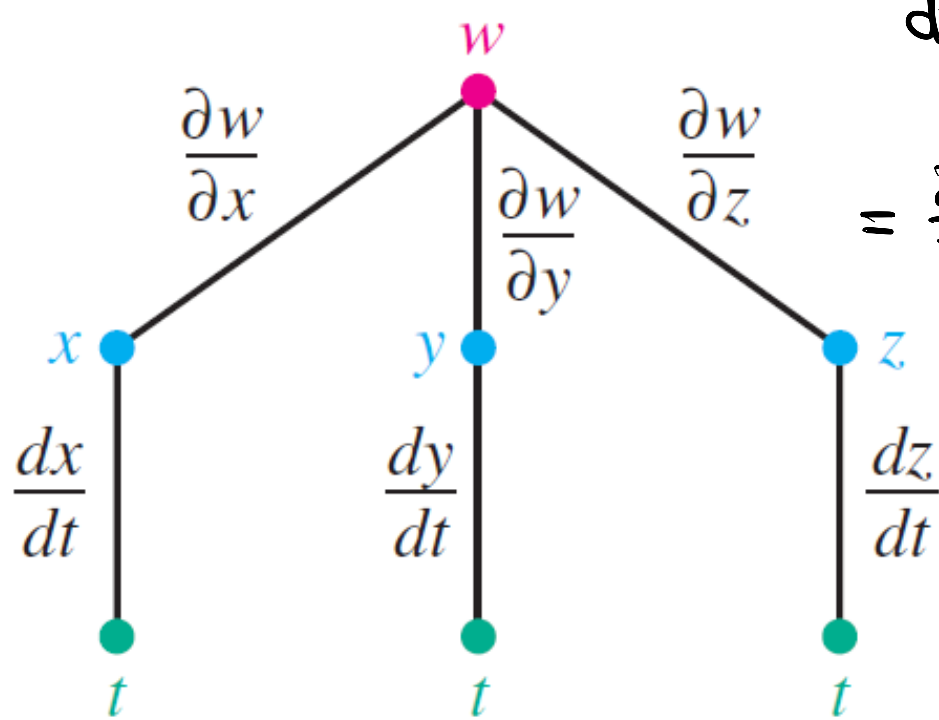
$$= 10(x+2y)^9\left[2\sin t\cos t + 30(3t+4)^4\right]$$

$w = f(x, y, z)$

$$\begin{cases} x = g(t) \\ y = h(t) \\ z = p(t) \end{cases}$$

Figure 15.37

$$\frac{dw}{dt} = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right\rangle \cdot \langle x', y', z' \rangle = \frac{\partial w}{\partial x} \cdot x' + \frac{\partial w}{\partial y} \cdot y' + \frac{\partial w}{\partial z} \cdot z'$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example 1 $w = x^2 - 3y^2 + 20 + e^{3z}$

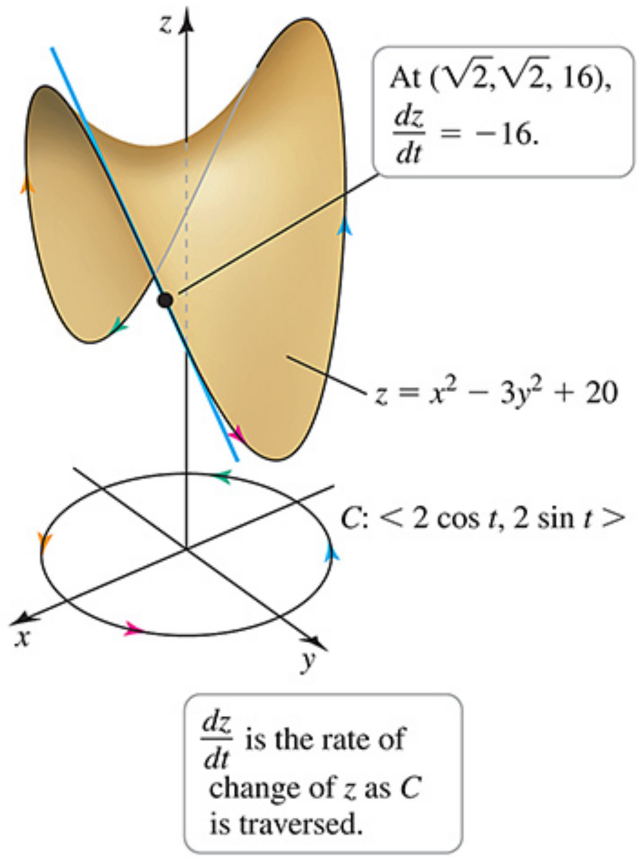
$$\begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = t^2 \end{cases}$$

$$\frac{dw}{dt} = 2x \cdot x' - 6y \cdot y' + 3e^{3z} \cdot z'$$

$$= 2x \cdot 2(-\sin t) - 6y \cdot 2\cos t + 3e^{3z} \cdot 2t$$

$$= -8\sin t\cos t - 24\sin t\cos t + 6te^{3t^2}$$

Figure 15.38



two independent variables

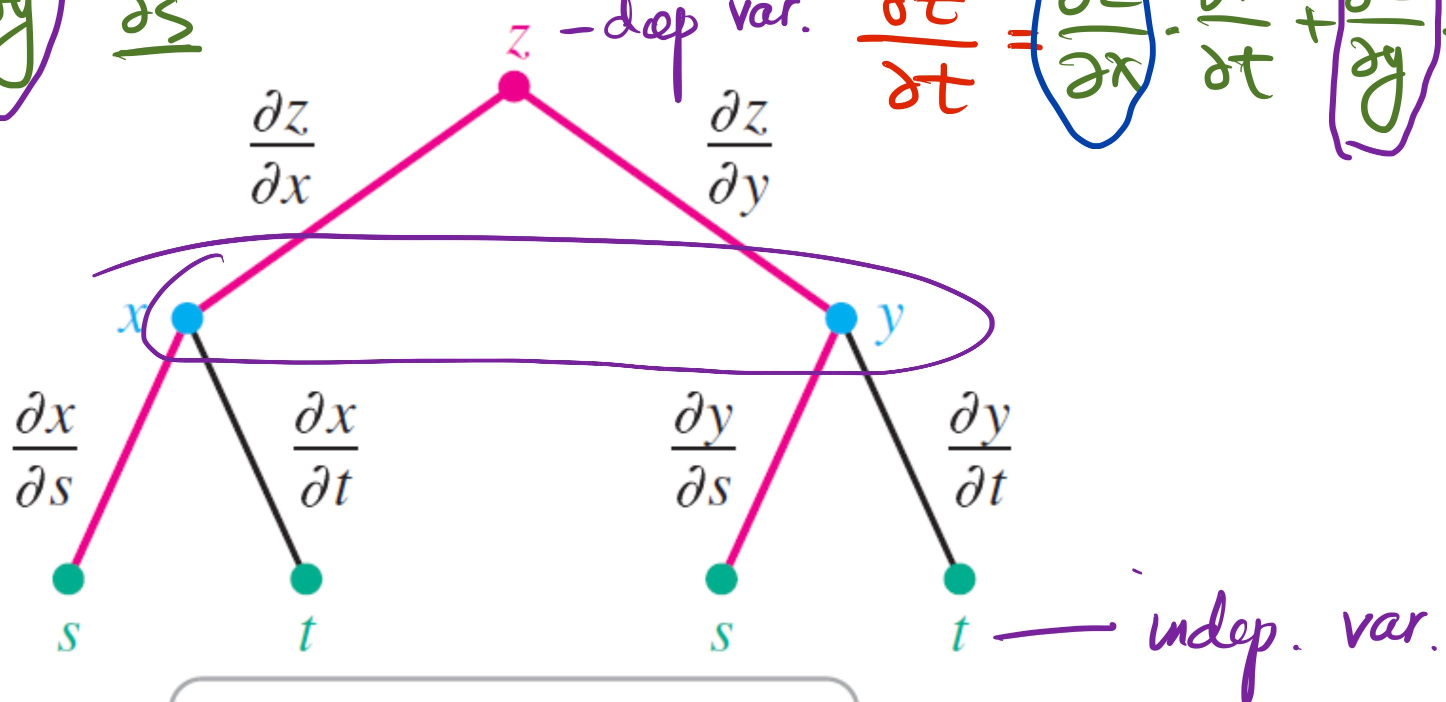
$z = f(x, y)$, $\begin{cases} x = g(s, t) \\ y = h(s, t) \end{cases}$

$= f(g(s,t), h(s,t))$

Figure 15.39

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$\frac{\partial z}{\partial x} = 2x\sin y$, $\frac{\partial z}{\partial y} = x^2 \cos y$

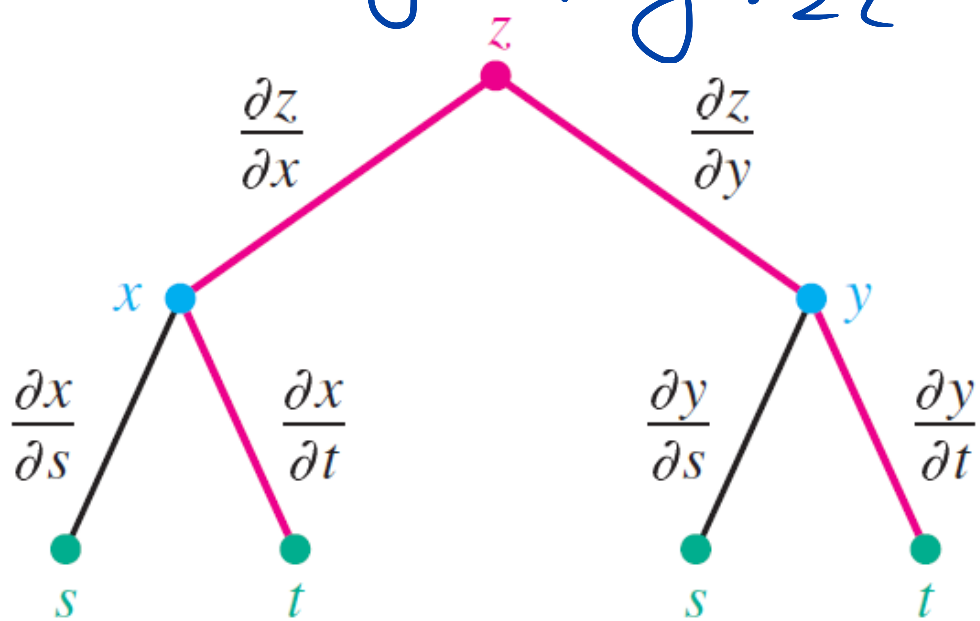
#19 $z = x^2 \sin y$, $\begin{cases} x = s - t \\ y = t^2 \end{cases}$

$\frac{\partial x}{\partial s} = 1$, $\frac{\partial x}{\partial t} = -1$

$\frac{\partial y}{\partial s} = 0$, $\frac{\partial y}{\partial t} = 2t$

Figure 15.40

$\frac{\partial z}{\partial s} = 2x\sin y$, $\frac{\partial z}{\partial t} = -2x\sin y + x^2\cos y \cdot 2t$



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Theorem 13.8 Chain Rule (Two Independent Variables)

Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \text{ and } \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

#22 $z = \sin x \cos(2y)$

$$\begin{cases} x = s^2 t \\ y = (s+t)^{10} \end{cases}$$

$\frac{\partial z}{\partial x} = \cos x \cos(2y)$, $\frac{\partial z}{\partial y} = -2 \sin x \sin(2y)$

$\frac{\partial x}{\partial s} = 2st$, $\frac{\partial x}{\partial t} = s^2$

$\frac{\partial y}{\partial s} = 10(s+t)^9$, $\frac{\partial y}{\partial t} = 10(s+t)^9$

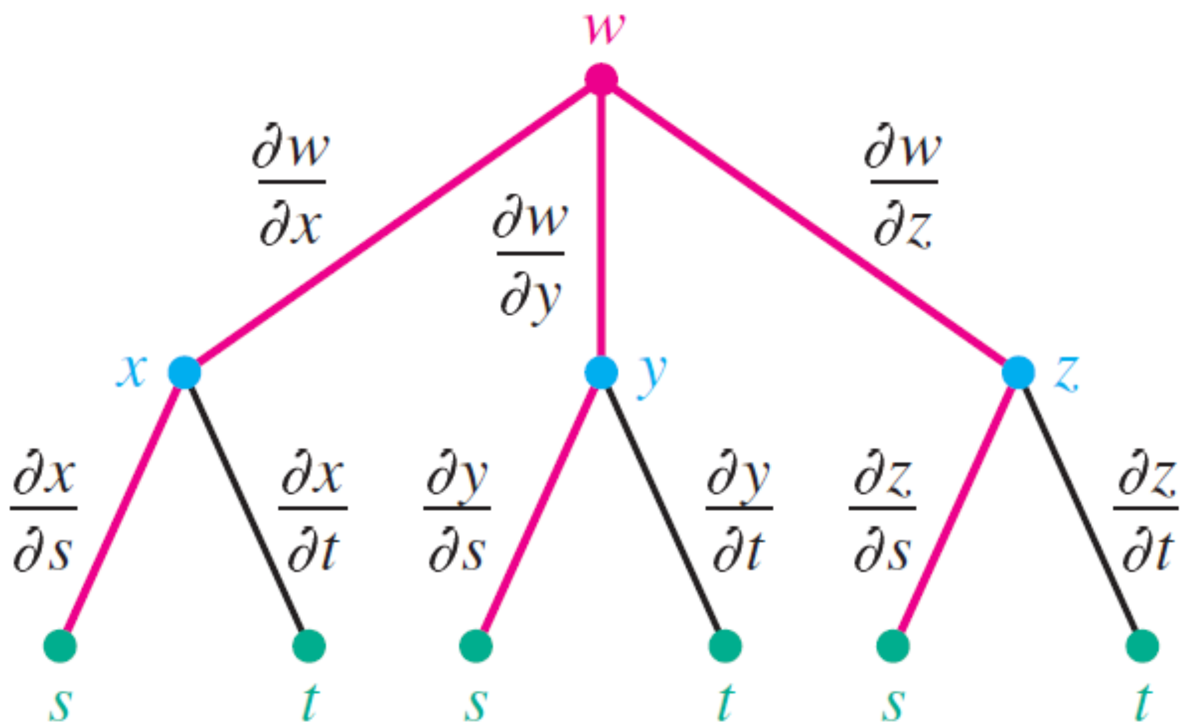
$\frac{\partial z}{\partial s} = 2st \cos x \cos(2y) - 20(s+t)^9 \sin x \sin(2y)$

$\frac{\partial z}{\partial t} = s^2 \cos x \cos(2y) - 20(s+t)^9 \sin x \sin(2y)$

$w = f(x, y, z)$

$$\begin{cases} x = g(s,t) \\ y = h(s,t) \\ z = p(s,t) \end{cases}$$

Figure 15.41

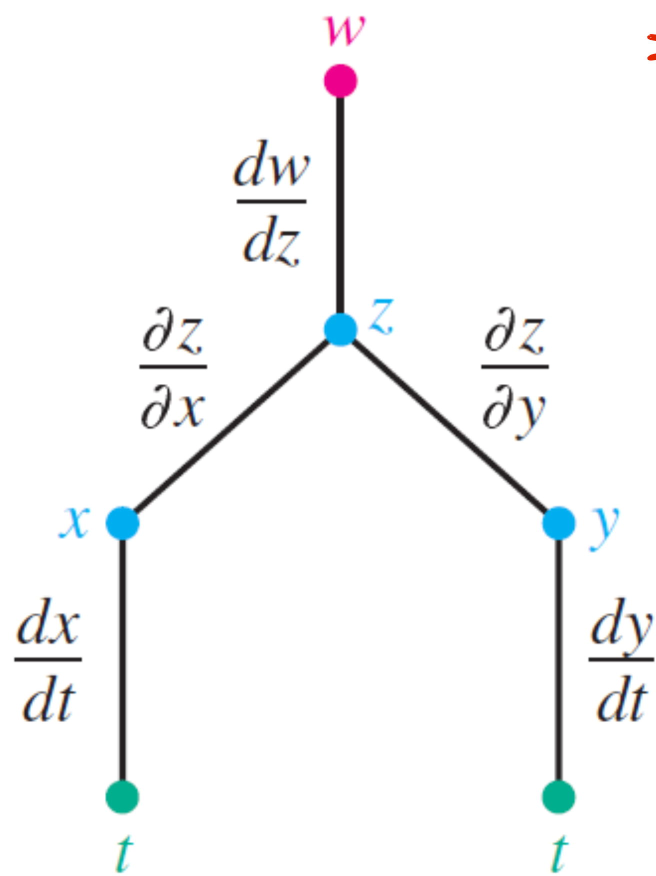


$w = f(z)$, $z = g(x, y)$, $\begin{cases} x = h(t) \\ y = p(t) \end{cases}$

$w = f(g(h(t), p(t)))$

$$\frac{dw}{dt} = \frac{dw}{dz} \cdot \frac{dz}{dt} = \frac{dw}{dz}\left[\frac{\partial z}{\partial x} \cdot x' + \frac{\partial z}{\partial y} \cdot y'\right]$$

Figure 15.42



Theorem 15.9 Implicit Differentiation

Let F be differentiable on its domain and suppose

$F(x, y) = 0$ defines y as a differentiable function of x.

