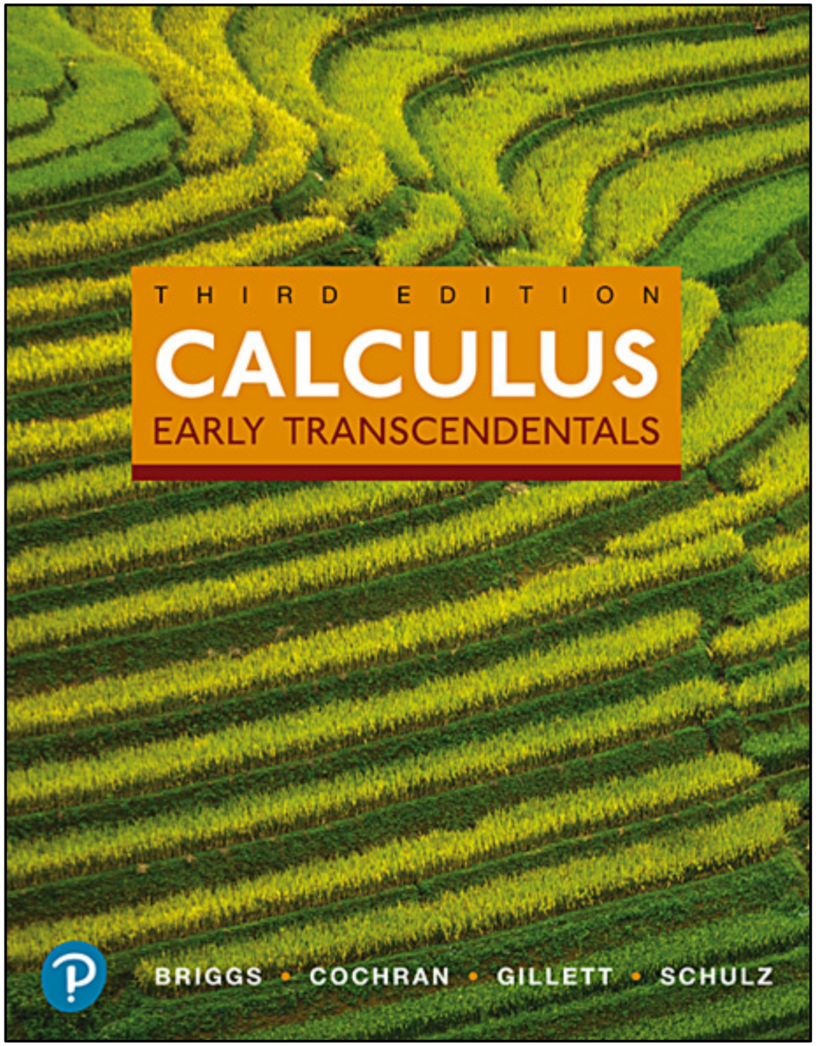
Calculus Early Transcendentals

Third Edition



Chapter 16

Multiple Integration

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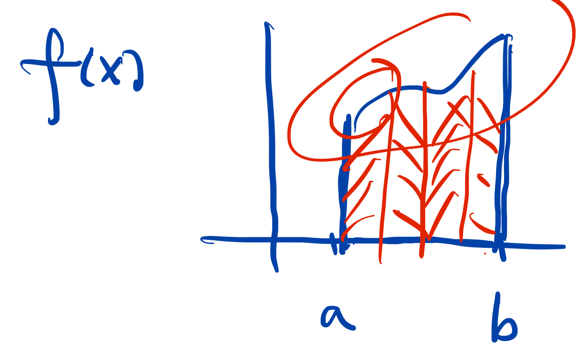
Section 16.1 Double Integrals over Rectangular Regions

Table 16.1

	Derivatives	Integrals
Single variable: $f(x)$	$f'(x)$	$\int_a^b f(x)dx$
Several variables: $f(x,y)$ and $f(x,y,z)$	$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$	$\iint_R f(x,y)dA, \iiint_D f(x,y,z)dV$



$\int_a^b f(x)dx = A$

Figure 16.1 (a & b)

A three-dimensional solid bounded by $z = f(x, y)$ and a region R in the xy-plane is approximated by a collection of boxes.

