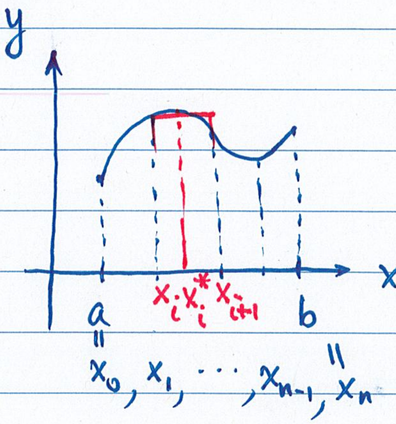
Chapter 16 Multiple Integration

(7 lectures)

§16.1 Double Integrals over Rectangular Regions

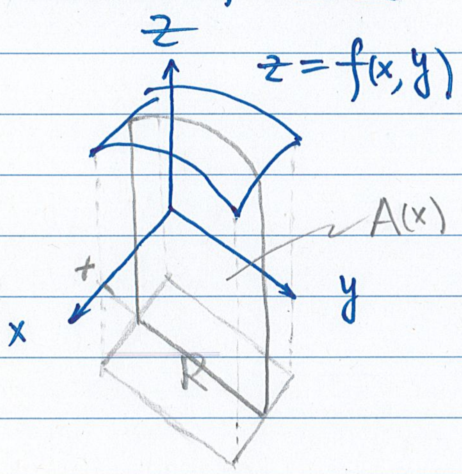
- **Review of the definite integral**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

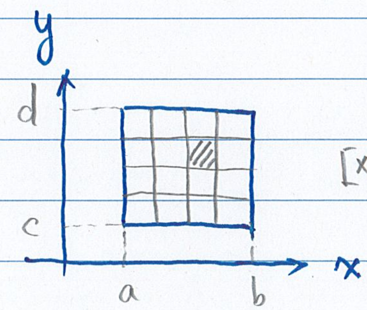


$$\Delta x = \frac{b-a}{n}$$

- **Volumes of Solids**



$R = [a,b] \times [c,d]$



$[x_i, x_{i+1}] \times [y_j, y_{j+1}]$

$$\Delta x = \frac{b-a}{m}$$

$$\Delta y = \frac{d-c}{n}$$

$$V = \iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

- **Iterated Integrals**

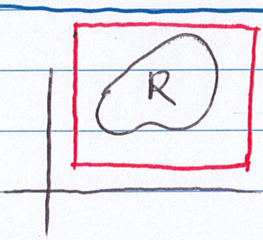
$$V = \int_a^b A(x)\,dx \qquad A(x) \text{ — area of cross-section}$$

Fubini Theorem $\displaystyle \iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

- **Average Value**

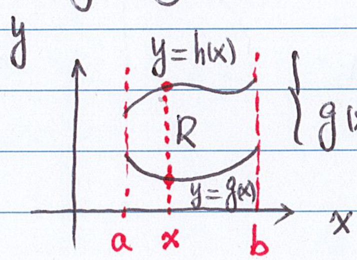
$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x,y)\,dA$$

§16.2 Double Integrals over General Regions

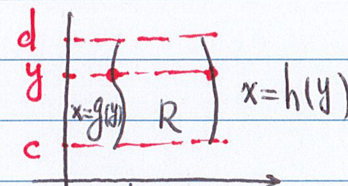


$$\iint_R f(x,y)\,dA = \iint_\Omega \tilde{f}(x,y)\,dA \,, \qquad \tilde{f} = \begin{cases} f, & \text{in } R \\ 0, & \text{in } \Omega \setminus R \end{cases}$$

elementary regions



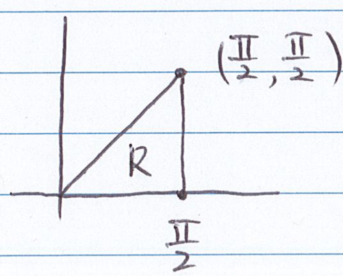
$$\begin{cases} a \le x \le b \\ g(x) \le y \le h(x) \end{cases}$$

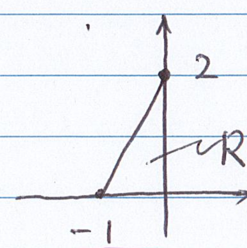


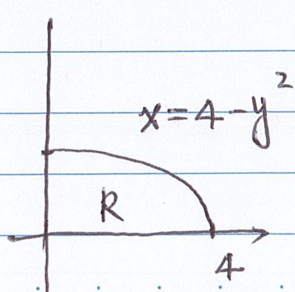
$$\begin{cases} c \le y \le d \\ g(y) \le x \le h(y) \end{cases}$$