Provided $F_y \neq 0$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

#38 $F(x,y) = ye^{xy} - 2 = 0$, compute $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y^2e^{xy}}{xye^{xy}} = -\frac{y}{x}$

$\frac{\partial F}{\partial x} = ye^{xy} \cdot y = y^2e^{xy}$

Figure 15.43

$\frac{\partial F}{\partial y} = ye^{xy} \cdot x = xye^{xy}$

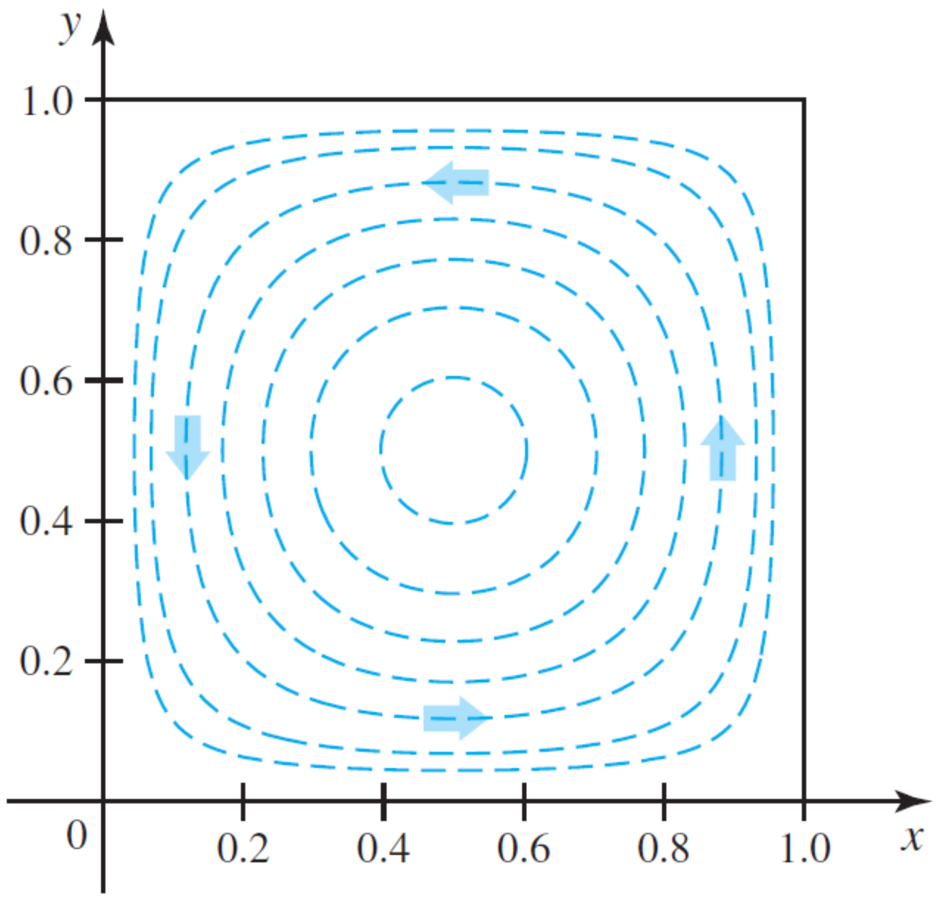
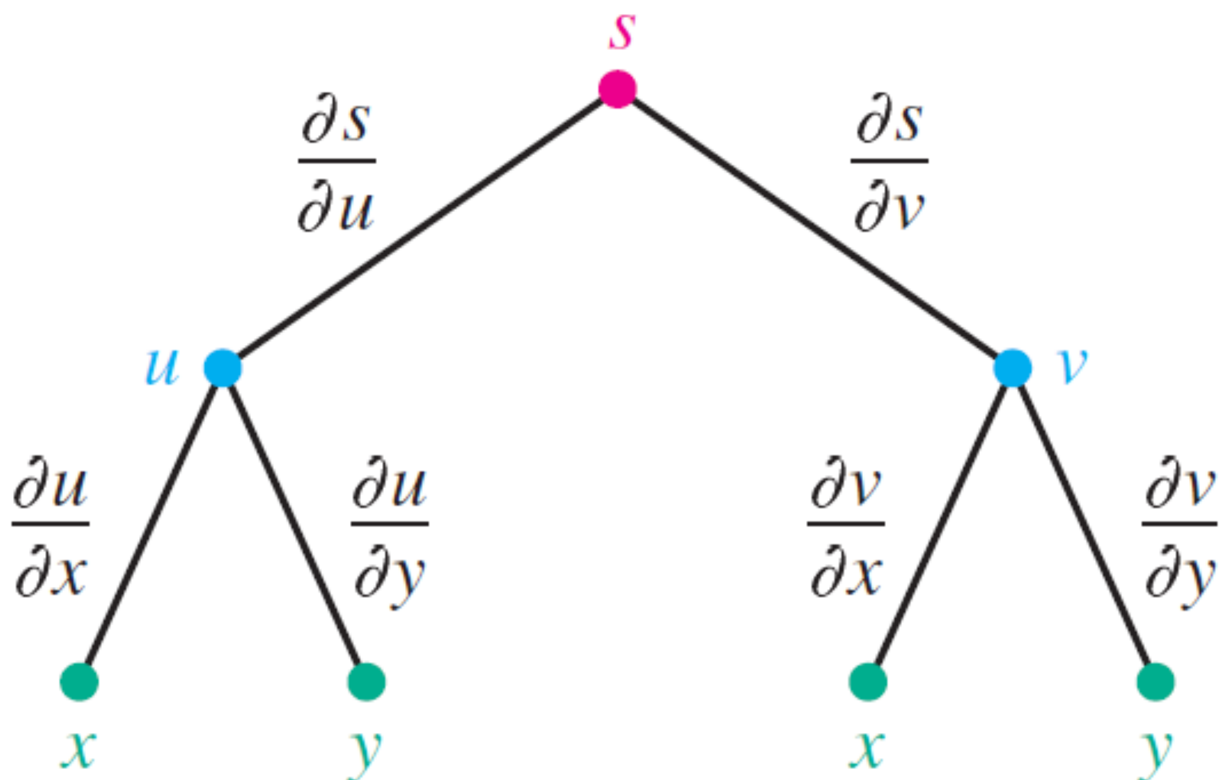


Figure 15.44

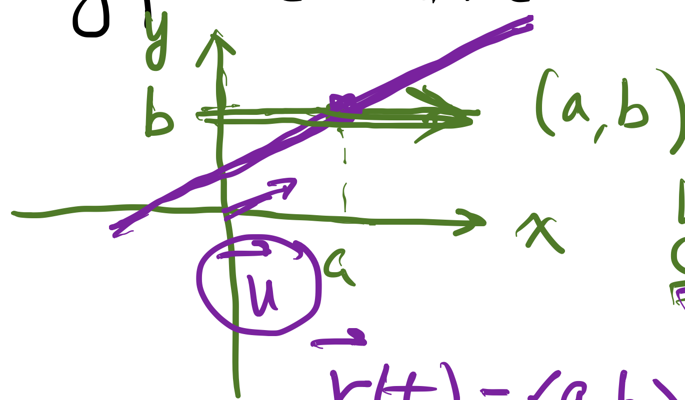

s
$\frac{\partial s}{\partial u}$
$\frac{\partial s}{\partial v}$
u
v
$\frac{\partial u}{\partial x}$
$\frac{\partial u}{\partial y}$
$\frac{\partial v}{\partial x}$
$\frac{\partial v}{\partial y}$
x
y
x
y

Section 15.5 Directional Derivatives and the Gradient

What is the rate of change of $z = f(x, y)$ along positive x-direction at (a, b)?

$$\frac{\partial f}{\partial x}(a, b) = \frac{d f(x, b)}{dx}\Big|_{x=a}$$

$\langle 1, 0 \rangle$



$y = b$

$$\begin{cases} z = f(x, y) \\ y = b \end{cases} \Rightarrow z = f(x, b)$$

$$\begin{cases} \vec{r}(t) = \langle a, b \rangle + t\langle u_1, u_2 \rangle \\ z = f(x, y) \end{cases}$$

What is the rate of change of $z = f(x,y)$ along the direction $\vec{u} = \langle u_1, u_2 \rangle$ at point (a,b)?

Figure 15.45

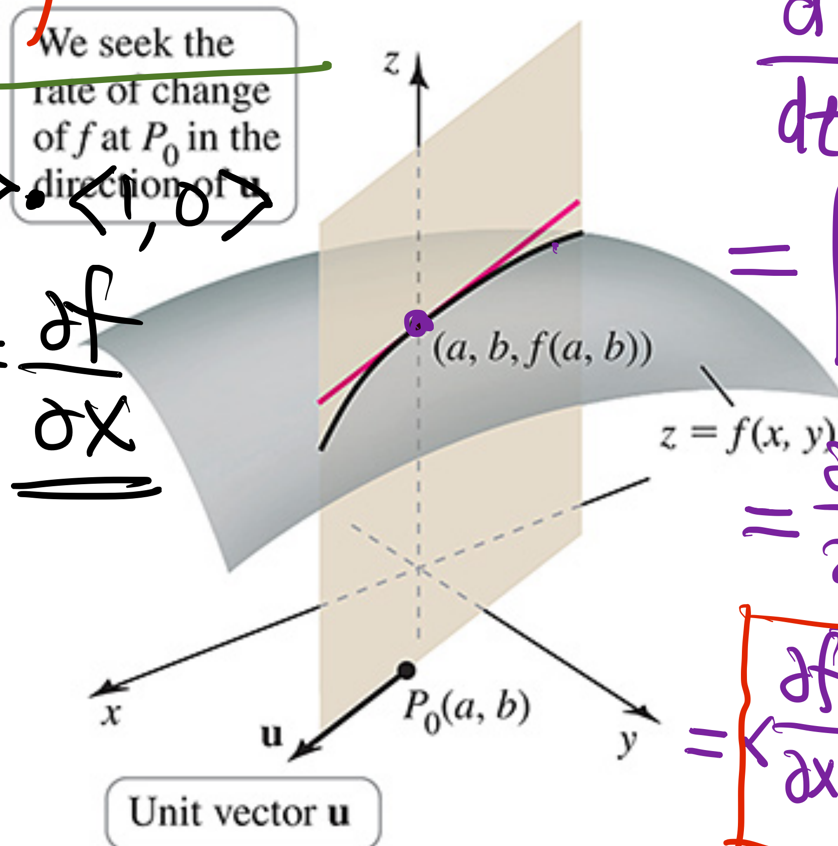
$$\begin{cases} z = f(x,y) \\ \vec{r}(t) = \langle x(t), y(t) \rangle = \langle a + t u_1, b + t u_2 \rangle \end{cases} \Longrightarrow z = f(a + t u_1, b + t u_2)$$

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

$$D_{\langle 1,0 \rangle} f(a,b) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle 1, 0 \rangle = \frac{\partial f}{\partial x}$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

gradient



$$\frac{d}{dt} f(a + t u_1, b + t u_2) \Big|_{t=0}$$

$$= \left[\frac{\partial f}{\partial x} \cdot x'(t) + \frac{\partial f}{\partial y} \cdot y'(t) \right]_{t=0}$$

$$= \frac{\partial f}{\partial x}(a,b) \cdot u_1 + \frac{\partial f}{\partial y}(a,b)\, u_2$$

$$= \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right\rangle \cdot \langle u_1, u_2 \rangle$$

gradient

Theorem 15.10 Directional Derivative

Let *f* be differentiable at (*a*, *b*) and let $\mathbf{u} = \langle u_1,\ u_2 \rangle$ be a unit vector in the xy-plane.

The **directional derivative of *f* at (*a*, *b*) in the direction of u** is

$$D_{\mathbf{u}} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle.$$

Example 1 $z = f(x,y) = \frac{1}{4}(x^2+2y^2)+2$, $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$, $\vec{v} = \frac{1}{2}\langle 1,-\sqrt{3}\rangle$

(a) $D_{\vec{u}} f(3,2) = \langle \frac{3}{2}, 2\rangle \cdot \frac{1}{\sqrt{2}}\langle 1,1\rangle = \frac{1}{\sqrt{2}}\left(\frac{3}{2}\cdot 1 + 2\cdot 1\right) = \frac{7}{2\sqrt{2}} > 0$ $\quad \nabla f = \langle \frac{1}{2}x, y\rangle$

Figure 15.48

$D_{\vec{v}} f(3,2) = \langle \frac{3}{2}, 2\rangle \cdot \frac{1}{2}\langle 1,-\sqrt{3}\rangle = \frac{1}{2}\left(\frac{3}{2}\cdot 1 + 2\cdot(-\sqrt{3})\right) = \frac{1}{4}(3-4\sqrt{3}) < 0$ $\quad \nabla f(3,2) = \langle \frac{3}{2}, 2\rangle$

(b) interpretation

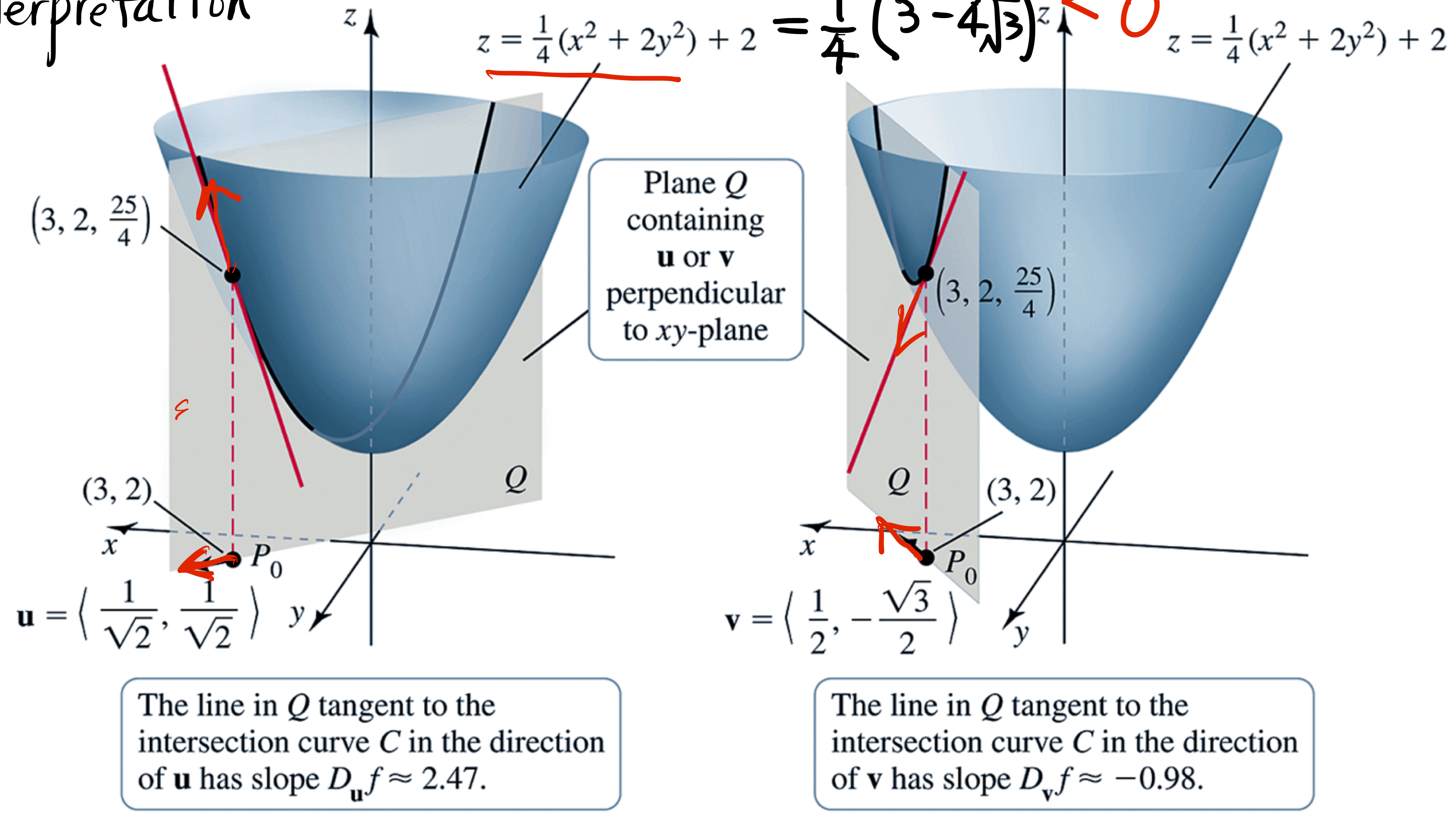


Figure 15.49

Example 3 $f(x,y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$

(a) $\nabla f(3,-1)$, (b) $\vec{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$, $D_{\vec{u}} f(3,-1) = ?$

(c) $\vec{v} = \langle 3, 4\rangle$ $D_{\vec{v}} f(3,-1) = ?$

(a) $\nabla f = \langle -\frac{2x}{10} + \frac{y^2}{10}, \frac{2xy}{10}\rangle$

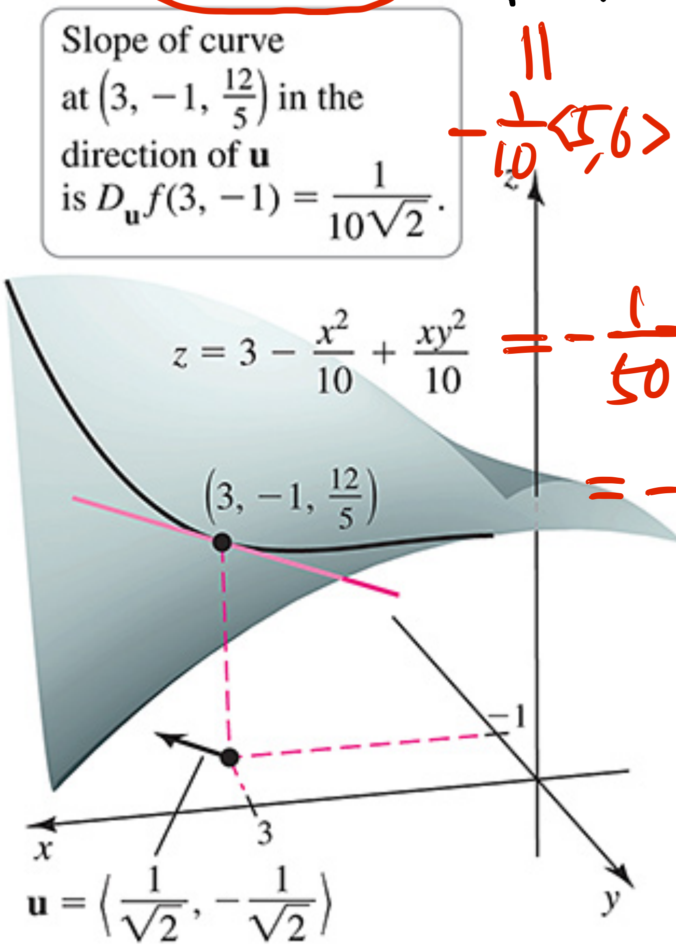
$= \frac{1}{10}\langle -2x + y^2, 2xy\rangle$

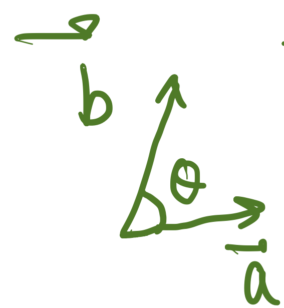
$\nabla f(3,-1) = \frac{1}{10}\langle -6+1, -6\rangle$

$= -\frac{1}{10}\langle 5, 6\rangle$

(b) $= -\frac{1}{10}\langle 5,6\rangle \cdot \frac{1}{\sqrt{2}}\langle 1,1\rangle = -\frac{11}{10\sqrt{2}}$

(c) $= -\frac{1}{10}\langle 5,6\rangle \cdot \frac{\langle 3,4\rangle}{\sqrt{3^2+4^2}} = -\frac{1}{50}(15+24) = -\frac{39}{50}$





$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Question: In which direction at (a,b) does the value of $f(x,y)$ increase most rapidly?

$\vec{u} = \langle u_1, u_2 \rangle = ?$

$$\max_{\vec{u}} D_{\vec{u}} f(a,b) = \max_{\vec{u}} \left(\nabla f(a,b) \cdot \vec{u} \right)$$

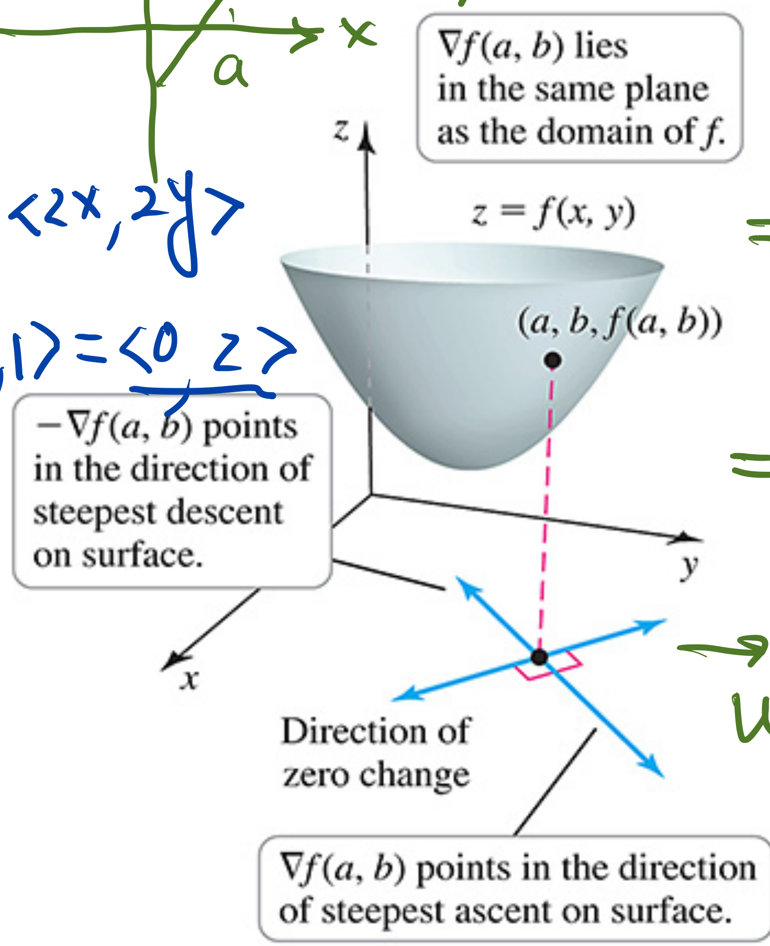
$$= \max_{\vec{u}} |\nabla f(a,b)| \, |\vec{u}| \cos\theta$$

$$= \max_{\vec{u}} |\nabla f(a,b)| \cos\theta$$

$$= |\nabla f(a,b)| \max_{\vec{u}} \cos\theta$$

$$\vec{u} = \frac{\nabla f(a,b)}{|\nabla f(a,b)|}$$

Figure 15.50



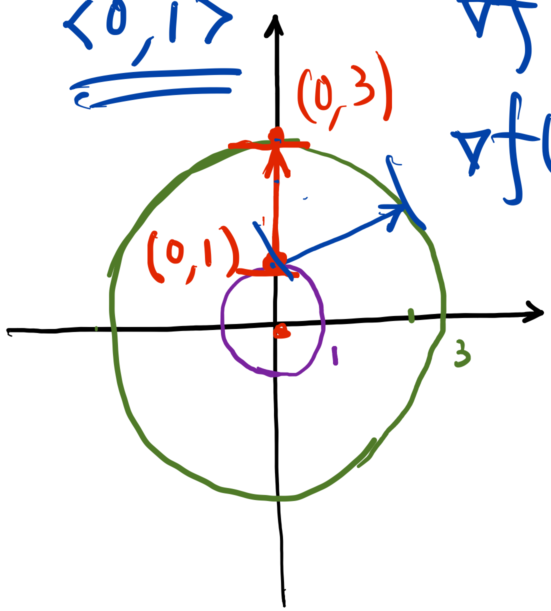
Example $z = f(x,y) = x^2 + y^2$

level curves $x^2 + y^2 = k^2$

$\nabla f = \langle 2x, 2y \rangle$

$\nabla f(0,1) = \langle 0, 2 \rangle$

$\langle 0, 1 \rangle$



Theorem 15.11 Directions of Change

Let *f* be differentiable at (*a*, *b*) with $\nabla f(a,b) \neq \mathbf{0}$.