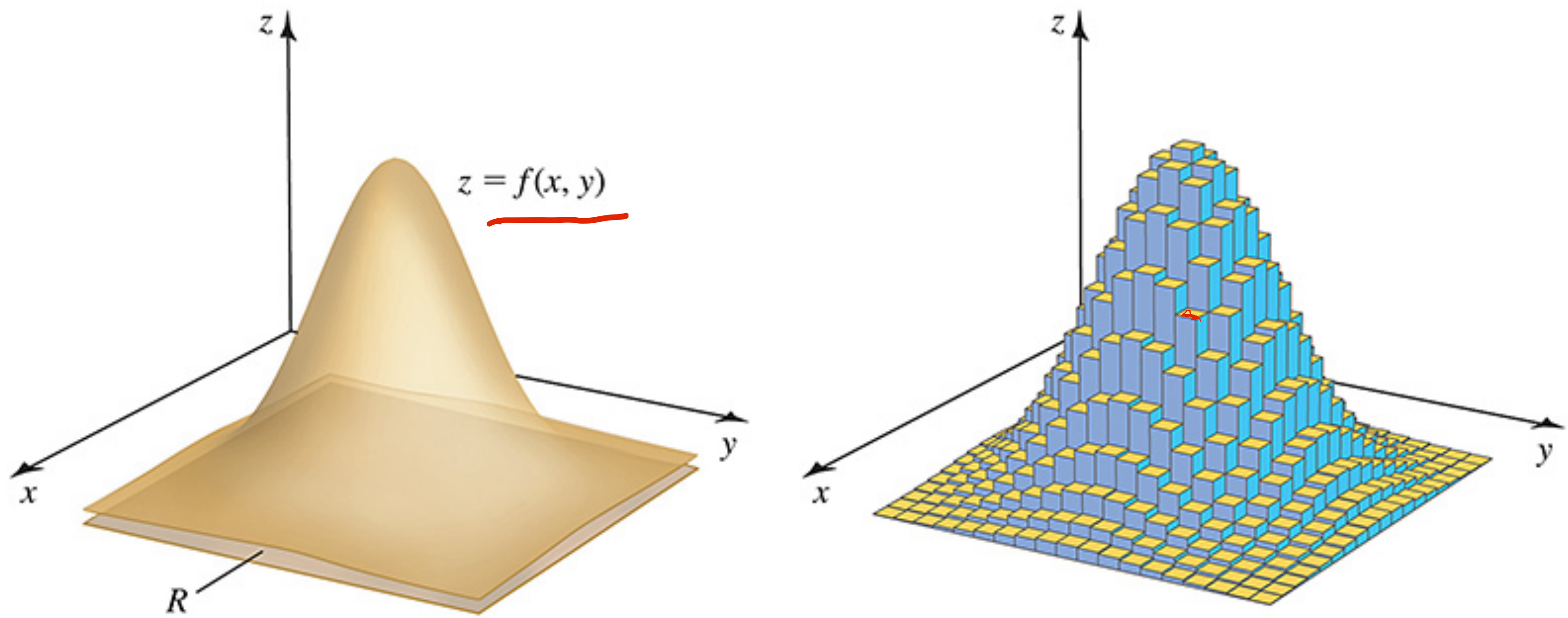
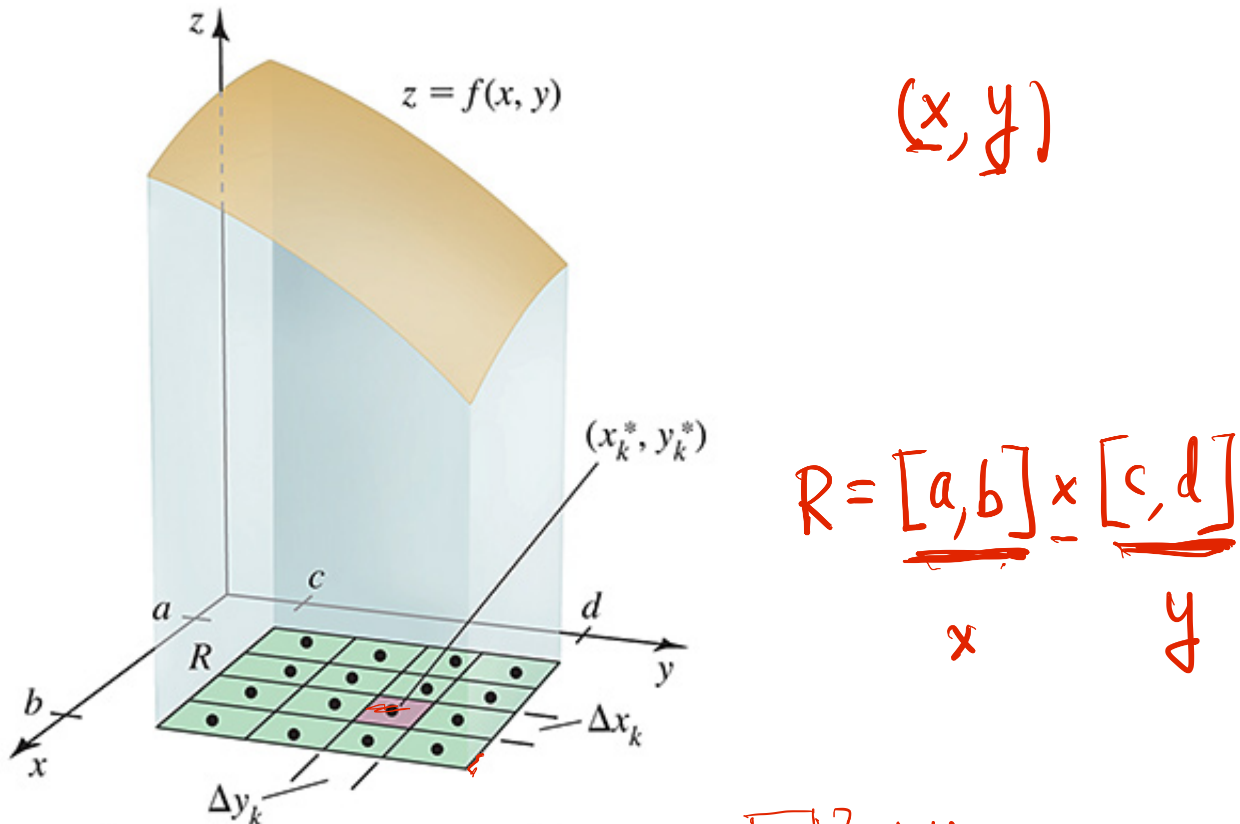