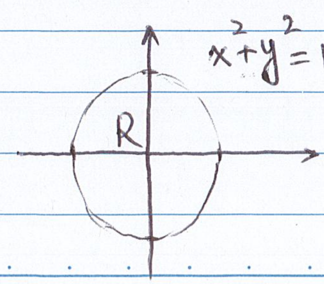
- $$\iint_R f\,dA = \int_a^b \int_{g(x)}^{h(x)} f\,dA = \int_c^d \int_{g(y)}^{h(y)} f\,dA$$

examples







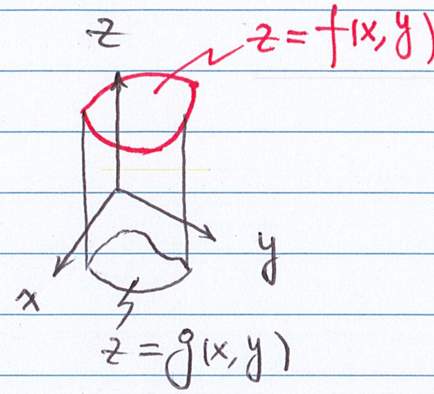


Ex. 1 Express $\iint_R 2x^2y \, dA$ as an iterated integral

R is the region bounded by the parabolas $y = 3x^2$ and $y = 16 - x^2$.

Ex. 2, 3, 4

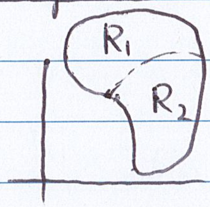
- Regions between 2 surfaces



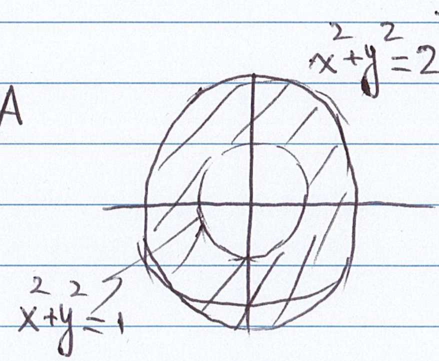
$$V = \iint_R (f - g) \, dA$$

Ex. 5

- Decomposition of regions



$$\iint_R f \, dA = \iint_{R_1} f \, dA + \iint_{R_2} f \, dA$$

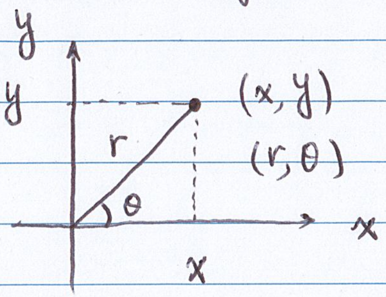


- areas of regions

$$\text{area of } R = \iint_R dA$$

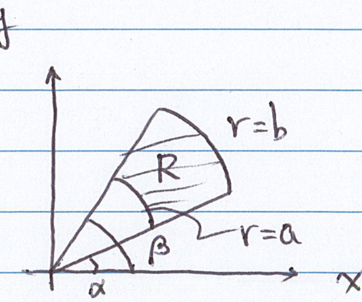
Ex. 6

§16.3 Double Integrals in Polar Coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

- <u>change of variables using polar coordinates</u>

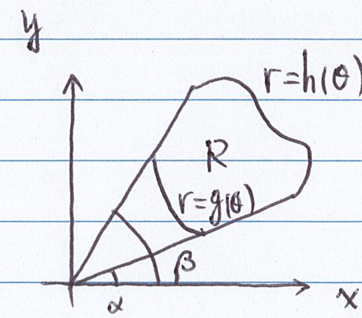


$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

where the region R is expressed in polar coordinates as $R = \left\{ (r,\theta) : \begin{array}{l} 0 \le a \le r \le b \\ \alpha \le \theta \le \beta \end{array} \right\}$

$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \text{absolute} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \text{abs.} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Ex. 1, 2, 3



$$R = \{ (r,\theta) : 0 \le g(\theta) \le r \le h(\theta),\ \alpha \le \theta \le \beta \}$$

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Ex. 4

- <u>areas of regions</u> $\quad A = \iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\,dr\,d\theta$

Ex. 5

- <u>average ~~valu~~ value</u>

Ex. 6

§16.4 Triple Integrals

Fubini's Theorem on $D=[a,b]\times[c,d]\times[r,s]$

$$\iiint_D f(x,y,z)\,dV = \int_a^b\int_c^d\int_r^s f(x,y,z)\,dz\,dy\,dx$$

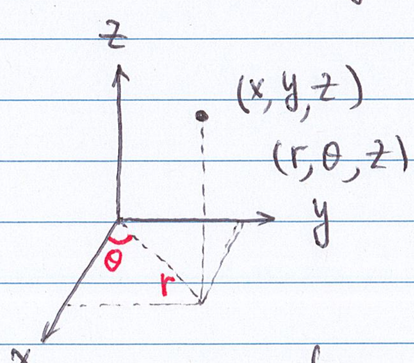
- elementary regions

$$\begin{cases} (x,y)\in R \\ G(x,y)\le z\le H(x,y) \end{cases},\quad \begin{cases} (y,z)\in R \\ G(y,z)\le x\le H(y,z) \end{cases},\quad \begin{cases} (x,z)\in R \\ G(x,z)\le y\le H(x,z) \end{cases}$$

$$\iiint_D f(x,y,z)\,dV = \iint_R \int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz =$$