1. *f* has its maximum rate of increase at *(a, b)* in the direction of the gradient $\nabla f(a,b)$. The rate of change in this direction is $|\nabla f(a,b)|$.

2. *f* has its maximum rate of decrease at (*a*, *b)* in the direction of $-\nabla f(a,b)$. The rate of change in this direction is $-|\nabla f(a,b)|$.

3. The directional derivative is zero in any direction orthogonal to $\nabla f(a,b)$.

Example 4 $z = f(x,y) = 4 + x^2 + 3y^2$

(a) At $(2, -\frac{1}{2}, \frac{35}{4})$, in which direction should you move in order to ascend at the max. rate? What is the rate of change?

(b) descend

(c) At (3, 1, 16), in which direction is there no change in the function values?

Figure 15.51 (a & b)

(a) $? = \nabla f(2, -\frac{1}{2}) = \langle 2x, 6y \rangle \big|_{(2,-\frac{1}{2})}$

$= \langle 4, -3 \rangle$

$|\nabla f(2, -\frac{1}{2})|$

$= \sqrt{4^2 + (-3)^2}$

$= 5$

(b) $-\nabla f(2, -\frac{1}{2}) = \langle -4, 3 \rangle$

$\langle a, b \rangle \cdot \langle -b, a \rangle$

$0 = \langle a, b \rangle \cdot \langle u_1, u_2 \rangle$

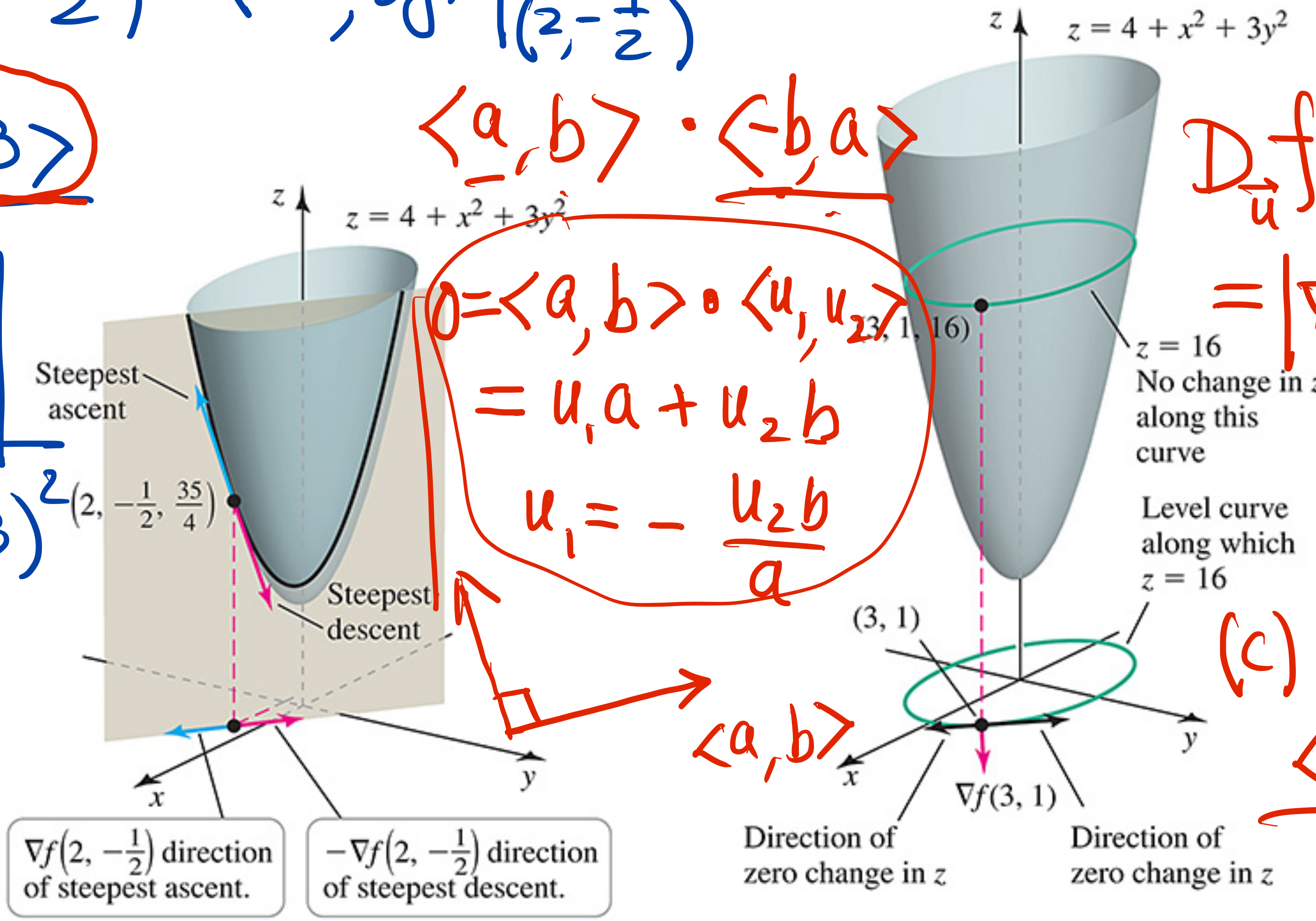
$= u_1 a + u_2 b$

$u_1 = -\frac{u_2 b}{a}$

$D_{\vec{u}} f(2, -\frac{1}{2})$

$= |\nabla f(2, -\frac{1}{2})| \cdot \cos\theta$

(c) $\langle 3, 4 \rangle$



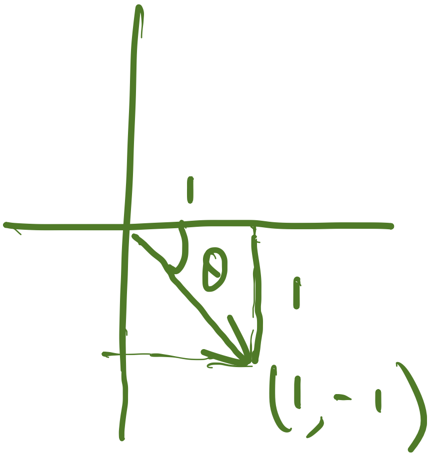
Example 5 $z = f(x,y) = 3x^2 - 2y^2$

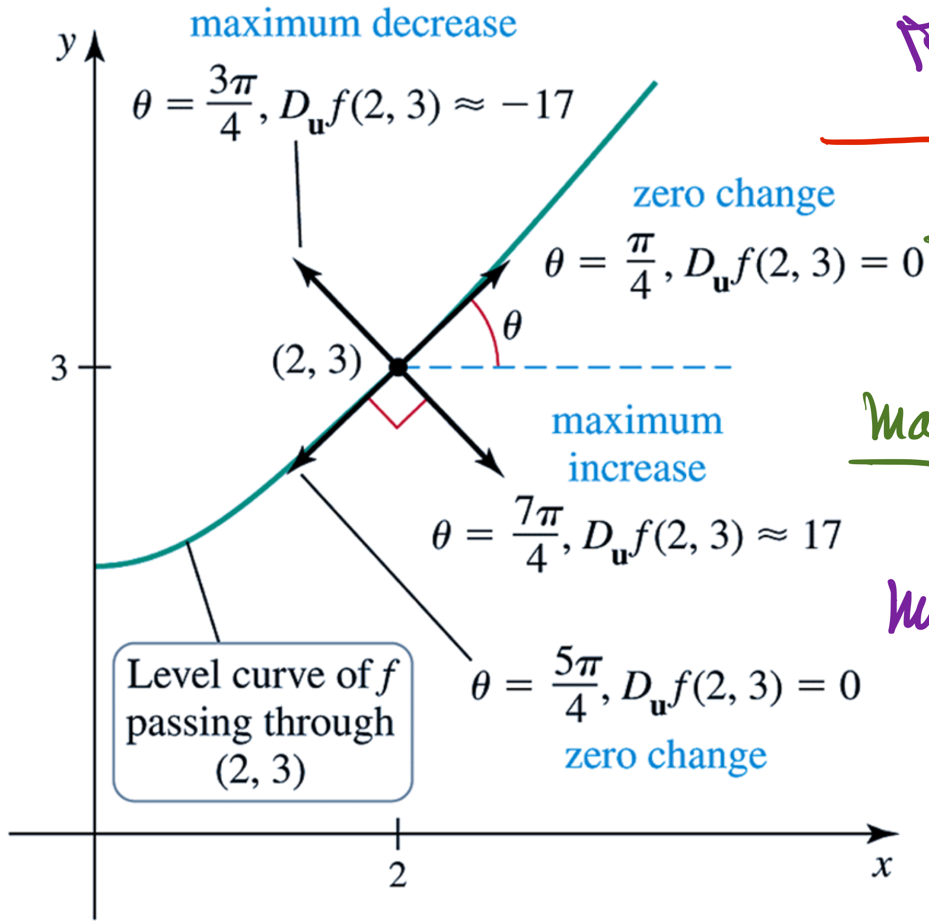
(a) $\nabla f(x,y) = \langle 6x, -4y \rangle$

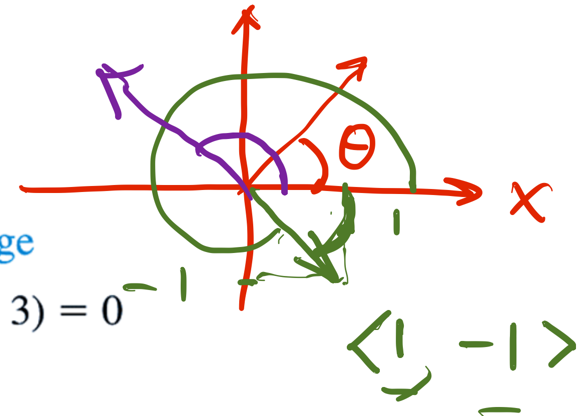
$\nabla f(2,3) = \langle 12, -12 \rangle = 12\langle 1, -1 \rangle$

Figure 15.52

(b) At $(2,3)$, $\vec{u} = \langle \cos\theta, \sin\theta \rangle$, for what values of θ (measured relative to the positive x-axis), with $0 \le \theta < 2\pi$, does the directional derivative have its maximum and minimum values? what are those values?







max $\frac{7\pi}{4}$

min $\frac{3\pi}{4}$

Theorem 15.12 The Gradient and Level Curves

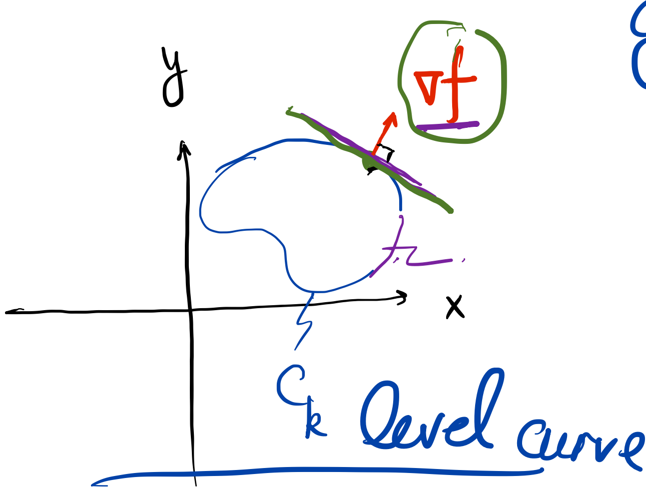
Given a function *f* differentiable at (a, b), the line tangent to the level curve of *f* at (a, b) is orthogonal to the gradient $\nabla f(a,b)$, provided $\nabla f(a,b) \neq \mathbf{0}$.

Review of Lesson 13 $f(x, y)$

gradient $\nabla f = \langle f_x, f_y \rangle$

(1)

(2) $\nabla f \perp$ level curve



directional der. $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$|\vec{u}| = 1$

Example 6 $z = f(x,y) = \sqrt{1+2x^2+y^2}$

(a) verify that $\nabla f(1,1) \perp$ level curve at (1,1).

(a) $\nabla f(1,1) = \langle f_x, f_y \rangle |_{(1,1)}$

$= \frac{1}{2}(1+2x^2+y^2)^{-\frac{1}{2}} \langle 4x, 2y \rangle |_{(1,1)}$

$= \frac{1}{4}\langle 4,2 \rangle = \langle 1, \frac{1}{2} \rangle$

$z = \sqrt{1+2+1} = 2 \Rightarrow \sqrt{1+2x^2+y^2} = 2$

$\frac{d}{dx}$ $2x^2 + y_{(x)}^2 = 3$

$4x + 2y \cdot y' = 0$

$y' = -\frac{2x}{y}$

$y'(1) = -2 \cdot 1/1 = -2$

$\vec{u} = \langle 1, -2 \rangle$

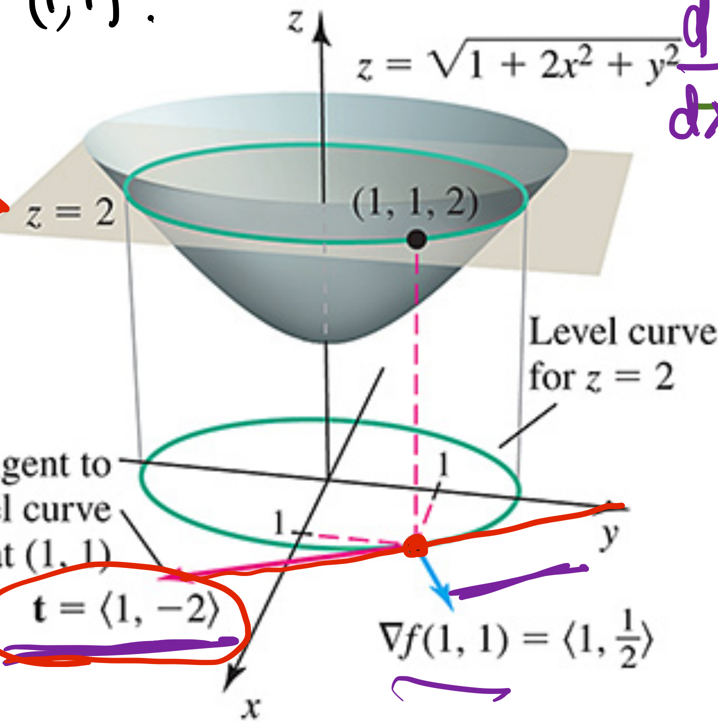
$\langle 1,-2 \rangle \cdot \langle 1, \frac{1}{2} \rangle = 0$

Figure 15.54

(b) Find an equation of the line tangent to the level curve at (1,1).

$\vec{r}(t) = \langle 1+t, 1-2t \rangle$

$= \langle 1,1 \rangle + t\vec{u}$



t · ∇f = 0
∇f is orthogonal to level curves.

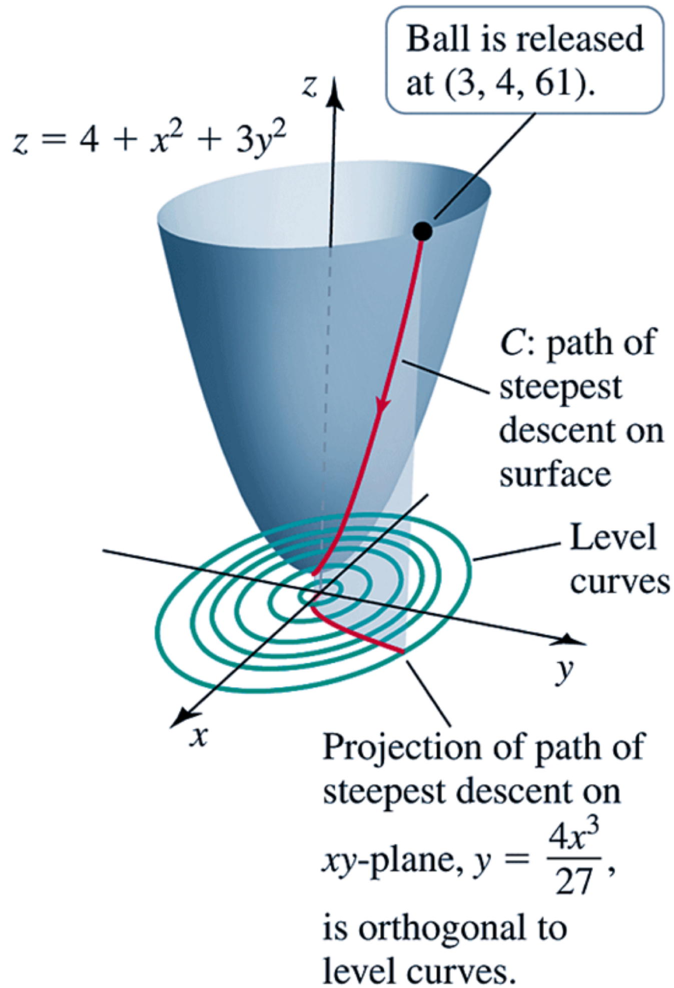
• Functions of three variables $f(x, y, z)$

$\nabla f = \langle f_x, f_y, f_z \rangle$ { • increasing most rapidly; $\nabla f \perp$ level surface }

$f(x,y,z) = 0$

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$

Figure 15.55



Definition Gradient and Directional Derivative in Three Dimensions

Let *f* be differentiable at (*a*, *b*, *c*) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The **directional derivative of *f* at (*a*, *b*, *c*) in the direction of u** is

$$D_{\mathbf{u}} f(a,b,c) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}.$$

provided this limit exists

The **gradient** of *f* at the point (*x*, *y*, *z*) is the vector-valued function

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

$$= f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}.$$

Theorem 15.13 Directional Derivative and Interpreting the Gradient

Let *f* be differentiable at (*a*, *b*, *c*) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector.

The directional derivative of *f* at (*a*, *b*, *c*) in the direction of **u** is

$$D_{\mathbf{u}} f(a,b,c) = \nabla f(a,b,c) \cdot \mathbf{u}$$

$$= \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle \cdot \langle u_1, u_2, u_3 \rangle.$$

Assuming $\nabla f(a,b,c) \neq \mathbf{0},$ the gradient in three dimensions has the following properties.