Figure 16.2



(x_k^*, y_k^*) is a point in the kth rectangle, which has area $\Delta A_k = \Delta x_k \Delta y_k$.

Figure 16.3

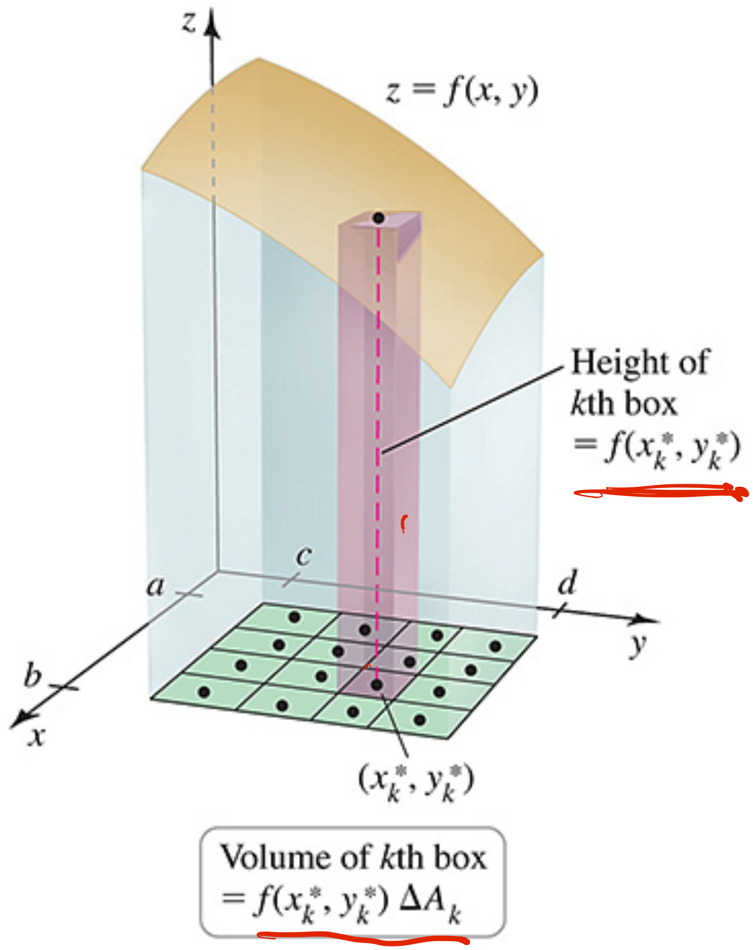

z
$z = f(x, y)$
Height of
kth box
$= f(x_k^*, y_k^*)$
a
b
c
d
x
y
(x_k^*, y_k^*)
Volume of kth box
$= f(x_k^*, y_k^*)\ \Delta A_k$

Figure 16.4 (1 of 2)