Ex. 1, 2, 3, 4

§16.5 Triple Integrals in Cylindrical and Spherical Coordinates



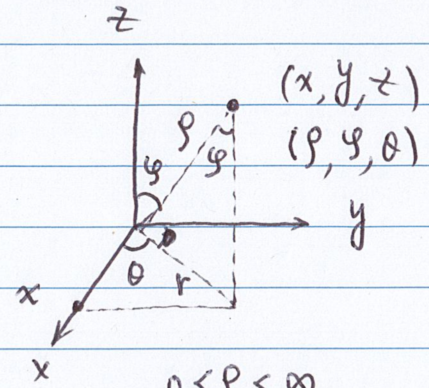
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2+y^2 \\ \tan\theta = y/x \\ z = z \end{cases}$$

$$D=\left\{(r,\theta,z):\ \begin{matrix} G(x,y)\le z\le H(x,y) \\ 0\le g(\theta)\le r\le h(\theta), \end{matrix}\quad \alpha\le\theta\le\beta\right\}$$

$$\iiint_D f(x,y,z)\,dV = \int_\alpha^\beta\int_{g(\theta)}^{h(\theta)}\int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\,r\,dr\,d\theta$$

Ex. 1, 2, 3, 4

• <u>spherical coordinates</u>



$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases} \qquad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\theta = y/x \\ \cos\varphi = z/\rho \end{cases}$$

$0 \leq \rho < \infty$
$0 \leq \varphi \leq 2\pi$
$0 \leq \theta \leq 2\pi$

Ex. 5 and descriptions of sets

$$D = \left\{ (\rho,\varphi,\theta) : 0 \leq g(\varphi,\theta) \leq \rho \leq h(\varphi,\theta),\ \begin{matrix} a \leq \varphi \leq b \\ \alpha \leq \theta \leq \beta \end{matrix} \right\}$$

$$\iiint_D f(x,y,z)\,dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} x_\rho & x_\varphi & x_\theta \\ y_\rho & y_\varphi & y_\theta \\ z_\rho & z_\varphi & z_\theta \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

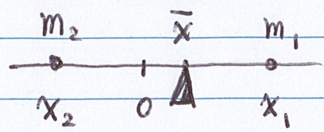
$$= \cos\varphi\ \rho^2\cos\varphi\sin\varphi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} + \rho\sin\varphi\ \rho\sin^2\varphi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$= \rho^2 \sin\varphi$$

Ex. 6, 7.

§16.6 Integrals for Mass Calculations (Center of Mass)

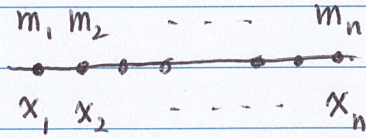
- discrete objects



Find the center of mass

let $\bar{x}$ be the center $\Longrightarrow m_1(x_1-\bar{x}) = m_2(\bar{x}-x_2)$

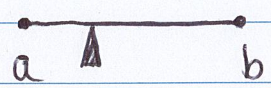
$$\Longrightarrow \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$$



$$m_1(x_1-\bar{x}) + \cdots + m_n(x_n-\bar{x}) = 0 \Longrightarrow \bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1+\cdots+m_n}$$

Ex. 1

- continuous objects in one dimension



a thin rod with density ρ

uniform partition: $a = x_0 < x_1 < \cdots < x_n = b$ $\qquad \Delta x = \frac{b-a}{n}$ $\qquad x_i = x_0 + i\,\Delta x$

on subinterval $[x_{k-1}, x_k]$ $\quad m_k \approx \rho(x_k)\,\Delta x$

$$\Longrightarrow \bar{x} = \lim_{\Delta x\to 0} \sum_{k=1}^{n}\left(\rho(x_k)\Delta x\right)x_k \Big/ \sum_{k=1}^{n}\rho(x_k)\Delta x = \underbrace{\int_a^b x\,\rho(x)\,dx}_{\substack{\| \\ M \\ \text{the total moment}}} \Big/ \underbrace{\int_a^b \rho(x)\,dx}_{\substack{\| \\ m \\ \text{the total mass}}}$$

Ex. 2

- 2 and 3 dimensions

2D $\quad \bar{x} = \overbrace{\iint_R x\rho\,dA}^{M_y} \Big/ \underbrace{\iint_R \rho\,dA}_{m}, \quad \bar{y} = \overbrace{\iint_R y\rho\,dA}^{M_x} \Big/ \iint_R \rho\,dA, \quad (\bar{x},\bar{y})$ — centroid if $\rho = 1$

3D $\quad \bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\rho\,dV, \quad \bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\rho\,dV, \quad \bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\rho\,dV$

Ex. 3, 4, 5, 6