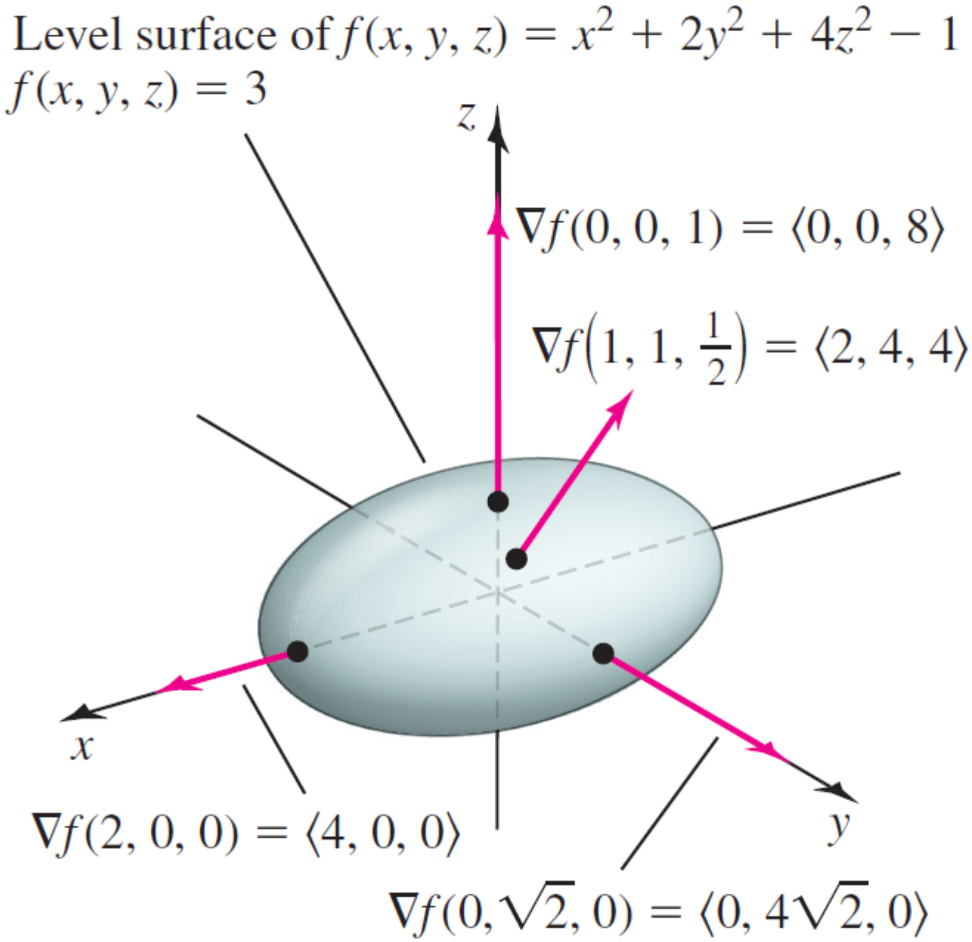
1. *f* has its maximum rate of increase at (*a*, *b*, *c*) in the direction of the gradient $\nabla f(a,b,c),$ and the rate of change in this direction is $|\nabla f(a,b,c)|.$
2. *f* has its maximum rate of decrease at (*a*, *b*, *c*) in the direction of $-\nabla f(a,b,c),$ and the rate of change in this direction is $-|\nabla f(a,b,c)|.$
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a,b,c).$

Example 8 $f(x,y,z) = x^2+2y^2+4z^2-1$, level surface $f(x,y,z)=3$

(a) Find ∇f at $P(2,0,0)$, $Q(0,\sqrt{2},0)$, $R(0,0,1)$, and $S(1,1,\frac{1}{2})$.

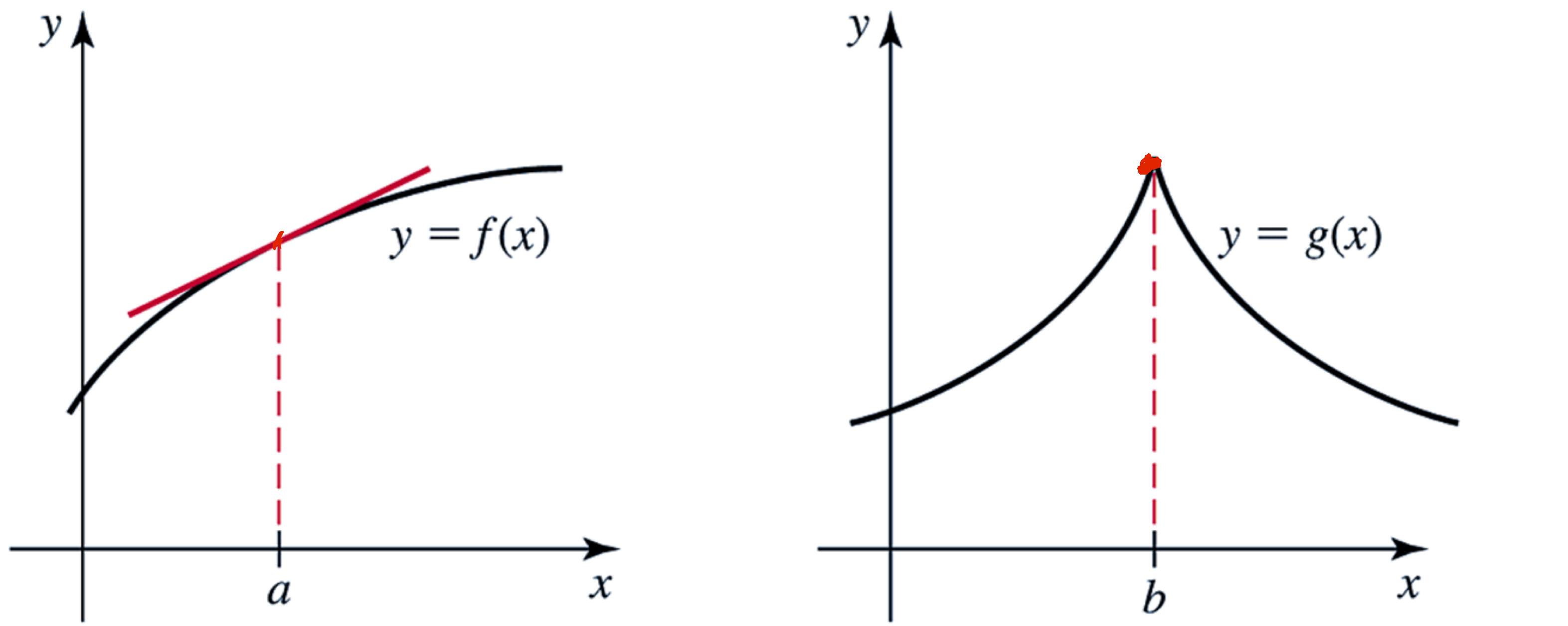
Figure 15.57

(b) What are the actual rates of change of f in the directions of the gradients in (a)?



Section 15.6 Tangent Planes and Linear Approximation

Figure 15.58 (1 of 2)



f differentiable at $a \Rightarrow$ tangent line at $(a, f(a))$

g not differentiable at $b \Rightarrow$ no tangent line at $(b, g(b))$

Figure 15.58 (2 of 2)

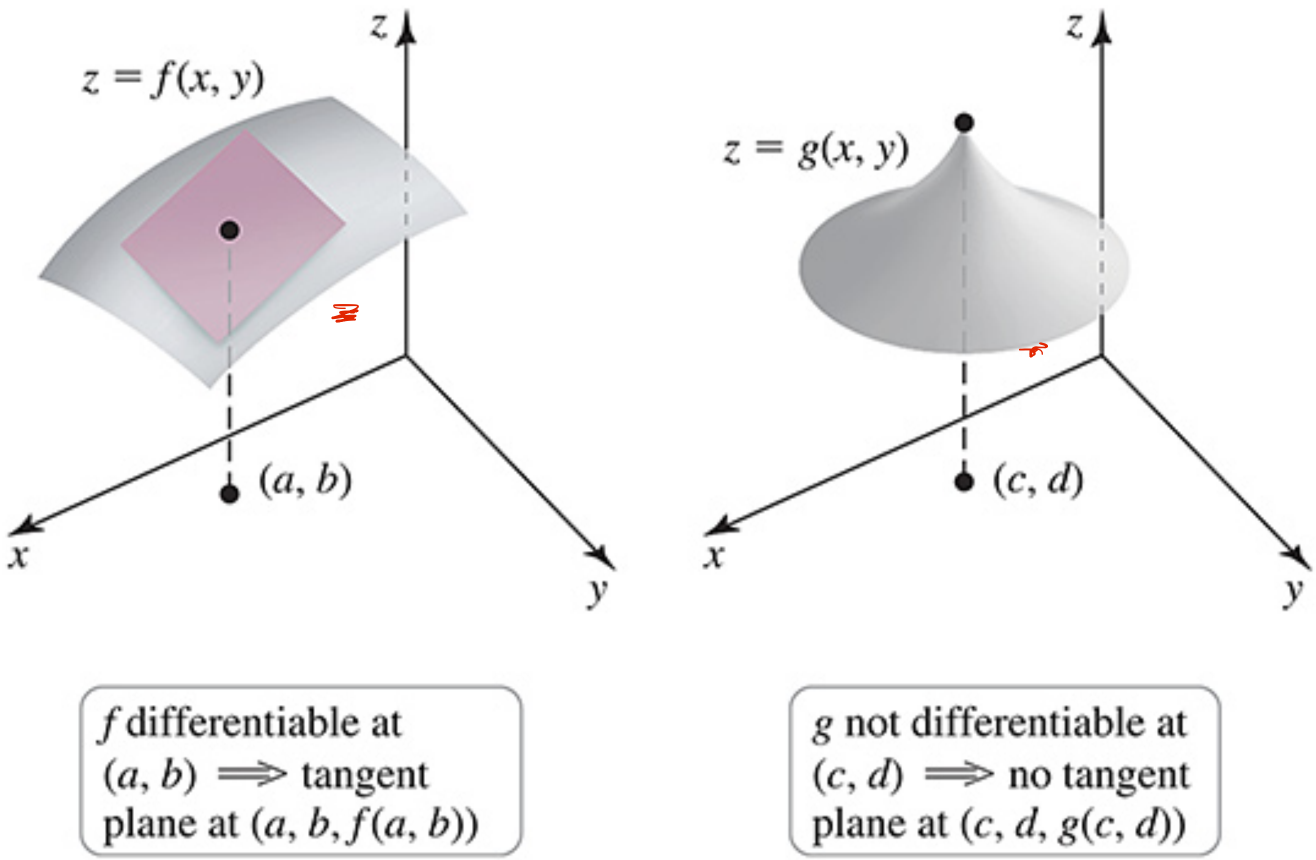


Figure 15.59 (1 of 2)

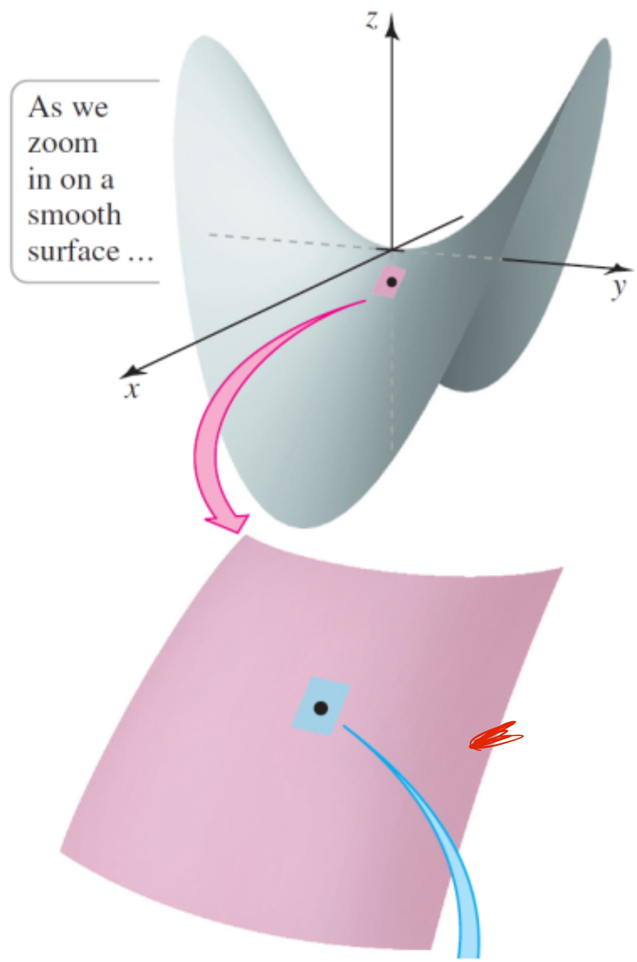
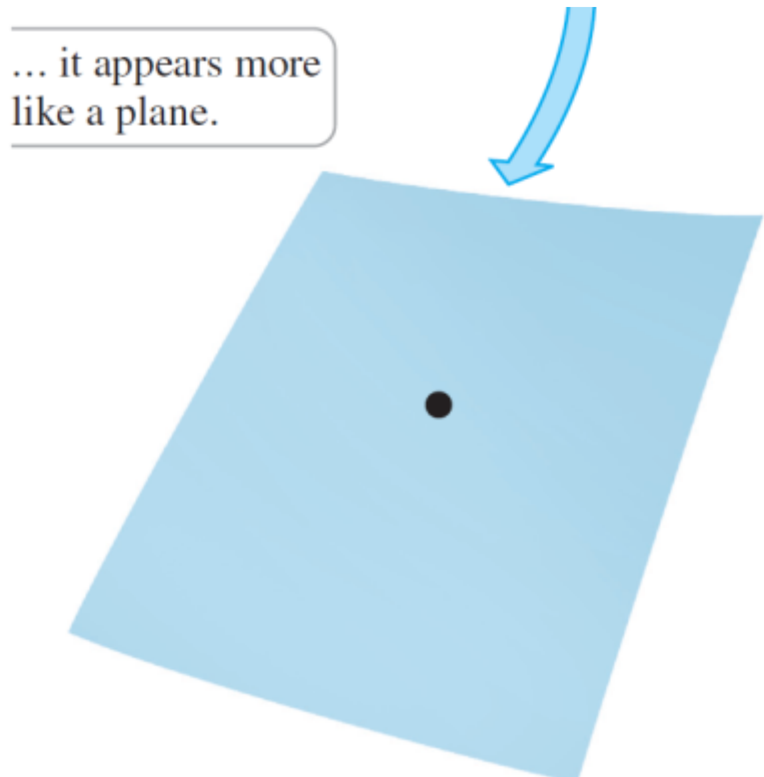


Figure 15.59 (2 of 2)



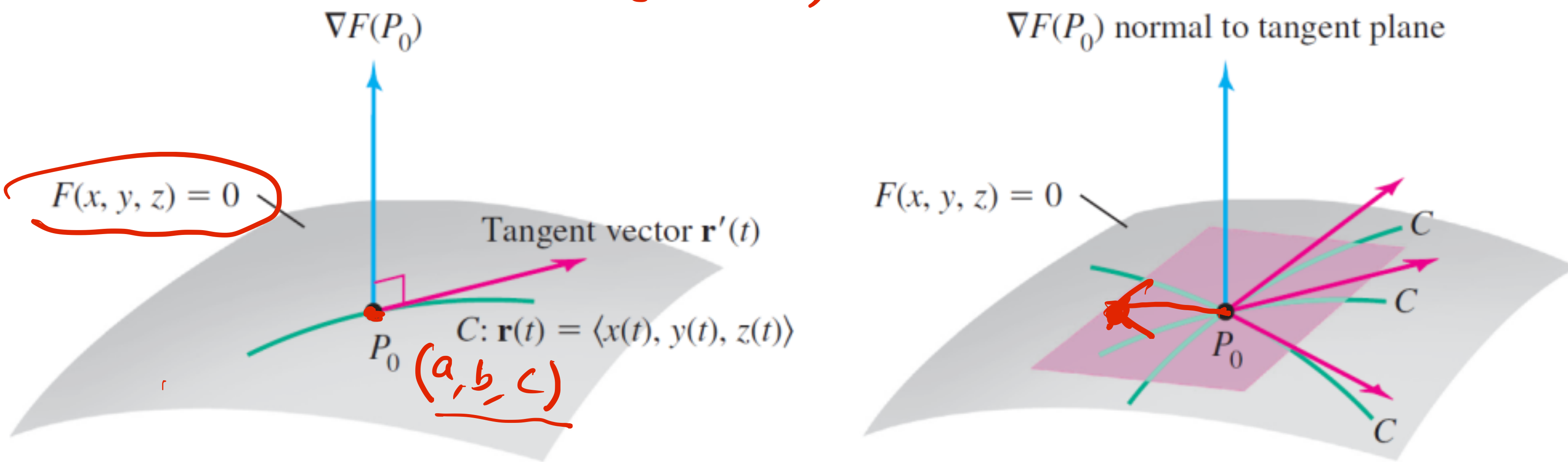
• Tangent Planes to surfaces

graph $z = f(x,y)$ ← $z - f(x,y) = 0$ → $F(x,y,z)$

level surface $F(x,y,z) = 0$

Figure 15.60 (a & b)

$\vec{n} = \nabla F(P_0) = \langle F_x(a,b,c), F_y(a,b,c), F_z(a,b,c) \rangle$



Vector tangent to C at P_0 is orthogonal to $\nabla F(P_0)$

Tangent plane formed by tangent vectors for all curves C on the surface passing through P_0

$0 = \nabla F(P_0) \cdot \langle x-a, y-b, z-c \rangle$

$$0 = F_x(P_0)(x-a) + F_y(P_0)(y-b) + F_z(P_0)(z-c)$$

Definition Equation of the Tangent Plane for *F* of *x*, *y* and *z* = 0

Let *F* be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$.

An equation of the tangent plane is

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0.$$

Example 1 $F(x,y,z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$

(a) Find an equation of the plane tangent to the ellipsoid at $(0, 4, \frac{3}{5})$.

(b) At what points on the ellipsoid is the tangent plane horizontal?