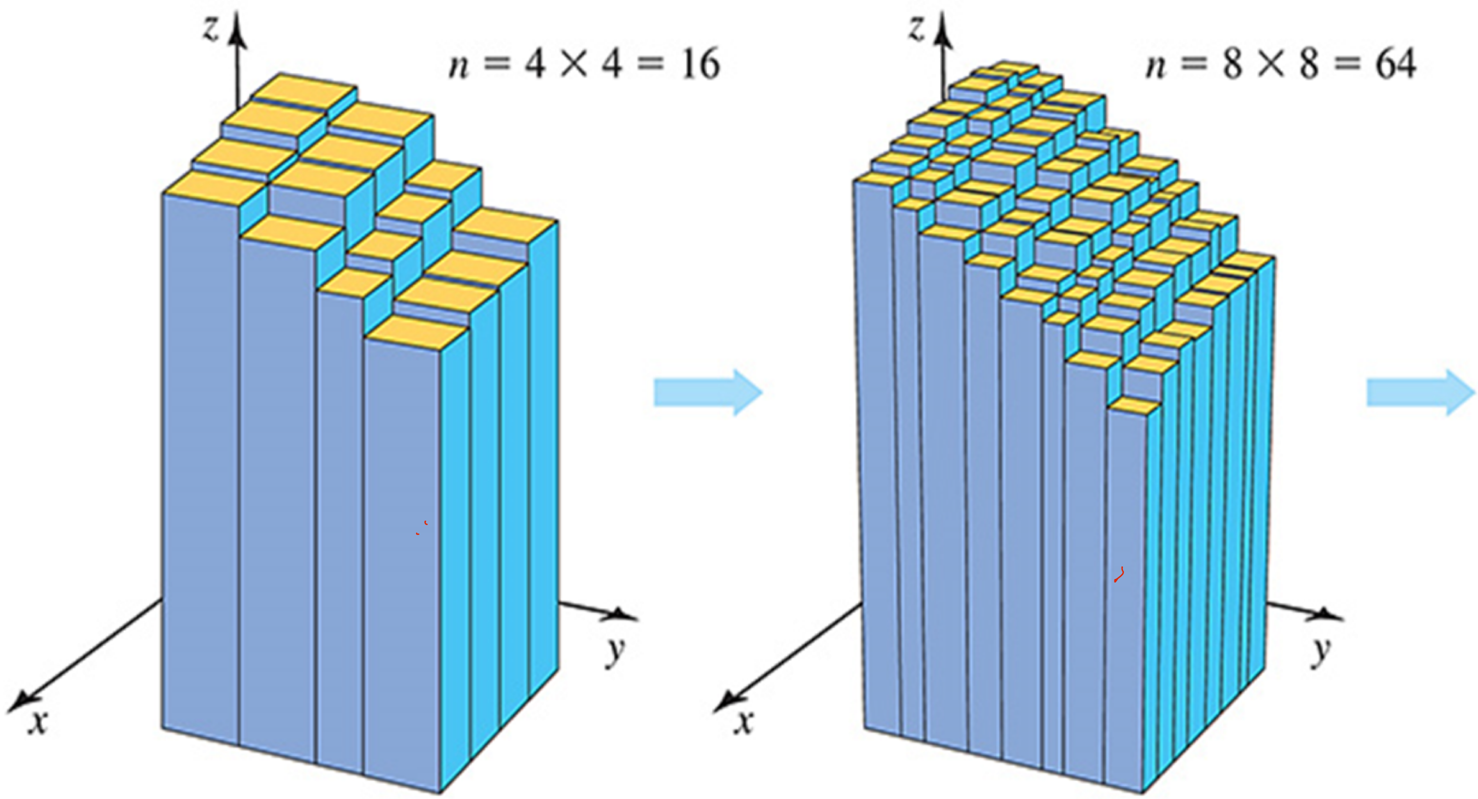
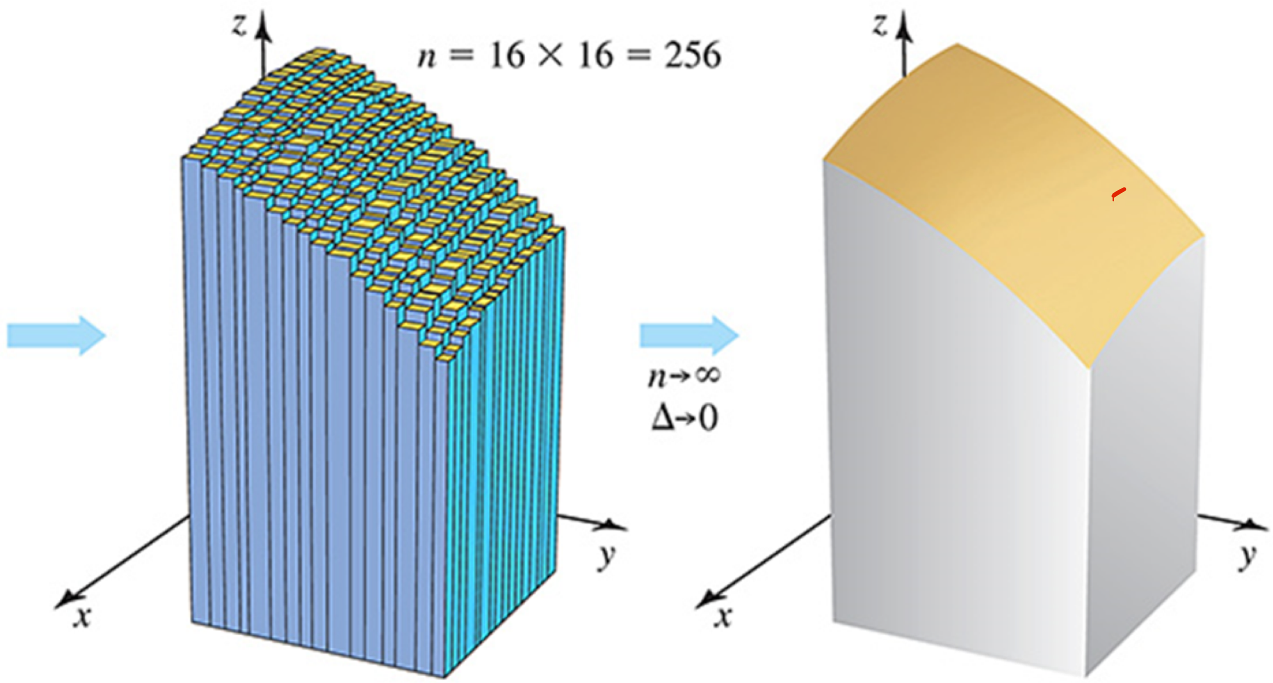


Figure 16.4 (2 of 2)



$$\text{Volume} = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k = \iint_R f(x, y)\, dA$$

Definition Double Integrals

A function *f* defined on a rectangular region *R* in the *xy*-plane is **integrable** on *R* if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$ exists for all partitions of *R* and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of *f* over *R***, which we write

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k.$$

How to compute double integral?