$\vec{n} = \nabla F(P_0) = \langle \frac{2x}{9}, \frac{2y}{25}, 2z \rangle \Big|_{P_0} = \langle 0, \frac{8}{25}, \frac{6}{5} \rangle$

$0 = \langle 0, \frac{8}{25}, \frac{6}{5} \rangle \cdot \langle x, y-4, z-\frac{3}{5} \rangle = \frac{8}{25}(y-4) + \frac{6}{5}(z - \frac{3}{5})$

Figure 15.61

$\vec{n} = \langle 0, 0, c \rangle$

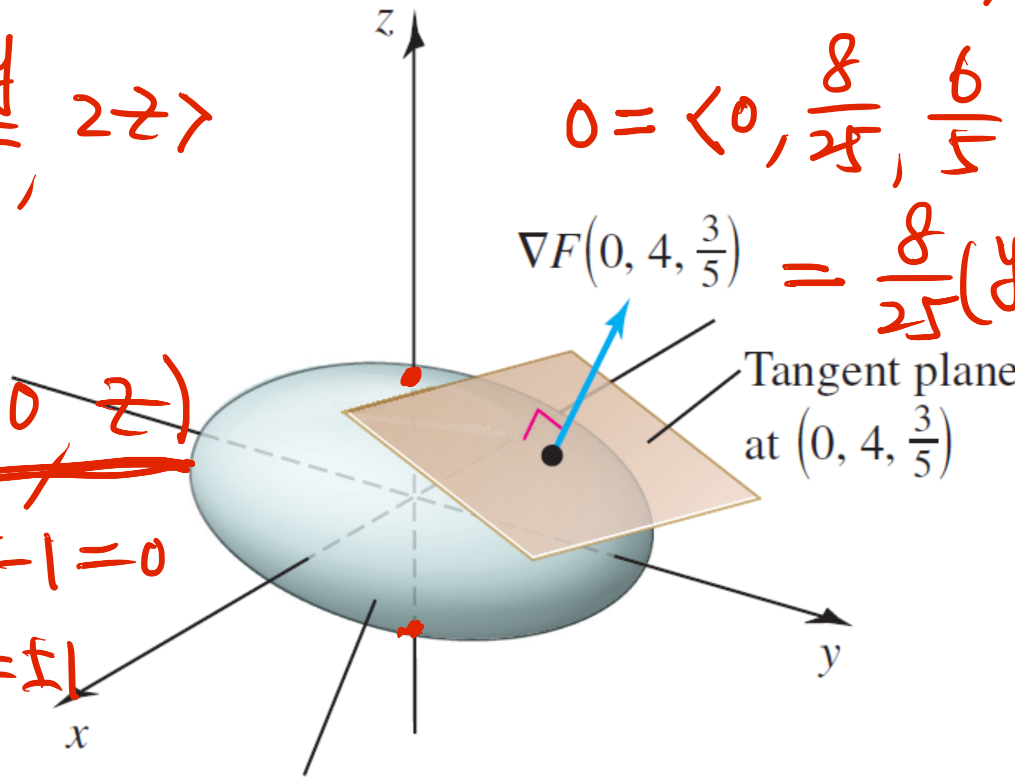
$\nabla F = \langle \frac{2x}{9}, \frac{2y}{25}, 2z \rangle$

$\vec{n} \parallel \nabla F \Rightarrow$

$\begin{cases} \frac{2x}{9} = 0 \\ \frac{2y}{25} = 0 \\ 2z = c \end{cases}$

$F(0,0,z) = z^2 - 1 = 0 \Rightarrow z = \pm 1$

$(0,0,1), (0,0,-1)$



$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$

Tangent Plane for *z* = *f* of *x* and *y*

Let *f* be differentiable at the point (*a*, *b*). An equation of the plane tangent to the surface $z = f(x,y)$ at the point $(a, b, f(a,b))$ is

$$F(x,y,z) = z - f(x,y) = 0$$

$$\nabla F = \langle -f_x, -f_y, 1 \rangle$$

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b).$$

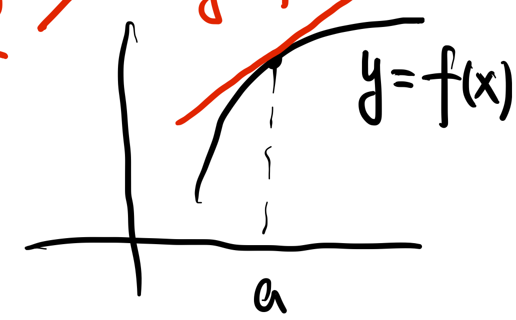
$$0 = \langle -f_x(a,b), -f_y(a,b), 1 \rangle \cdot \langle x-a, y-b, z-f(a,b) \rangle$$

$$= z - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)$$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$= f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle$$

$$y = f(a) + f'(a)(x-a)$$



Example 2 Find an equation of the plane tangent to the paraboloid

$$z = f(x,y) = 32 - 3x^2 - 4y^2 \text{ at } (2,1,16).$$

Figure 15.62

$$z = f(2,1) + \nabla f(2,1) \cdot \langle x-2, y-1 \rangle$$

$$= f(2,1) + \langle -6x, -8y \rangle \Big|_{(2,1)} \cdot \langle x-2, y-1 \rangle$$

$$= 16 - \langle 12, 8 \rangle \cdot \langle x-2, y-1 \rangle$$

$$= 16 - 12(x-2) - 8(y-1)$$

Tangent plane at (a,b,c)

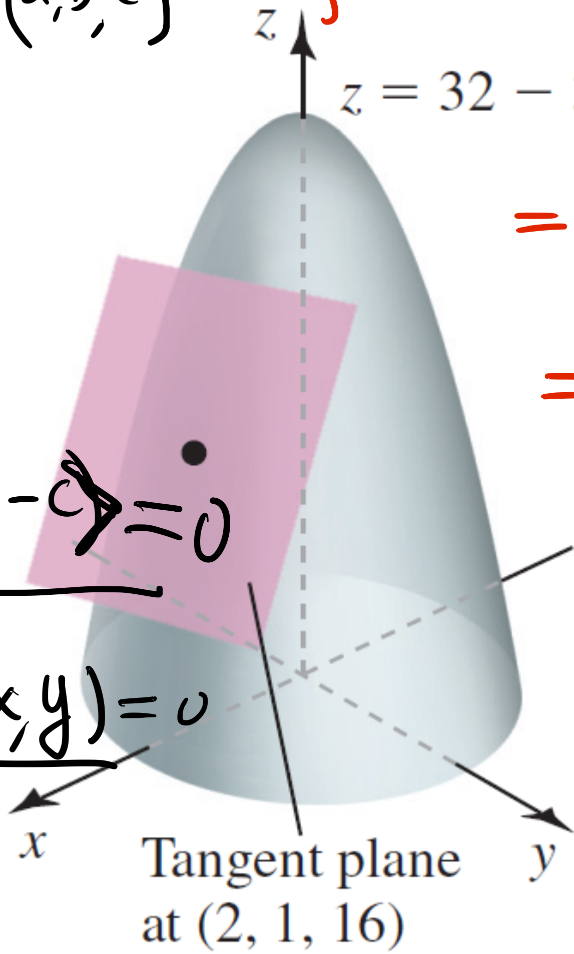
(a) Level surface

$$F(x,y,z) = 0$$

$$\nabla F(a,b,c) \cdot \langle x-a, y-b, z-c \rangle = 0$$

(b) $F(x,y,z) = z - f(x,y) = 0$

$$\Leftrightarrow z = f(x,y)$$



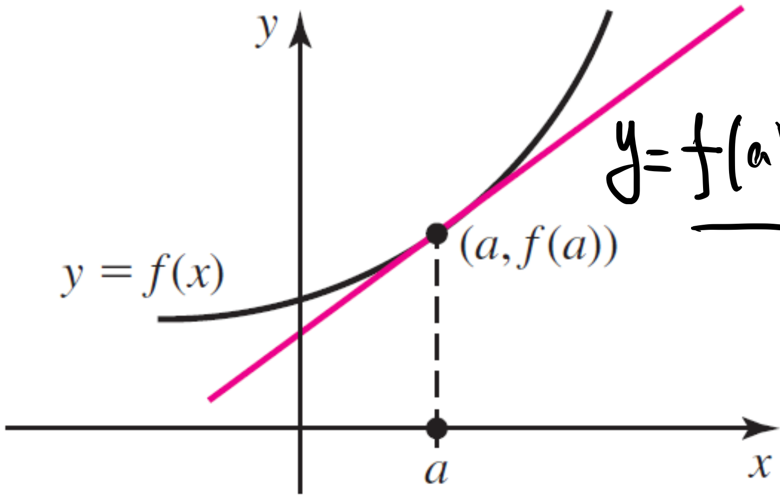
- Linear Approximation

$f(x,y) \approx L(x,y)$

$= f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle$

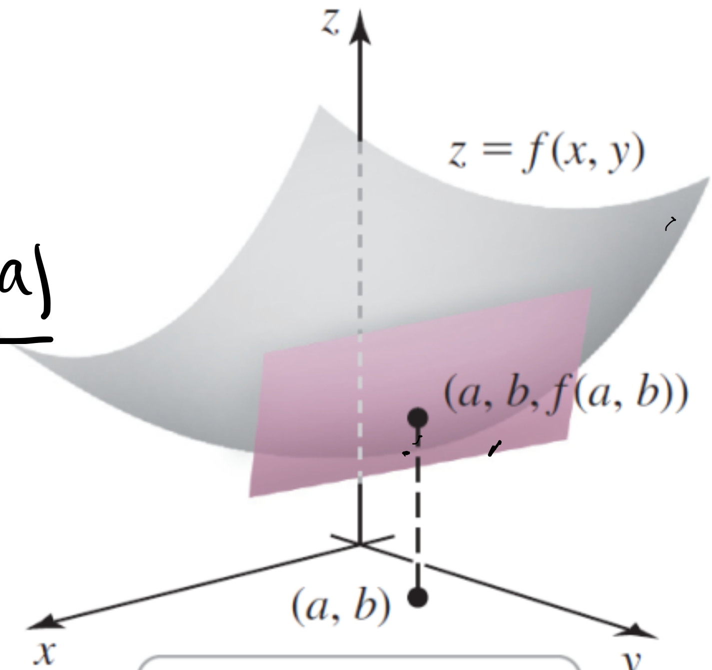
Figure 15.63

$f(x) \approx L(x) = f(a) + f'(a)(x-a)$



$y = f(a) + f'(a)(x-a)$

Tangent line
linear approximation
at $(a, f(a))$



Tangent plane
linear approximation
at $(a, b, f(a, b))$

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Definition Linear Approximation

Let *f* be differentiable at (*a*, *b*). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

For a function of three variables, the linear approximation to $w = f(x, y, z)$ at the point $(a, b, c, f(a, b, c))$ is given by

$$L(x, y, z) = f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) + f(a, b, c).$$

Example 3 $f(x,y)=\frac{5}{x^2+y^2}$

(a) Find the linear approx to the function at $(-1,2,1)$

(b) Use the linear approx. to estimate $f(-1.05, 2.1)$

Figure 15.64

$f(-1,2)=\frac{5}{1+4}=1$

$f_x(-1,2)=-5(x^2+y^2)^{-2}\cdot 2x\Big|_{(-1,2)}=\frac{10}{5^2}=\frac{2}{5}$

$f_y(-1,2)=-\frac{10y}{(x^2+y^2)^2}\Big|_{(-1,2)}$

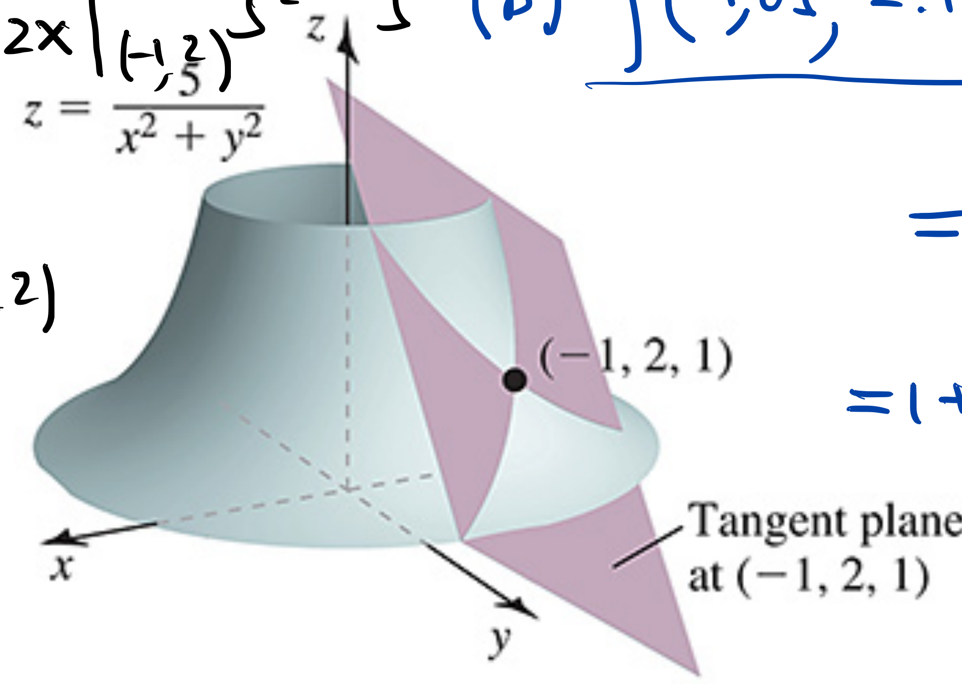
$=-\frac{20}{5^2}=-\frac{4}{5}$

(a) $L(x,y)=1+\frac{2}{5}(x+1)-\frac{4}{5}(y-2)$

(b) $f(-1.05, 2.1)\approx L(-1.05, 2.1)$

$=1+\frac{2}{5}(-1.05+1)-\frac{4}{5}(2.1-2)$

$=1+\frac{2}{5}\times(-0.05)-\frac{4}{5}\times 0.1$



- Differentials $\approx \Delta z$

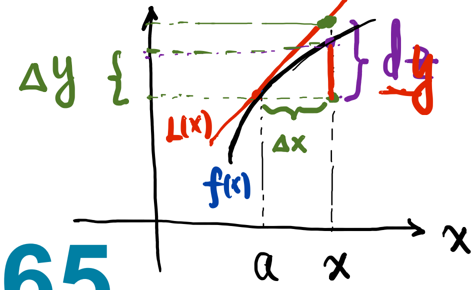


Figure 15.65

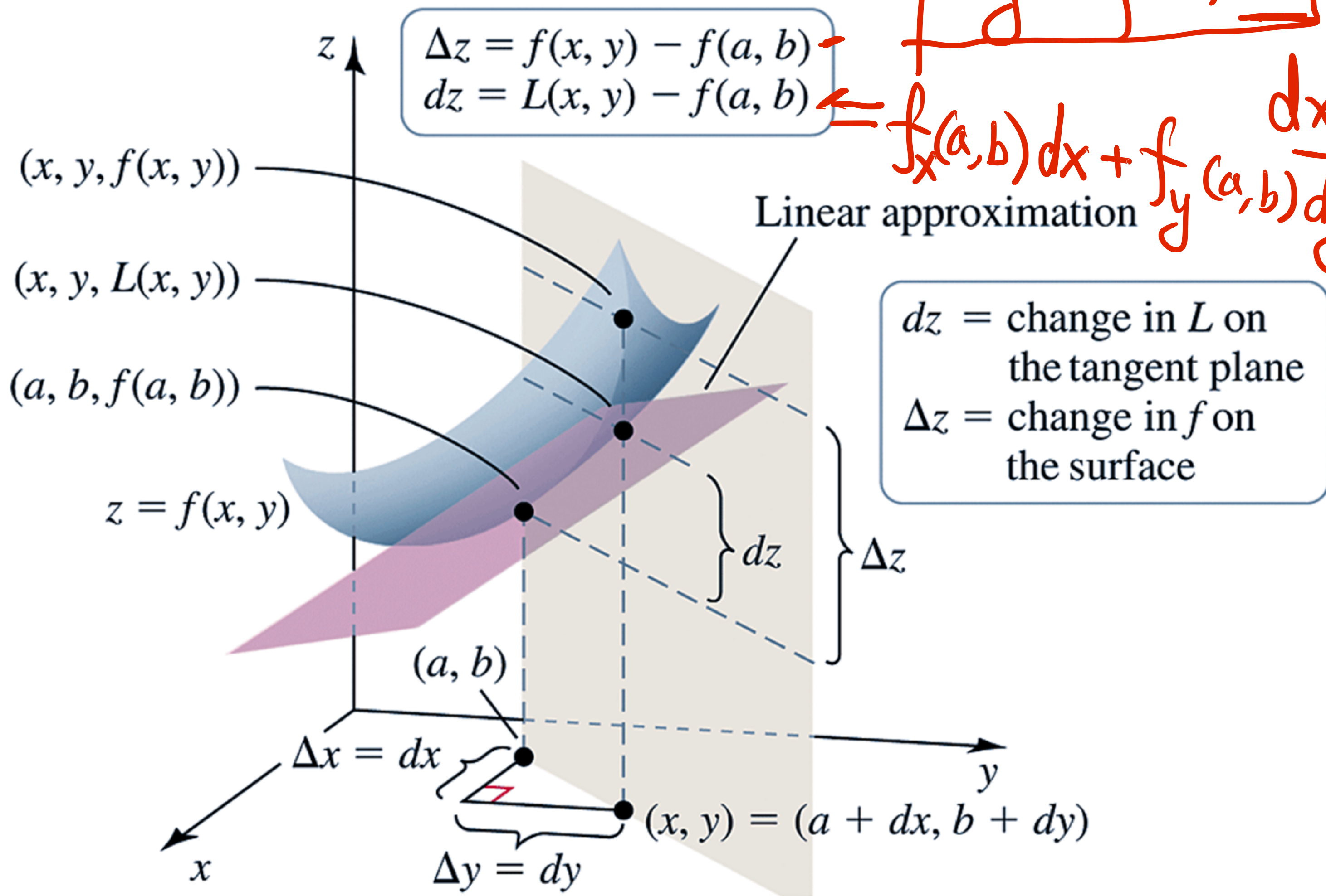
$\Delta y = f(x) - f(a)$

$dy = L(x) - f(a)$

$= f'(a)\Delta x$

$dy = f'(a)\,dx$

$dx = \Delta x$



Definition The differential *d z*

Let *f* be differentiable at the point (*x*, *y*). The change in $z = f(x, y)$ as the independent variables change from (*x*, *y*) to $(x + dx,\ y + dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(x,y)dx + f_y(x,y)dy.$$

Example 4 $z = f(x,y) = \frac{5}{x^2+y^2}$

Approximate the change in z, when indep variables change from $(-1,2)$ to $(-0.93, 1.94)$

$f_x(-1,2) = \frac{2}{5}$, $f_y(-1,2) = -\frac{4}{5}$

$dx = -0.93 - (-1) = 0.07$

$dy = 1.94 - 2 = -0.06$

$\Delta z \approx dz = f_x(-1,2)dx + f_y(-1,2)dy$

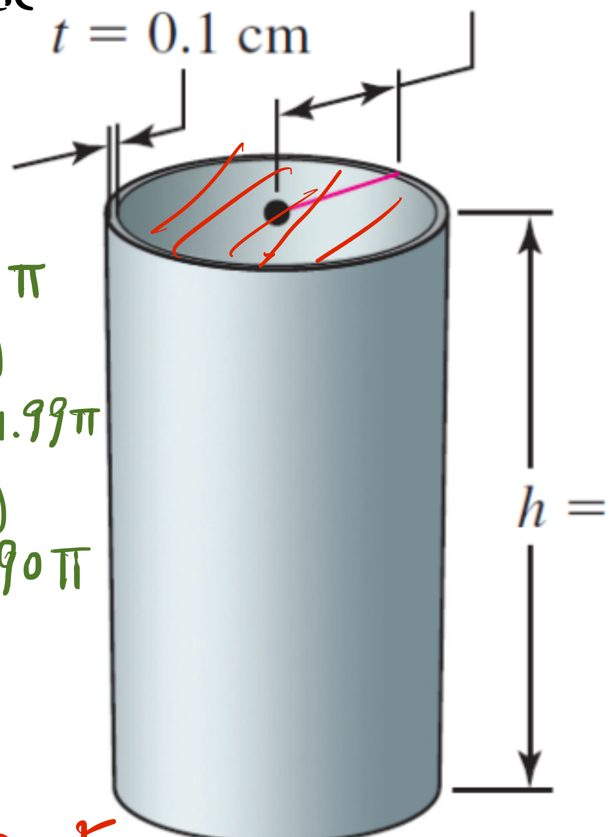
$= \frac{2}{5} \times 0.07 - \frac{4}{5} \times (-0.06)$

Example 6 A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of r = 10 cm, a height of h = 50 cm, a thickness of t = 0.1 cm. The manufacturing process produces tubes with a maximum error of ± 0.05 cm in the radius and height, and a maximum error of ± 0.0005 cm in the thickness.

Figure 15.66

$V(r,h,t) = \pi h t (2r - t)$

Use differentials to estimate the maximum error in the volume of a tube.



$\Delta V \approx dV$

$= V_r(10,50,0.1)\cdot dr + V_h(\ )\cdot dh + V_t(\ )\cdot dt$

$V_r(10,50,0.1) = 2\pi h t \big|_{(10,50,0.1)} = 10\pi$

$V_h(10,50,0.1) = \pi t(2r-t)\big|_{(10,50,0.1)} = 1.99\pi$

$V_t = 2\pi h(r-t)\big|_{(10,50,0.1)} = 990\pi$

$= 10\pi \times 0.05 + 1.99\pi \times 0.05 + 990\pi \times 0.0005$

0.05

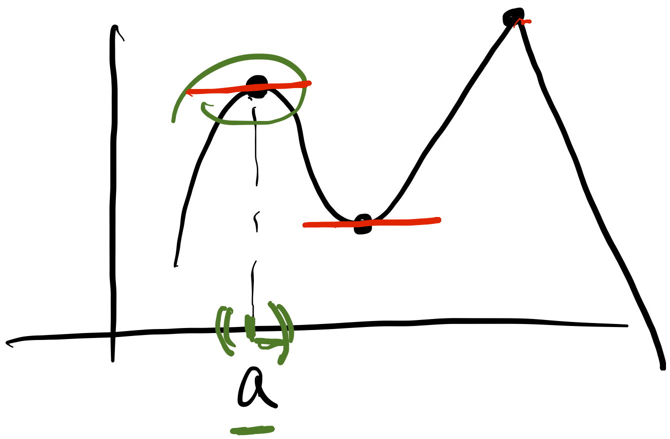
$r-0.05 \quad r \quad r+0.05$

Section 15.7 Maximum/Minimum Problems

1st der. test: f(a) is a l. max/min & $f'(a)$ exists $\Rightarrow f'(a) = 0$

critical pts: $f'(a) = 0$ or $f'(a)$ DNE

2nd der. test: $f''(a) < 0$ l. max; $f''(a) > 0$ l. min



f(a) is a l. max $\Leftrightarrow f(a) \geq f(x), \forall x \in O_r(a)$

l. min $\Leftrightarrow f(a) \leq f(x) \ \forall x \in O_r(a)$

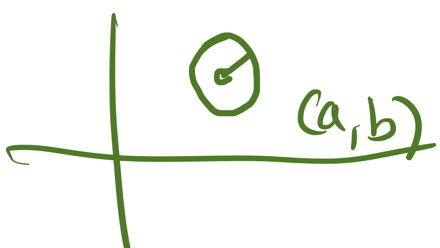
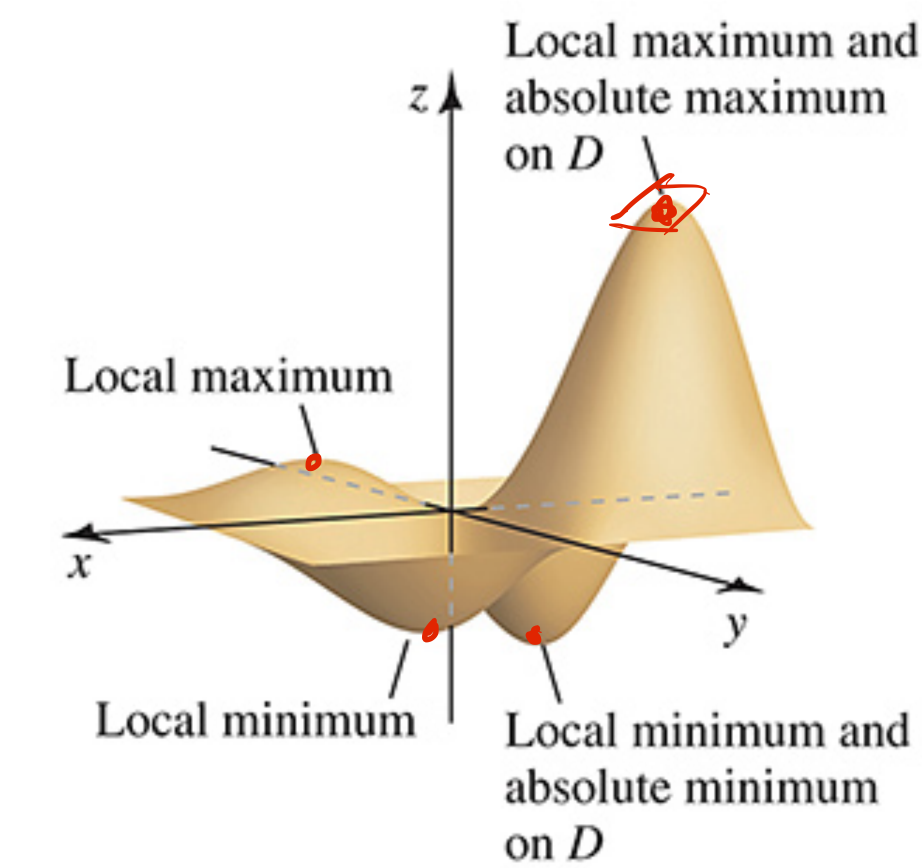


Figure 15.67


Local maximum and
absolute maximum
on D
z
Local maximum
x
y
Local minimum
Local minimum and
absolute minimum
on D

Definition Local Maximum / Minimum Values

Suppose (a, b) is a point in a region R on which f is defined. If $f(x,y) \le f(a,b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a,b)$ is a **local maximum value** of f. If $f(x,y) \ge f(a,\ b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a,b)$ is a **local minimum value** of f. Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

First Derivative Test

Theorem 15.14 Derivatives and Local Maximum / Minimum Values

① If f has a local maximum or minimum value at (a, b) and the partial derivatives

② f_x and f_y exist at (a, b), ~~then~~ ⟹ $f_x(a,b) = f_y(a,b) = 0.$

$(a,b) \to \langle h_1, h_2 \rangle$

$$g(t) = f(a + th_1, b + th_2)$$

$$0 = g'(t) = \frac{d}{dt} f(\quad)\Big|_{t=0}$$

$$= \nabla f(a,b) \cdot \langle h_1, h_2 \rangle$$

Definition Critical Point

An interior point (a, b) in the domain of f is a **critical point** of f if either

1. $f_x(a,b) = f_y(a,b) = 0,$ or

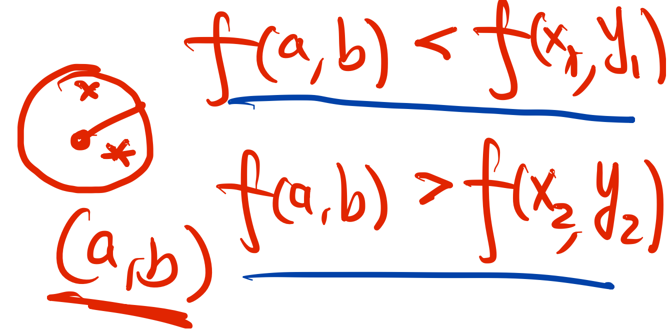
2. at least one of the partial derivatives f_x and f_y does not exist at (a, b).

Example 1 Find critical points of $f(x,y) = xy(x-2)(y+3)$

$f_x = y(y+3)[x-2+x] = 2y(y+3)(x-1) = 0 \Rightarrow y=0,\ y=-3,$ or $x=1$

$f_y = x(x-2)[y+3+y] = x(x-2)(2y+3) = 0 \Rightarrow x=0,\ x=2,$ or $y=-\frac{3}{2}$

$(0,0),\ (2,0),\ (0,-3),\ (2,-3),\ (1,-\frac{3}{2})$



Definition Saddle Point

Consider a function *f* that is differentiable at a critical point (*a*, *b*). Then *f* has a **saddle point** at (*a*, *b*) if, in every open disk centered at (*a*, *b*), there are points (*x*, *y*) for which $f(x,y) > f(a,b)$ and points for which $f(x,y) < f(a,b).$

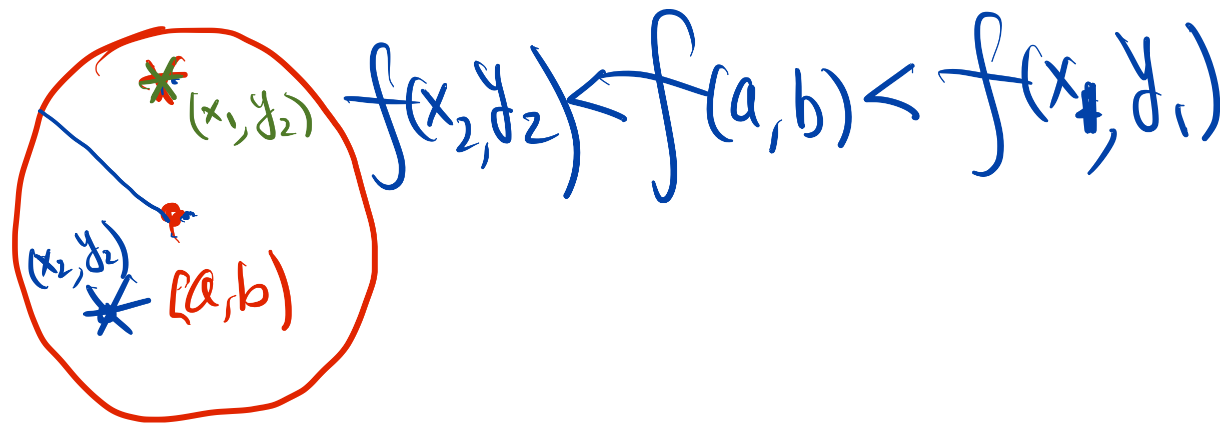


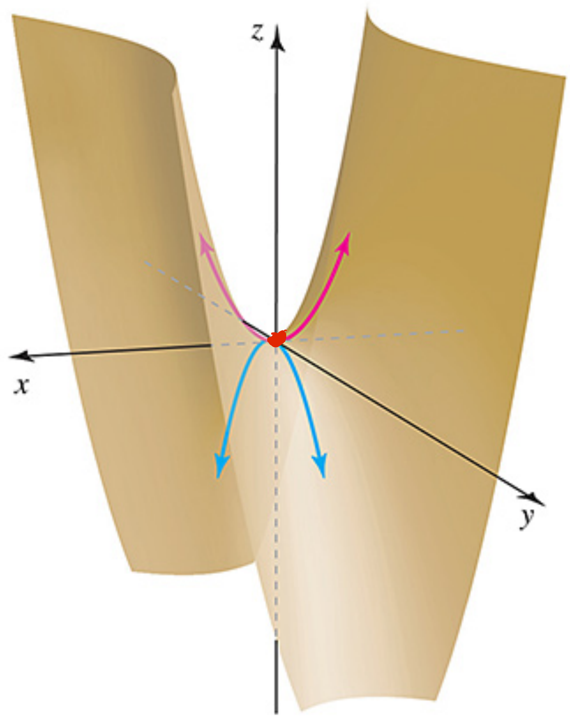
Figure 15.68

min/max $f(x, y)$

→ critical pts (a,b)

$\nabla f(a,b) = \langle 0, 0 \rangle$

or $\nabla f(a,b)$ DNE



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at (0, 0).

Theorem 15.15 Second Derivative Test

Suppose that the second partial derivatives of *f* are continuous throughout an open disk centered at the point (*a*, *b*), where $f_x(a,b) = f_y(a,b) = 0$. Let

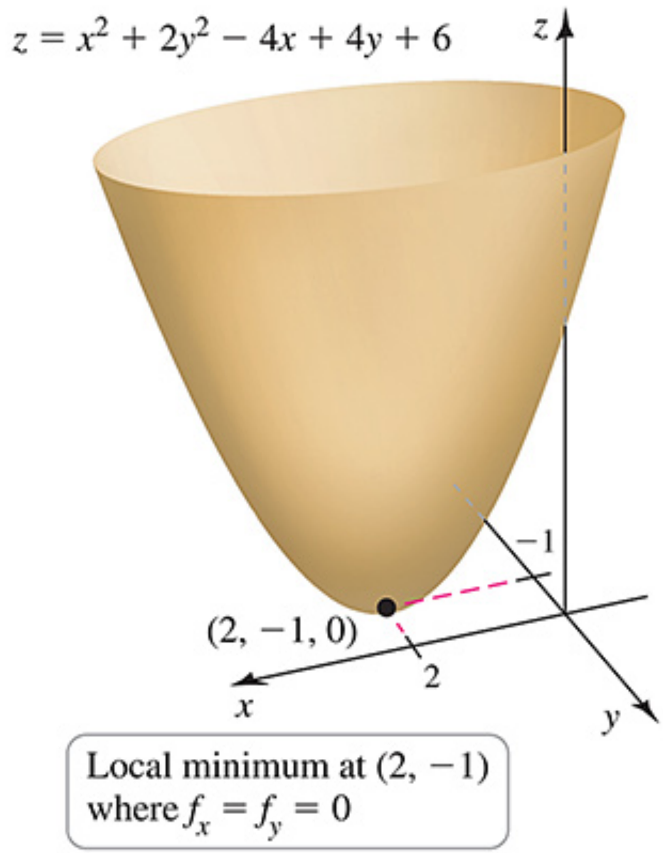
$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2.$$

1. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then *f* has a local maximum value at (*a*, *b*).
2. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then *f* has a local minimum value at (*a*, *b*).
3. If $D(a,b) < 0$, then *f* has a saddle point at (*a*, *b*).
4. If $D(a,b) = 0$, then the test is inconclusive.

Example 2 Use the Second Derivative Test to classify the critical points of $f(x,y) = x^2 + 2y^2 - 4x + 4y + 6$

X

Figure 15.69



Example 3 Classify the critical points of $f(x,y) = xy(x-2)(y+3)$.

$f_x = 2(x-1)y(y+3)$ $\quad f_{xx} = 2y(y+3)$ $\quad f_{xy} = 2(x-1)[y+3+y]$

$f_y = x(x-2)(2y+3)$ $\quad f_{yy} = 2x(x-2)$ $\quad = 2(x-1)(2y+3)$

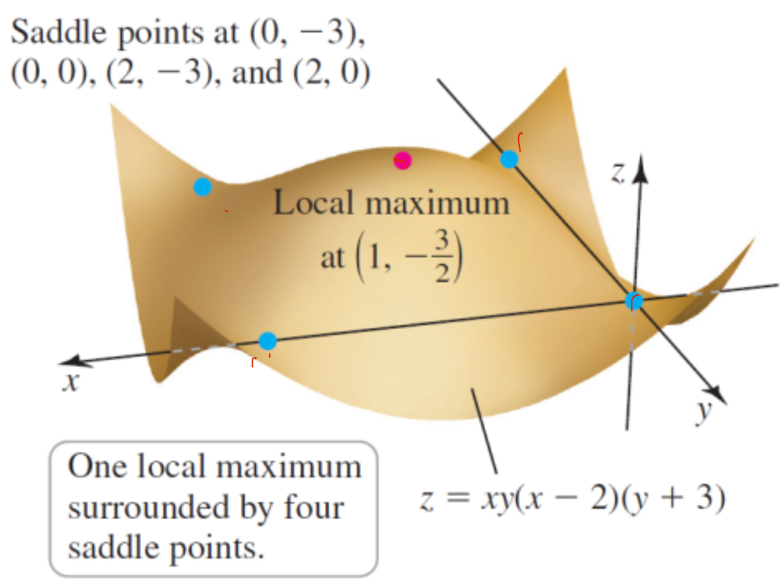
Table 15.4

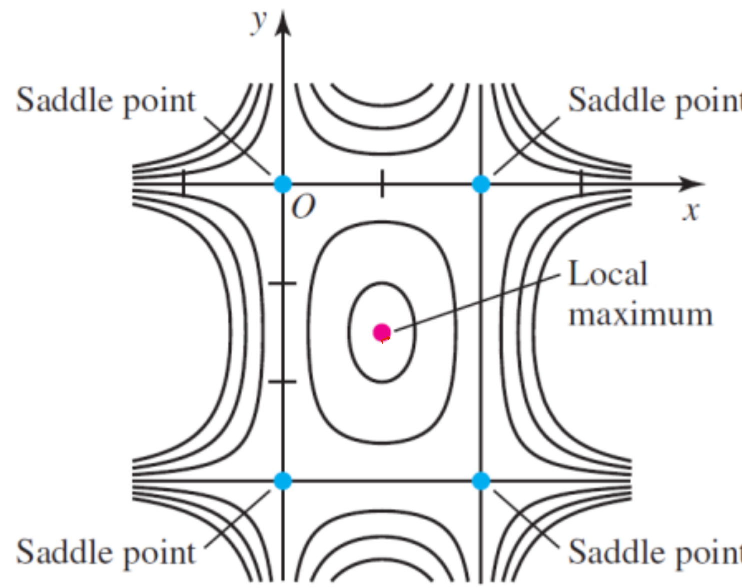
$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4\{xy(x-2)(y+3) - (x-1)^2(2y+3)^2\}$$

(x, y)	$D(x, y)$	f_{xx}	Conclusion
(0,0)	−36	0	Saddle point
(2,0)	−36	0	Saddle point
$(1, -\frac{3}{2})$	9	$-\frac{9}{2}$	Local maximum
(0, −3)	−36	0	Saddle point
(2, −3)	−36	0	Saddle point

Handwritten annotations next to the table: < 0 (for $D = -36$), > 0 (for $D = 9$), < 0 (for $f_{xx} = -\frac{9}{2}$).

Figure 15.70 (a & b)





Example 4 (Inconclusive Tests) Apply the SDT and interpret the results.

(a) $f(x,y) = 2x^4 + y^4 \geq 0$

$f_x = 8x^3 = 0$

$f_y = 4y^3 = 0$

$\Rightarrow (0,0)$

$f_{xx} = 24x^2$

$f_{yy} = 12y^2$

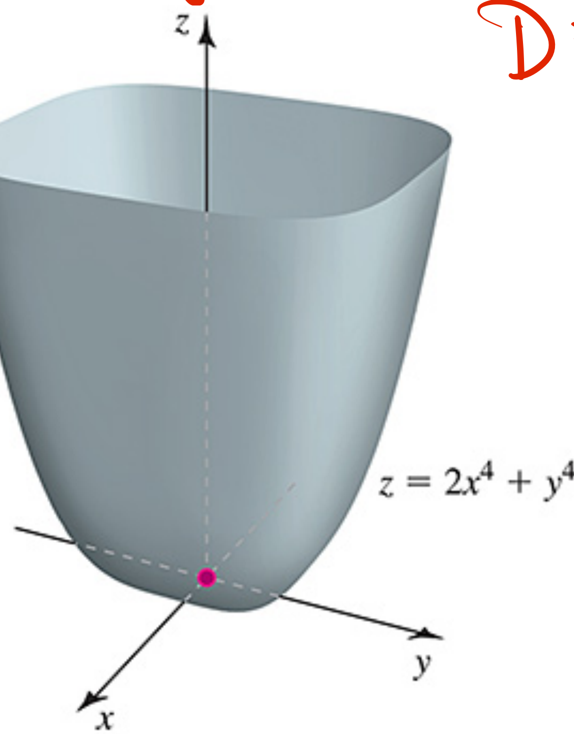
$f_{xy} = 0$

$D = f_{xx} f_{yy} - f_{xy}^2$

$= 0 - 0 = 0$

$f(0,0) = 0$

Figure 15.71

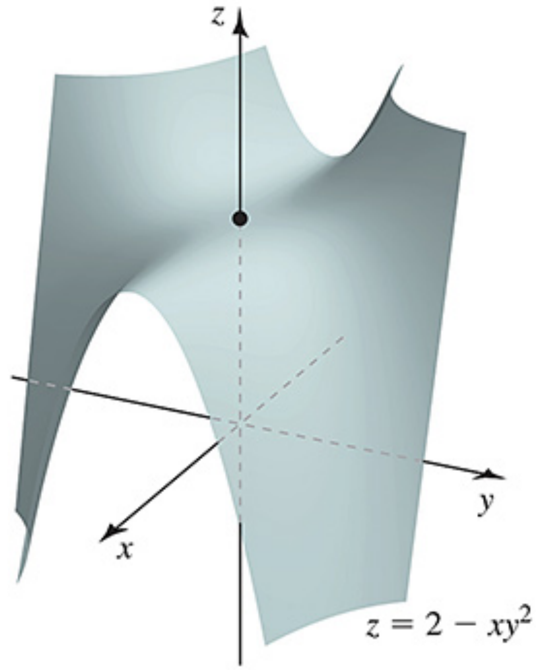


Local minimum at (0, 0), but the Second Derivative Test is inconclusive.

(b) $f(x,y) = 2 - xy^2$

X

Figure 15.72



Second derivative test fails to detect saddle point at (0, 0).