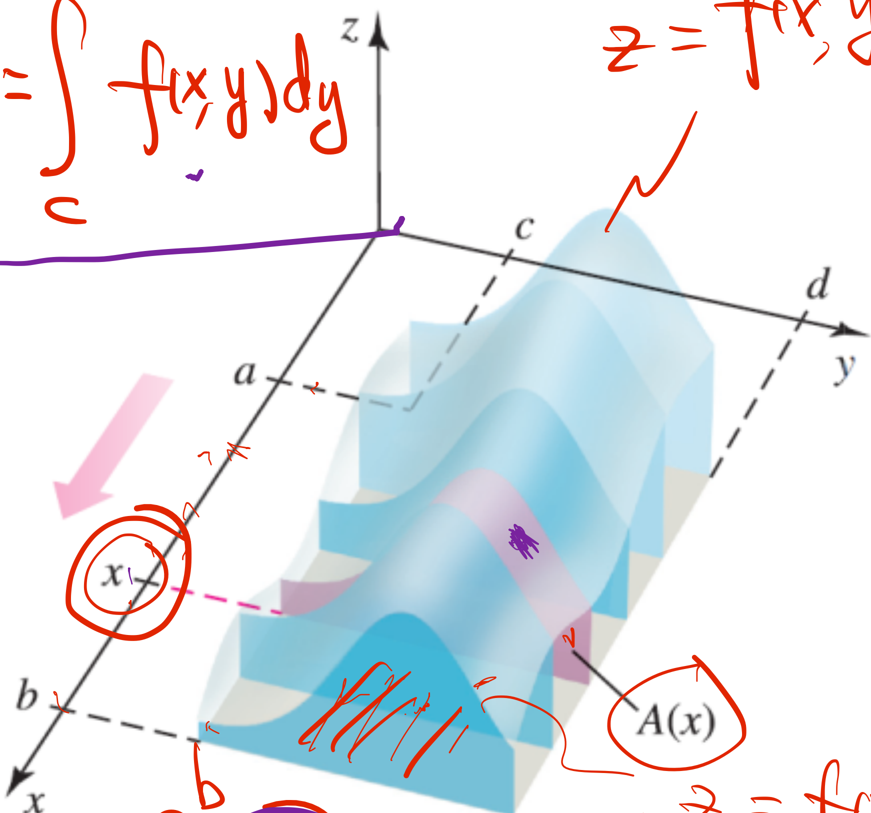
$$\int_0^1\int_0^2 (6-2x-y)\,dA \qquad R=[0,1]\times[0,2]$$

$$=\int_0^1\left(\int_0^2 (6-2x-y)\,dy\right)dx$$

Figure 16.5

$$A(x)=\int_c^d f(x,y)\,dy$$

$z=f(x,y)$



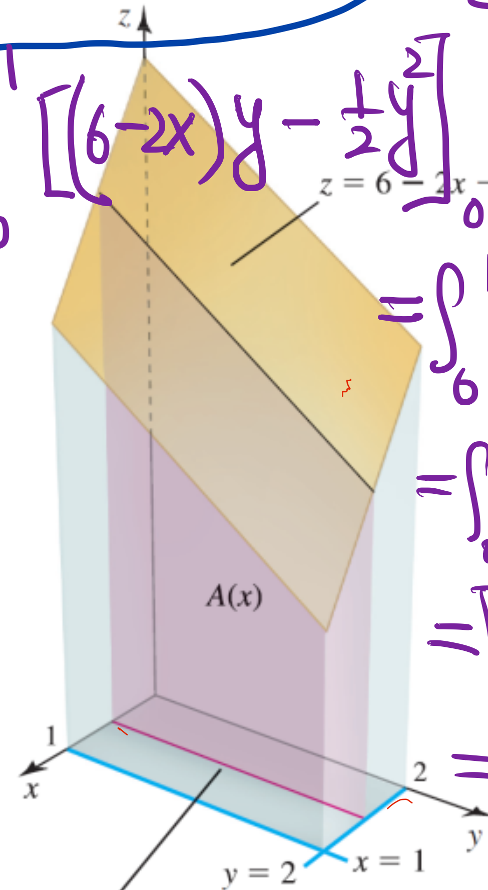
$$=\int_0^1\left[(6-2x)y-\tfrac{1}{2}y^2\right]_0^2 dx$$

$$=\int_0^1\left[2(6-2x)-2\right]dx$$

$$=\int_0^1 (10-4x)\,dx$$

$$=\left[10x-2x^2\right]_0^1$$

$$=10-2=8$$



A slice at a fixed value of x has area $A(x)$, where $0 \le x \le 1$.

$$V=\int_a^b A(x)\,dx$$

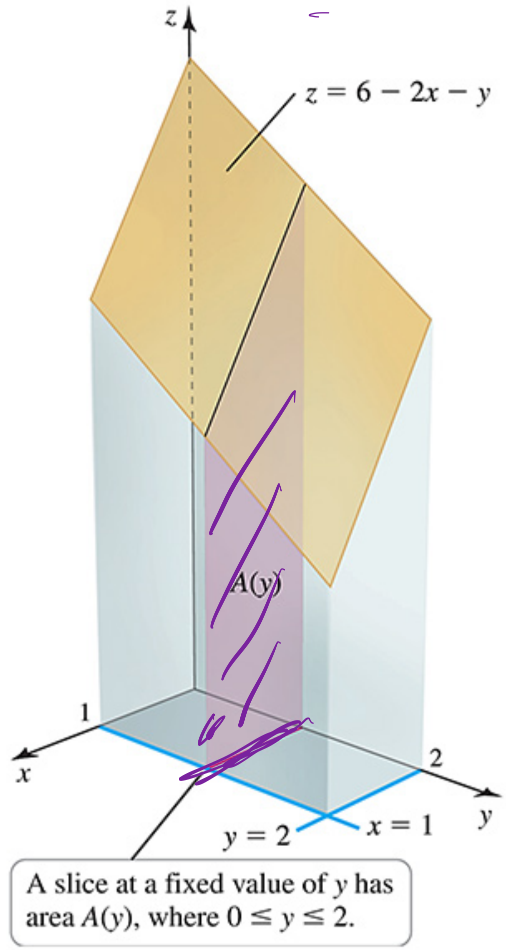
$$\left\{\begin{array}{l} z=f(x,y) \\ x=b \end{array}\right.$$

$$=\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

the same integral in a different order.

$$V = \int_0^2 \left(\int_0^1 (6-2x-y)\,dx \right) dy$$

Figure 16.6



y = const

Theorem 16.1 (Fubini) Double Integrals on Rectangular Regions

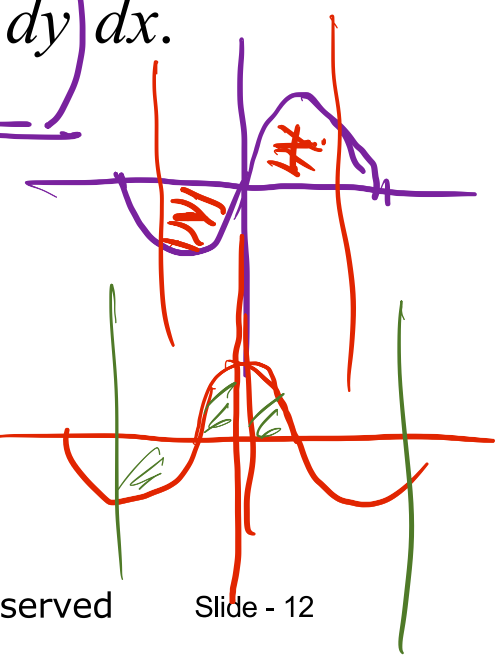
Let *f* be continuous on the rectangular region

$R = \{(x, y) : a \le x \le b, c \le y \le d\}$. The double integral of *f* over

R may be evaluated by either of two iterated integrals:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & f \text{ is odd} \\ 2\int_0^a f(x)\,dx & f \text{ is even} \end{cases}$$



Example 3 Find the volume of the solid bounded by the surface $f(x,y)=4+9x^2y^2$ over the region $R=[-1,1]\times[0,2]$.