Definition Absolute Maximum / Minimum Values

global

Let *f* be defined on a set *R* in $\mathbb{R}^2$ containing the point (*a*, *b*). If $f(a,b) \geq f(x,y)$ for every (*x*, *y*) in *R*, then $f(a,b)$ is an **absolute maximum value** of *f* on *R*. If $f(a,b) \leq f(x,y)$ for every (*x*, *y*) in *R*, then $f(a,b)$ is an **absolute minimum value** of *f* on *R*.

Example 5 A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box has the largest volume.

Figure 15.73

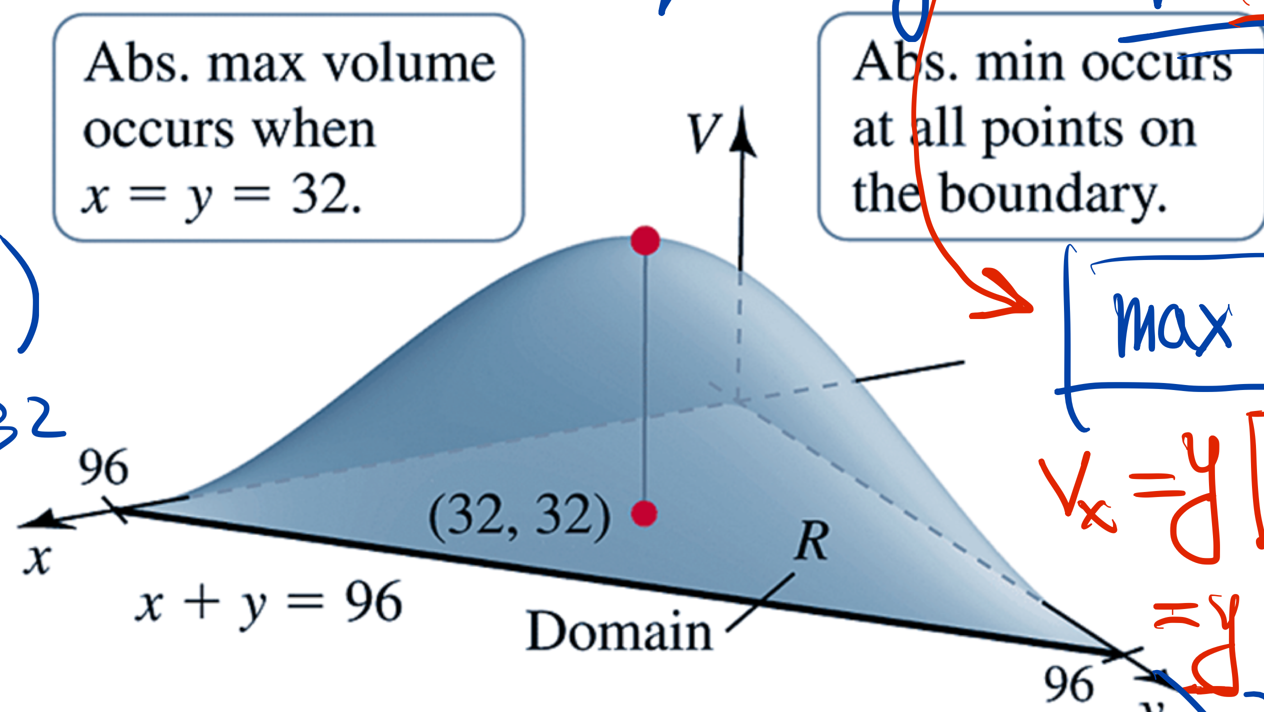
$x+y+z \leq 96$ (crossed out; replaced by $x+y+z=96$)

$\max V(x,y,z)$, subject to $x+y+z=96$

$V = xyz$

$z = 96 - x - y \Rightarrow V = xy(96-x-y) = V(x,y)$

$\max V(x,y)$



Volume $V = xy(96 - x - y)$

$(x,y) = (32,32)$

$z = 96 - x - y = 32$

$V = 32^3$

$V_x = y[96 - x - y - x] = y(96 - 2x - y) = 0$

$\Rightarrow y = 0$ (crossed out) or $2x + y = 96$

$V_y = x(96 - 2y - x) = 0$

$x = 0$ (crossed out) or $x + 2y = 96$

Procedure Finding Absolute Maximum / Minimum Values on Closed Bounded Sets

Let *f* be continuous on a closed bounded set *R* in $\mathbb{R}^2$. To find the absolute maximum and minimum values of *f* on *R*:

1. Determine the values of *f* at all critical points in *R*.
2. Find the maximum and minimum values of *f* on the boundary of *R*.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of *f* on *R*, and the least function value found in Steps 1 and 2 is the absolute minimum value of *f* on *R*.

$$\min/\max_{x\in[a,b]} f(x) = \min/\max \left\{ f(a), f(b), \min/\max_{x\in(a,b)} f \right\}$$

Example 6 Find the absolute max. and min. values of $f(x,y) = xy - 8x - y^2 + 12y + 160$ over the triangular region $\bar{R} = \{(x,y) \mid 0 \le x \le 15,\ 0 \le y \le 15 - x\}$.

Figure 15.74

(1) min/max $f(x,y)$

$(x,y) \in R$

$f_x = y - 8 = 0 \Rightarrow y = 8$

$f_y = x - 2y + 12 = 0 \Rightarrow x = 2y - 12 = 2 \cdot 8 - 12 = 4$

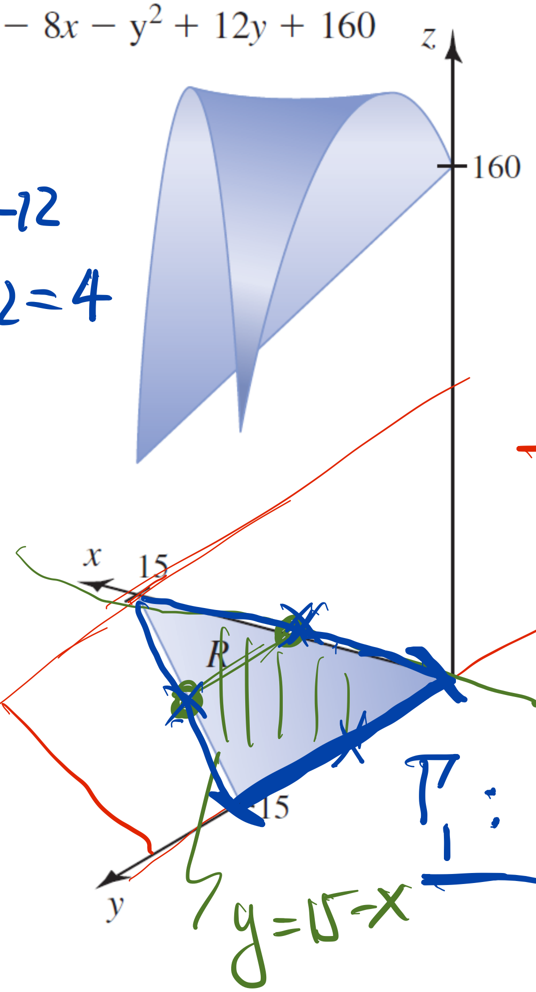
$(4, 8)$

(2) On boundary

On Γ_1 $f(0,y) = -y^2 + 12y + 160$

$y \in [0, 15]$

$f(0,0)$, $f(0,15)$, $f(0,6)$



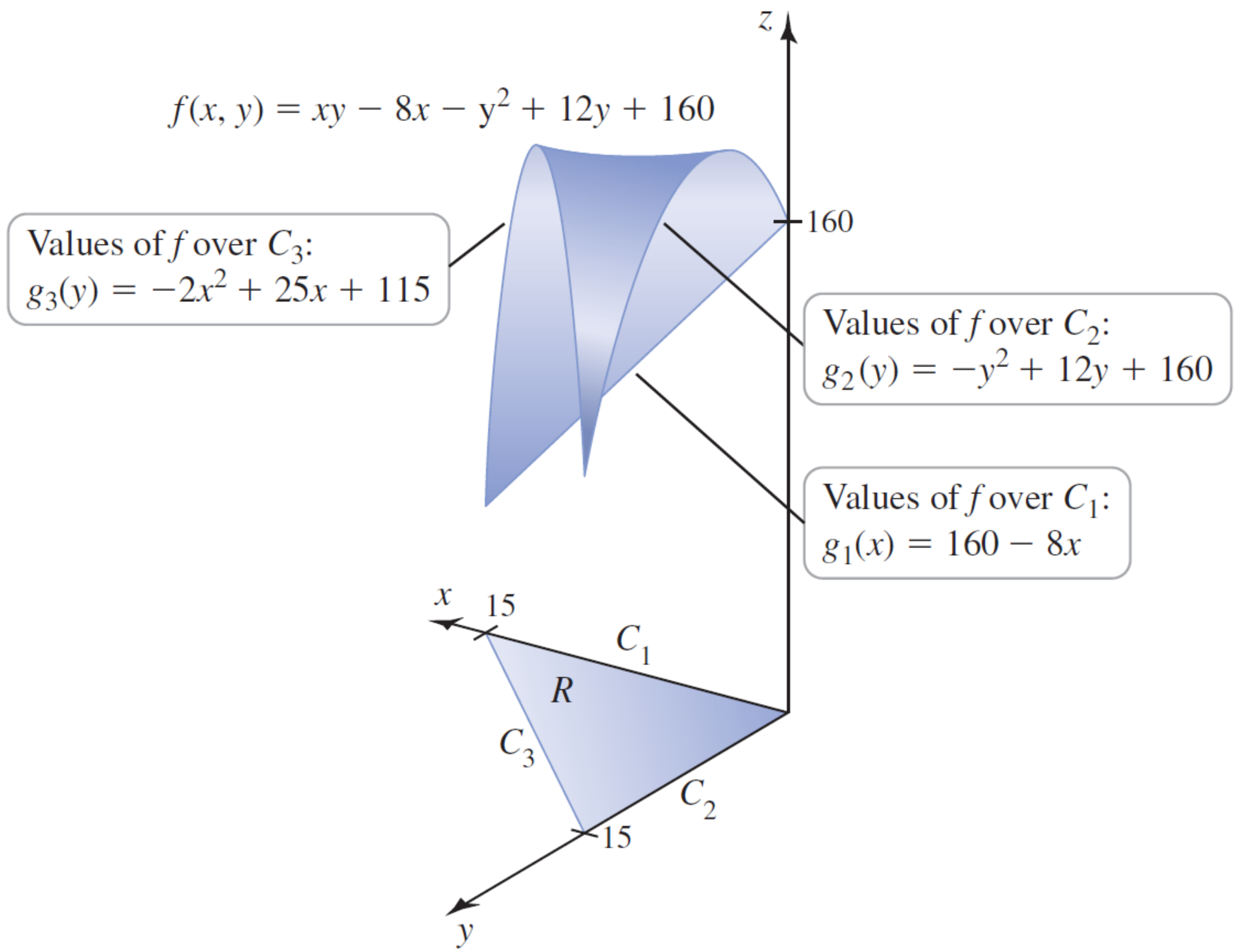
$f'(0,y) = -2y + 12 = 0$

$\Rightarrow y = 6$

$(0, 6)$

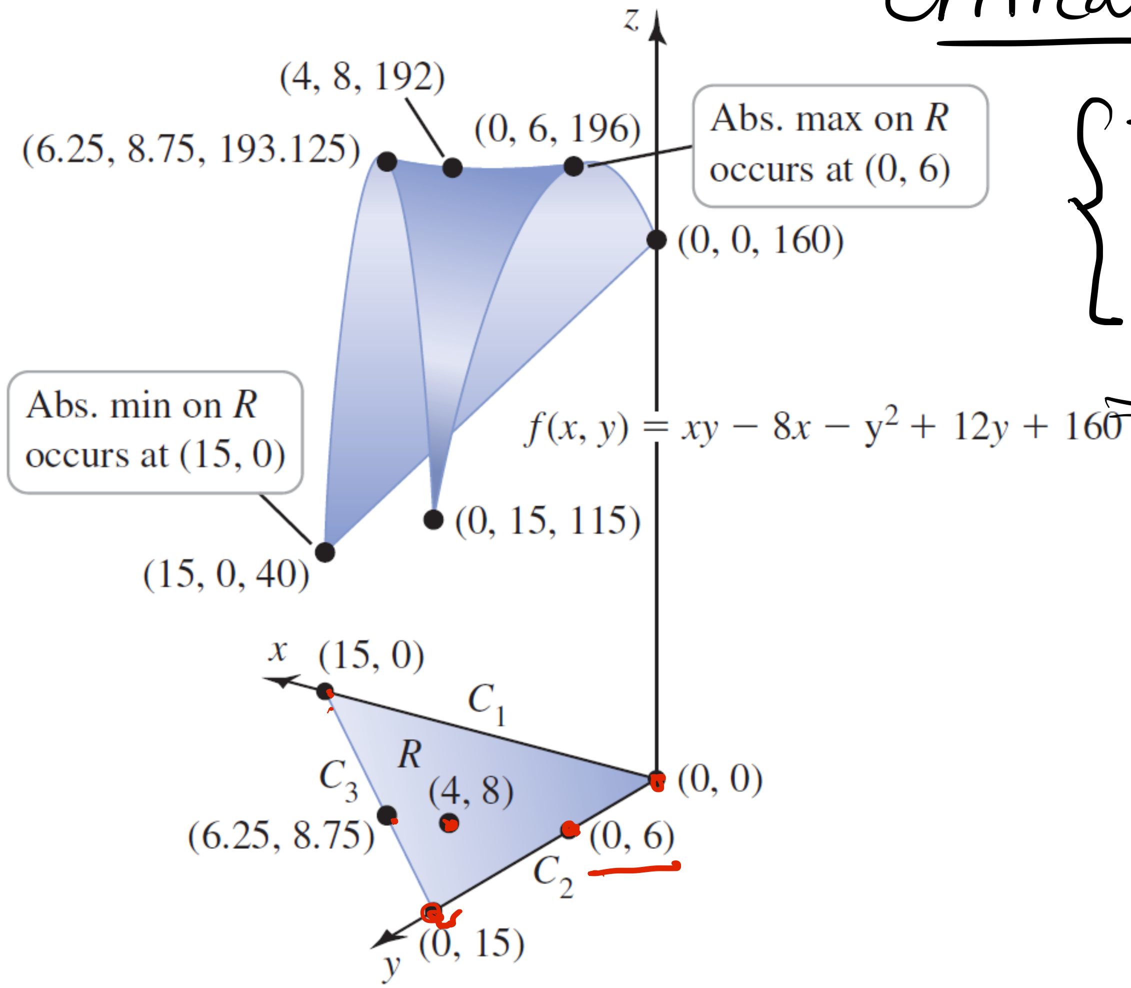
Γ_1: $x = 0$, $0 \le y \le 15$

Figure 15.75



$f(x, y) = xy - 8x - y^2 + 12y + 160$
z
160
Values of f over C_3:
$g_3(y) = -2x^2 + 25x + 115$
Values of f over C_2:
$g_2(y) = -y^2 + 12y + 160$
Values of f over C_1:
$g_1(x) = 160 - 8x$
x
15
C_1
R
C_3
C_2
15
y

Figure 15.76



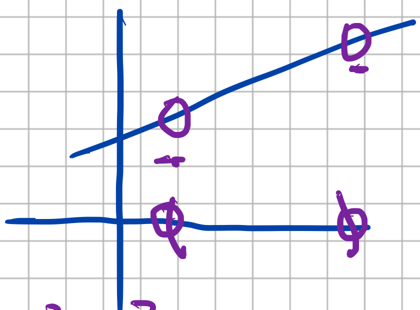
min/max $f(x,y)$

critical pts

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases}$$

Example 8 Find the absolute max. and min. values of $f(x,y) = 4 - x^2 - y^2$ on the open disk $R = \{(x,y) \mid x^2+y^2<1\}$. $= 4-(x^2+y^2) > 4-1=3$

$f(x) = ax+b$



$x^2+y^2=1$

$f(x,y)=4-1=3$

$f_x = -2x = 0$

$f_y = -2y = 0$

$\Rightarrow (x,y)=(0,0)$

$D = f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - 0^2 = 4 > 0$

$f_{xx}(0,0) = -2 < 0$ $\Rightarrow$ l. max

Example 9 Find the point(s) on the plane $x+2y+z=2$ closest to the point $P(2,0,4)$

Let (x,y,z) on the plane

$$d = \sqrt{(x-2)^2 + y^2 + (z-4)^2}$$

$$\min d(x,y,z) = \min d(x, y, 2-x-2y)$$

$x+2y+z=2$

$z = 2-x-2y$

$$= \min \sqrt{(x-2)^2 + y^2 + (x+2y+2)^2}$$

Section 15.8 Lagrange Multipliers

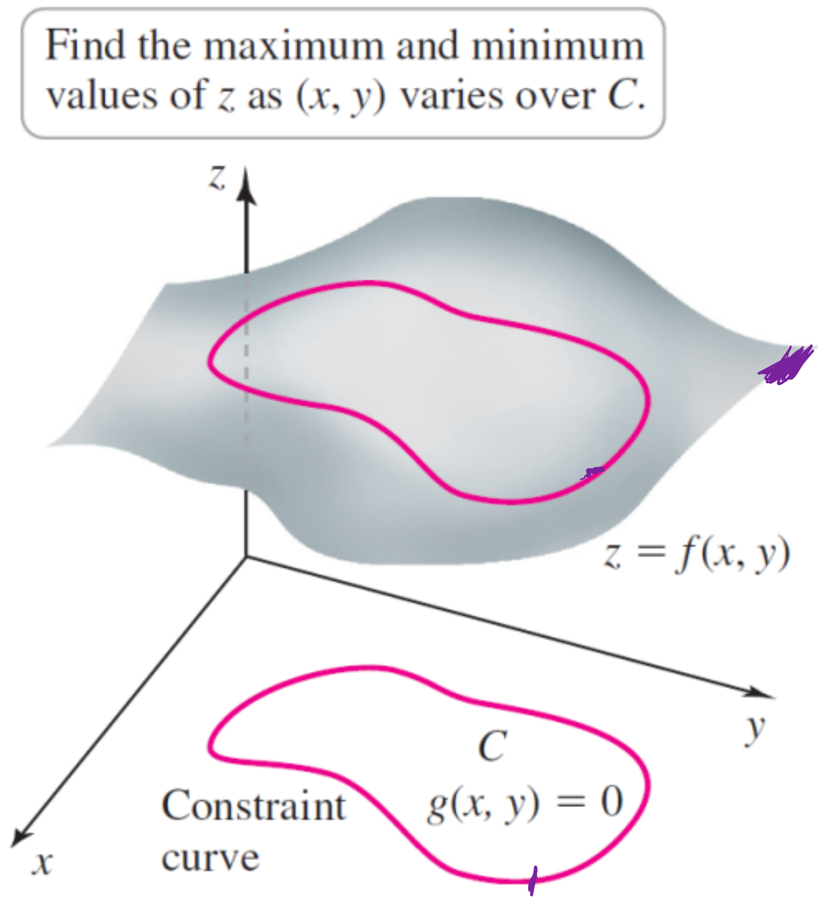
- constrained maximum/minimum problems

max/min $f(x, y)$ objective function

$g(x, y) = 0$ constraint

Figure 15.79

min f(x, y)



Example $\min_{2x+y=5} (x^2+y^2)$

Solution 1 $y=5-2x$, $= \min \left(x^2+(5-2x)^2\right)$

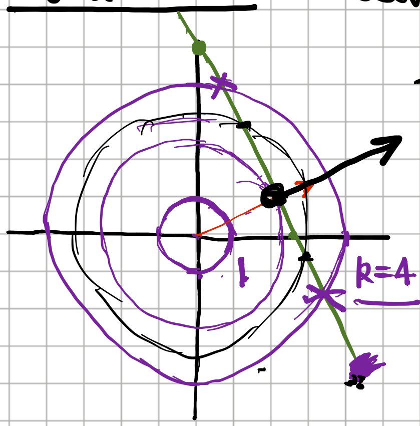
$f_x = 2x$, $g_x = 2$ $\quad \begin{cases} 2x = 2\lambda \\ 2y = \lambda = x \\ 2x+y=5 \end{cases}$ $\quad 4y+y=5 \Rightarrow y=1$, $x=2$