$$V=\iint_R (4+9x^2y^2)\,dA$$

$$=\int_{-1}^{1}\int_0^2 (4+9x^2y^2)\,dy\,dx$$

$$=\int_{-1}^{1}\left[4y+3x^2y^3\right]_{y=0}^{2}dx$$

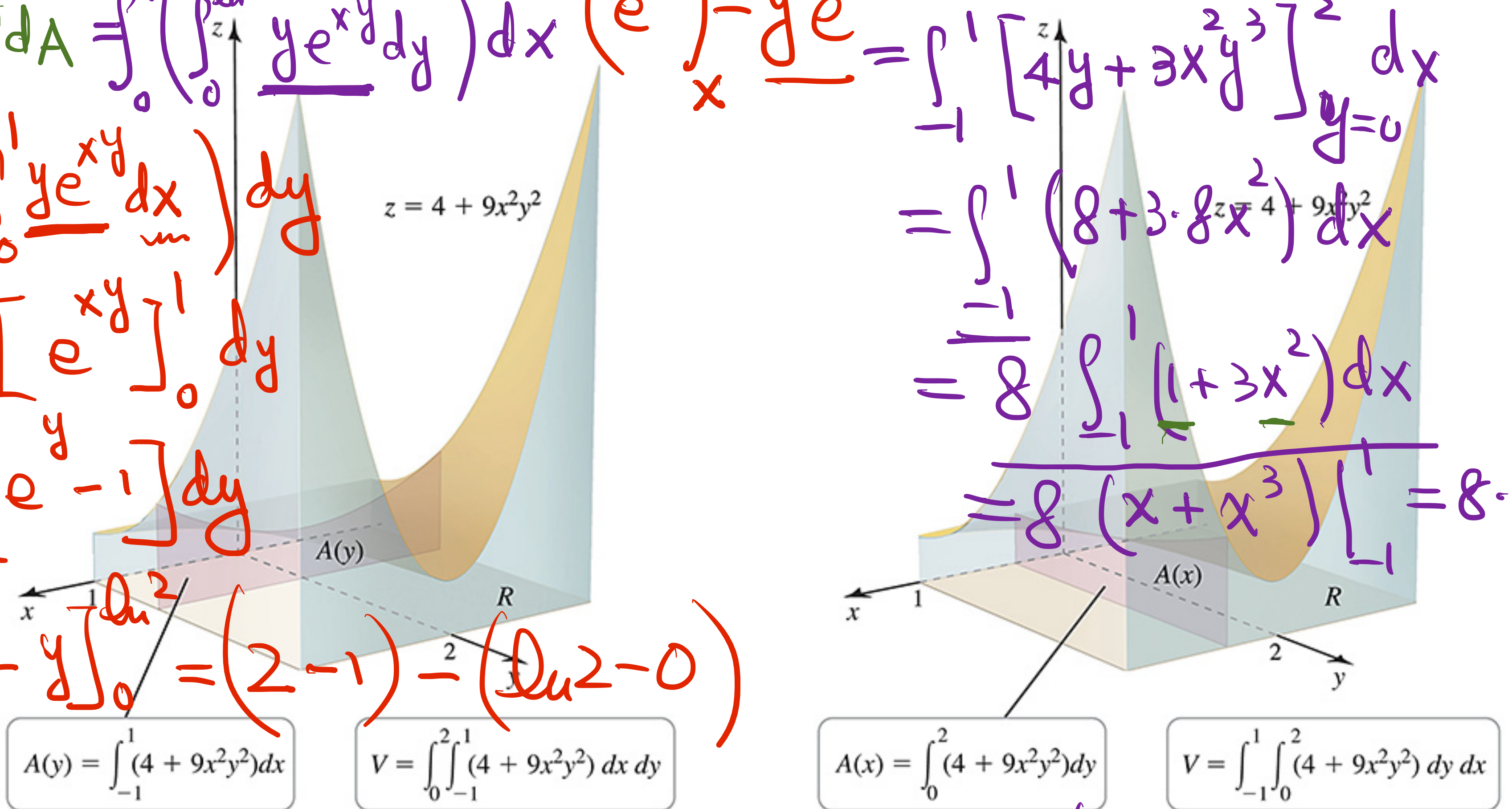
$$=\int_{-1}^{1}(8+3\cdot 8x^2)\,dx$$

$$=8\int_{-1}^{1}(1+3x^2)\,dx$$

$$=8(x+x^3)\Big|_{-1}^{1}=8\cdot 4=32$$

$$(1+1)-(-1-1)=4$$

Figure 16.7 (a & b)



Example 4 $R=[0,1]\times[0,\ln 2]$

$$\iint_R ye^{xy}\,dA=\int_0^1\left(\int_0^{\ln 2} ye^{xy}\,dy\right)dx$$

$$\left(e^{xy}\right)_x=ye^{xy}$$

$$=\int_0^{\ln 2}\left(\int_0^1 ye^{xy}\,dx\right)dy$$

$$=\int_0^{\ln 2}\left[e^{xy}\right]_0^1 dy$$

$$=\int_0^{\ln 2}\left[e^y-1\right]dy$$

$$=\left[e^y-y\right]_0^{\ln 2}=(2-1)-(\ln 2-0)$$

$$=1-\ln 2$$

Definition Average Value of a Function over a Plane Region

The **average value** of an integrable function *f* over a region *R* is

$$\bar{f} = \frac{1}{\text{area of } R}\iint_R f(x, y)\,dA.$$

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 5 Find the average value of the quantity $f(x,y)=2-x-y$ over $R=[0,2]\times[0,2]$