$f_y = 2y$, $g_y = 1$

$f(2,1) = 4+1 = 5$

system of nonlinear equations

Solution 2 level curves : $f(x,y) = x^2+y^2 = k^2$



∇f ∇g constraint : $g(x,y) = 2x+y = 5$

$\nabla f \parallel \nabla g \Rightarrow \begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 5 \end{cases}$

Example 2 Find point(s) on the surface $g(x,y,z) = x^2 - z^2 - 1 = 0$ that are closest to $(0,0,0)$.

$$d = \sqrt{x^2+y^2+z^2}$$

$$\min_{g(x,y,z)=0} d(x,y,z) \neq \min_{g=0} d^2$$

Solution 1 $x^2 - z^2 - 1 = 0 \Rightarrow z^2 = x^2 - 1$

$$d^2 = x^2 + y^2 + z^2$$

$$d^2\Big|_{z^2=x^2-1} = 2x^2 + y^2 - 1 = f(x,y)$$

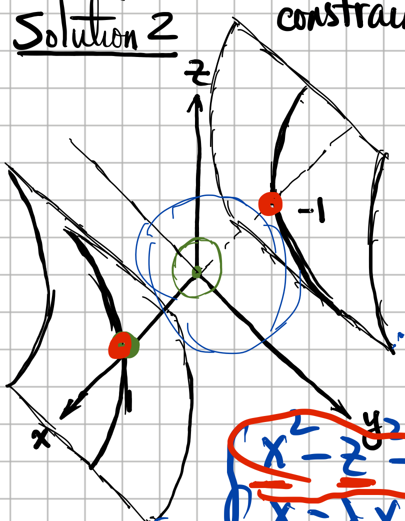
$f_x = 4x = 0 \Rightarrow (x,y) = (0,0)$ $f(0,0) = -1$

$f_y = 2y = 0$

$x^2 = z^2 + 1$ $\quad d^2\Big|_{x^2=z^2+1} = y^2 + 2z^2 + 1$ $\quad (y,z) = (0,0)$

$$f(x,y,z) = d^2(x,y,z) = x^2 + y^2 + z^2 = k^2$$

Solution 2 constraint: $x^2 = z^2 + 1$



$\nabla f \parallel \nabla g$ $\quad \begin{cases} \nabla f = \lambda \nabla g \\ x^2 - z^2 - 1 = 0 \end{cases}$

$f_x = 2x$, $g_x = 2x$

$f_y = 2y$, $g_y = 0$

$f_z = 2z$, $g_z = -2z$

$x = 0$ $\quad z^2 = -1$ ✗

or $z = 0$ $\quad x^2 = 1 \Rightarrow x = \pm 1$

$f(\pm 1, 0, 0) = 1 \Rightarrow d = \sqrt{f} = 1$

$$\begin{cases} x^2 - z^2 - 1 = 0 \\ x = \lambda x \\ y = 0 \\ z = -\lambda z \end{cases}$$

$\dfrac{x}{z} = -\dfrac{x}{z}$ $\quad xz = -xz$, $\quad \boxed{xz = 0}$

Theorem 15.16 Parallel Gradients

Let *f* be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve *C* given by $g(x, y) = 0$. Assume that *f* has a local extreme value on *C* at a point $P(a,b)$. Then $\nabla f(a,b)$ is orthogonal to the line tangent to *C* at *P*. Assuming $\nabla g(a,b) \neq \mathbf{0}$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

Procedure Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Curves

Let the objective function *f* and the constraint function *g* be differentiable on a region of R^2 with $\nabla g(x, y) \neq \mathbf{0}$ on the curve $g(x, y) = 0.$ To locate the absolute maximum and minimum values of *f* subject to the constraint $g(x, y) = 0,$ carry out the following steps.

1. Find the values of *x*, *y*, and λ (if they exist) that satisfy the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0.$$

2. Evaluate *f* at the values (*x*, *y*) found in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of *f* subject to the constraint.

Example Find the absolute maximum and minimum values of the objective function $f(x,y) = x^2 + y^2 + 2$, where x and y lie on the ellipse C given by $g(x,y) = x^2 + xy + y^2 - 4 = 0$.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}, \quad f_x = 2x, \; g_x = 2x + y, \quad f_y = 2y, \; g_y = x + 2y$$

$$\begin{cases} 2x = \lambda(2x+y) \\ 2y = \lambda(x+2y) \\ x^2 + xy + y^2 - 4 = 0 \end{cases}$$

$$\frac{x}{y} = \frac{2x+y}{x+2y} \Rightarrow x(x+2y) = y(2x+y)$$

$$x^2 + 2xy = 2xy + y^2 \Rightarrow x^2 = y^2$$

$x = \pm y$

$\underline{x = y}$ $\quad 0 = g(x,y) = g(y,y) = 3y^2 - 4 \Rightarrow y = \pm\frac{2}{\sqrt{3}}$

$$\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right), \; -\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$$

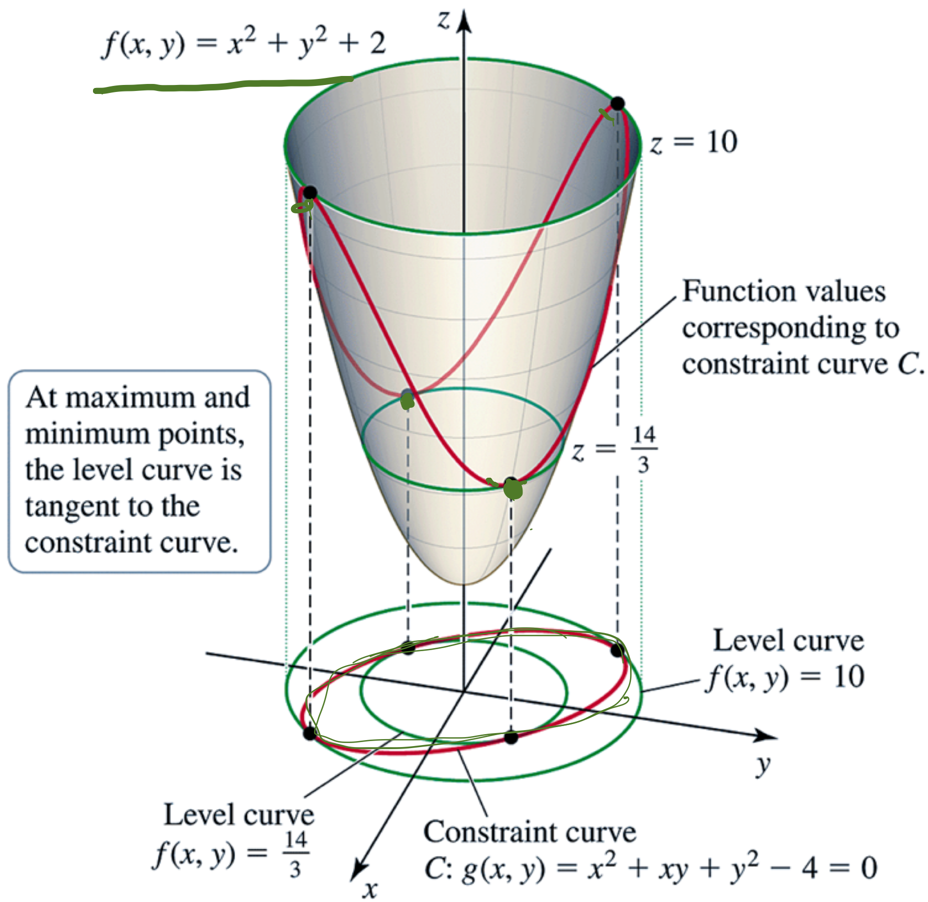
$\underline{x = -y}$ $\quad y^2 - 4 = 0 \Rightarrow y = \pm 2$

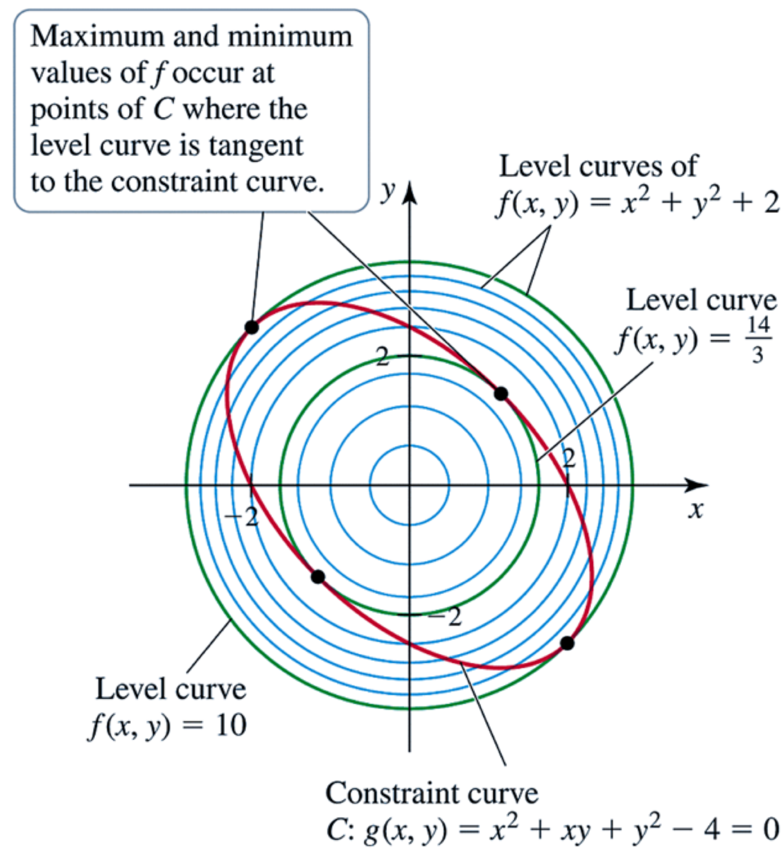
$$(-2, 2), \; (2, -2)$$

$f\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ $\qquad f\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$

$f(-2, 2)$ $\qquad f(2, -2)$

Figure 15.81 (a & b)





Procedure Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Surfaces

Let *f* and *g* be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x,y,z) \neq \mathbf{0}$ on the surface $g(x,y,z)=0$. To locate the absolute maximum and minimum values of *f* subject to the constraint $g(x,y,z)=0$, carry out the following steps.

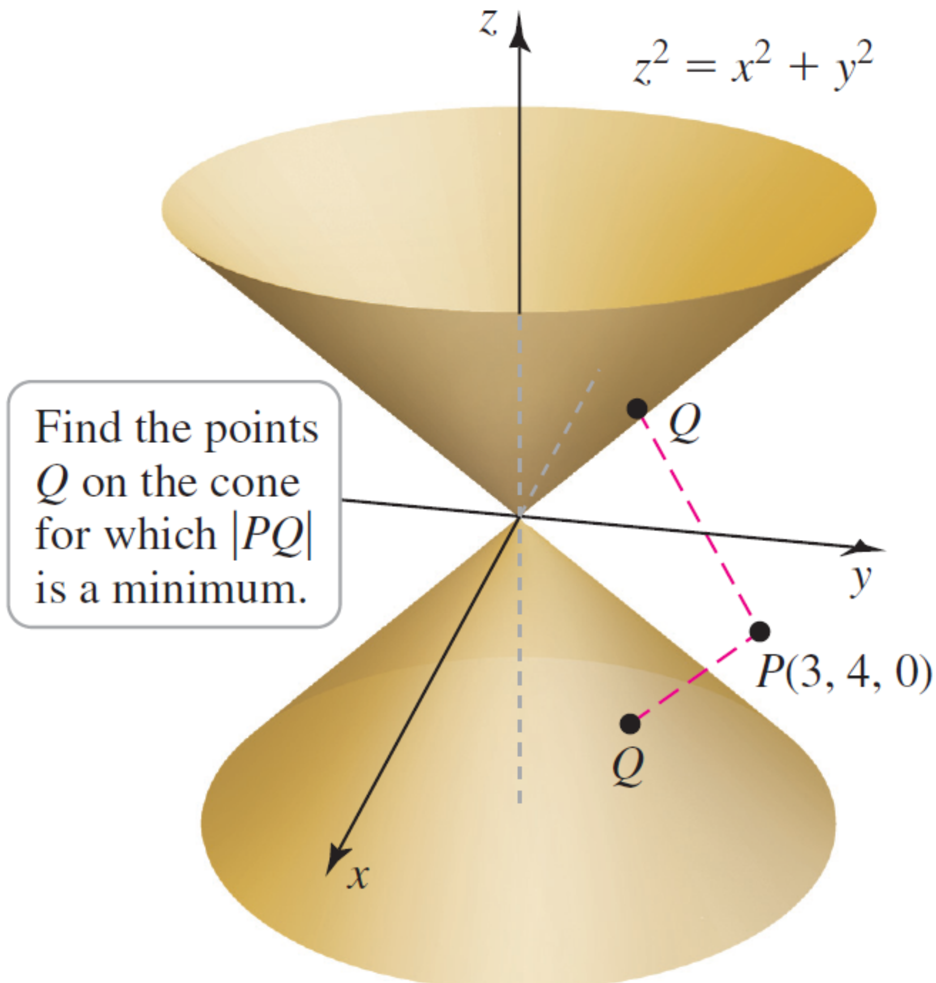
1. Find the values of *x*, *y*, *z*, and λ that satisfy the equations

$$\nabla f(x,\ y,z)=\lambda\nabla g(x,\ y,z) \quad \text{and} \quad g(x,y,z)=0$$

2. Among the values (*x*, *y*, *z*) found in Step 1, select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of *f* subject to the constraint.

<u>Example</u> Find the least distance between the point $P(3,4,0)$ and the surface of the cone $z^2 = x^2 + y^2$.

Figure 15.83



z
$z^2 = x^2 + y^2$
Find the points Q on the cone for which $|PQ|$ is a minimum.
Q
y
$P(3, 4, 0)$
Q
x

Figure 15.84

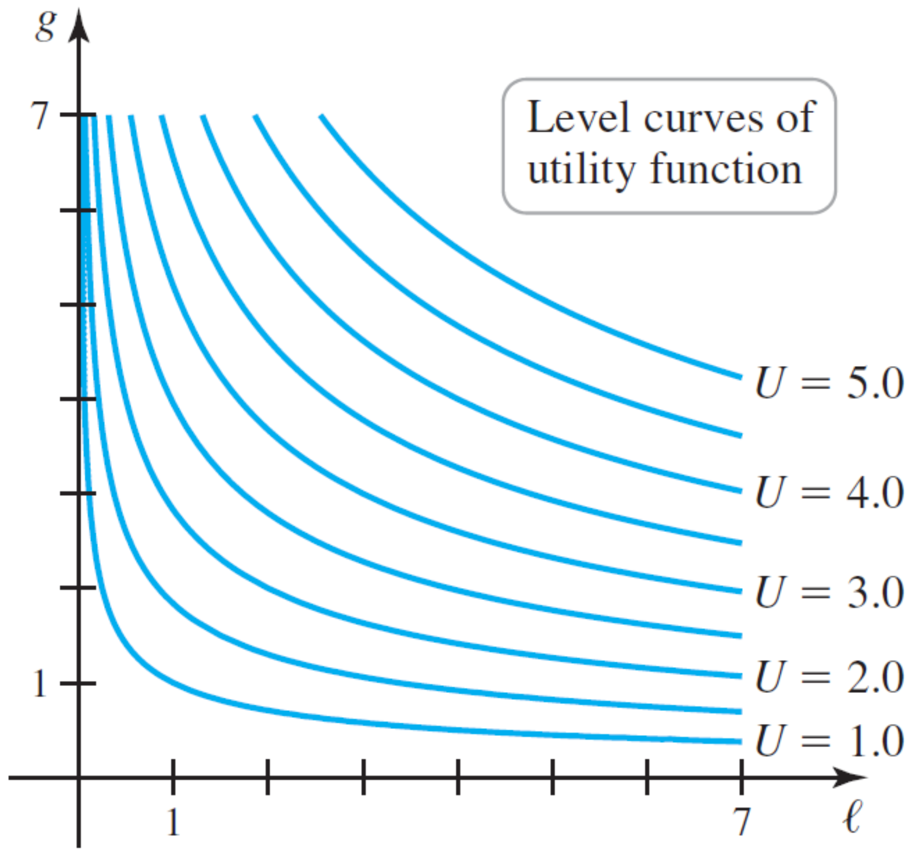
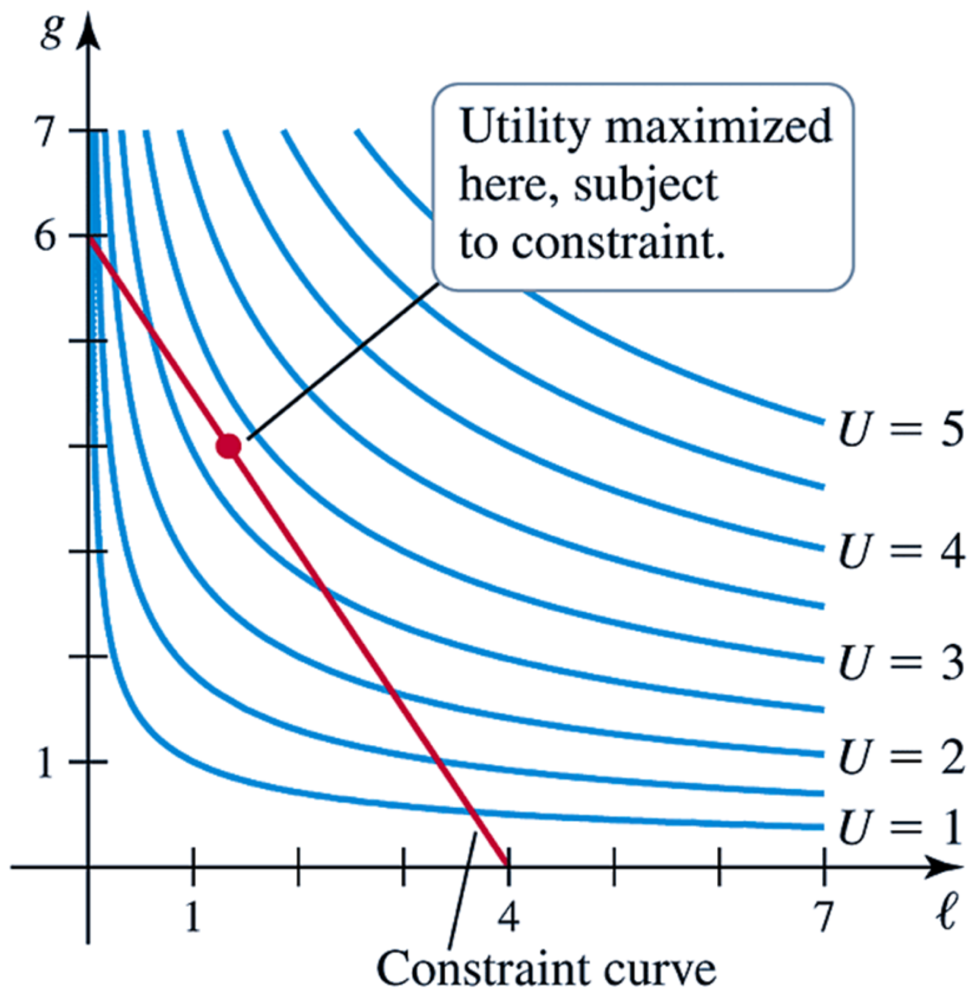


Figure 15.85


g
7
6
1
1
4
7
ℓ
Utility maximized here, subject to constraint.
U = 5
U = 4
U = 3
U = 2
U = 1
Constraint curve