Figure 16.8

area of $R = 2\times 2 = 4$

$$\iint_R f\,dA = \int_0^2\left(\int_0^2 (2-x-y)\,dx\right)dy$$

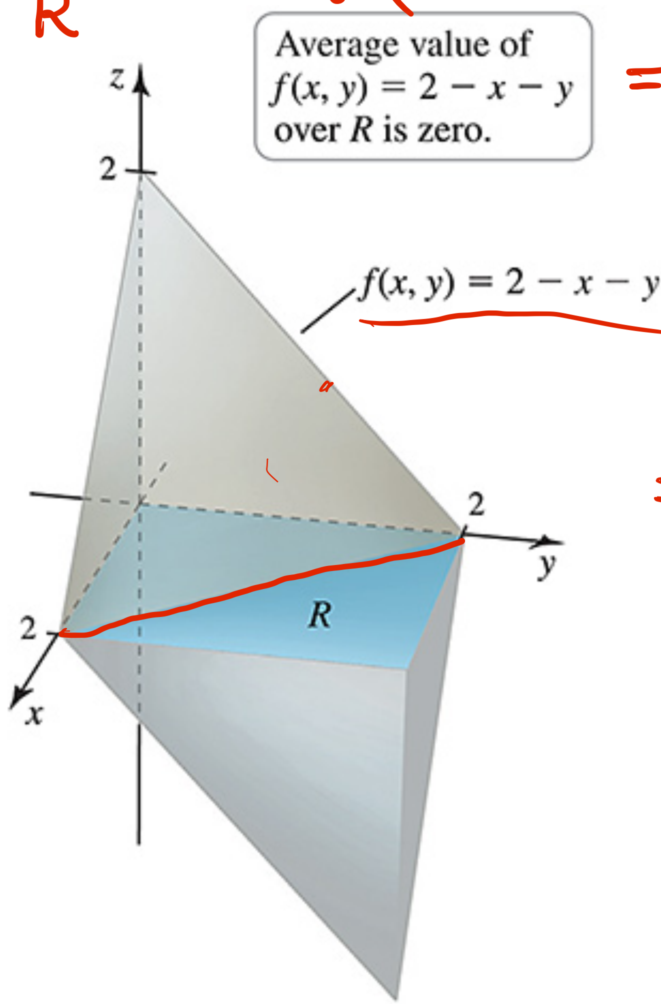
$$= \int_0^2 \left((2-y)x - \tfrac{1}{2}x^2\right)\Big|_0^2\,dy$$

$$= \int_0^2 \left[2(2-y) - 2\right]dy$$

$$= \int_0^2 \left[2-2y\right]dy$$

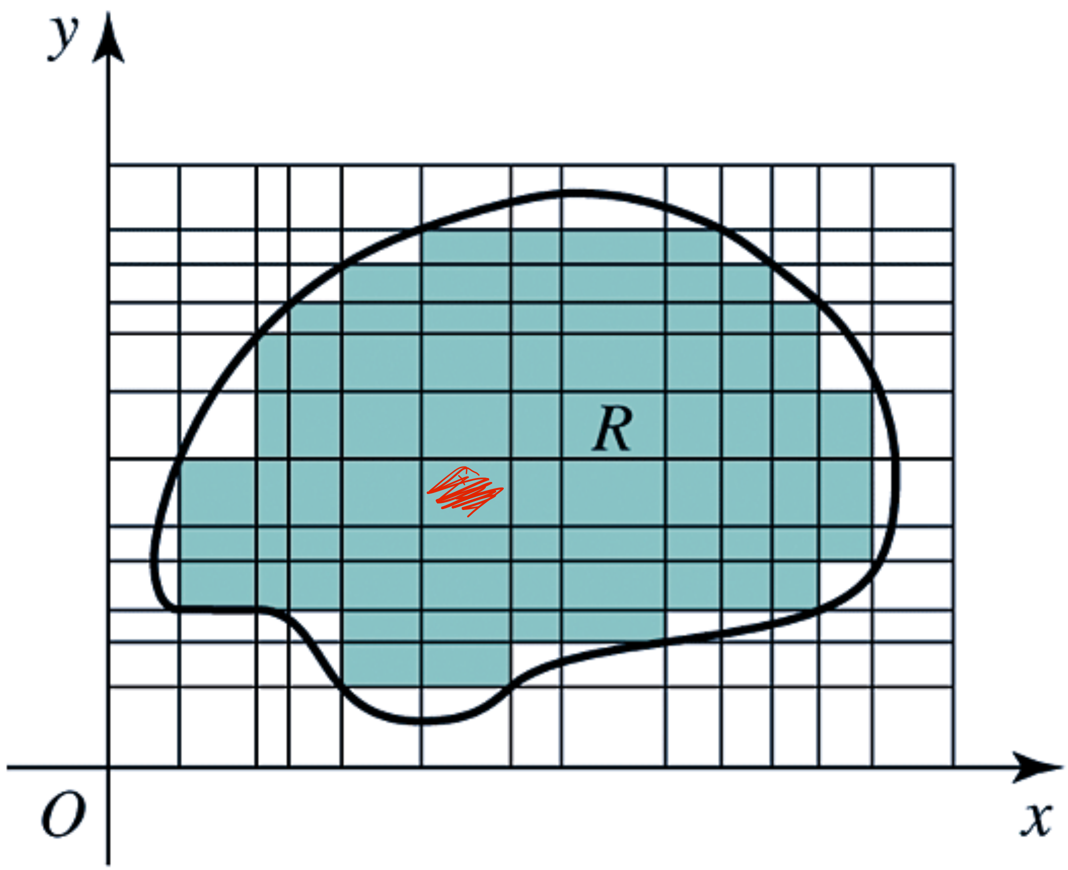
$$= \left[2y - y^2\right]_0^2 = 0$$

$$\bar{f} = \frac{0}{4} = 0$$



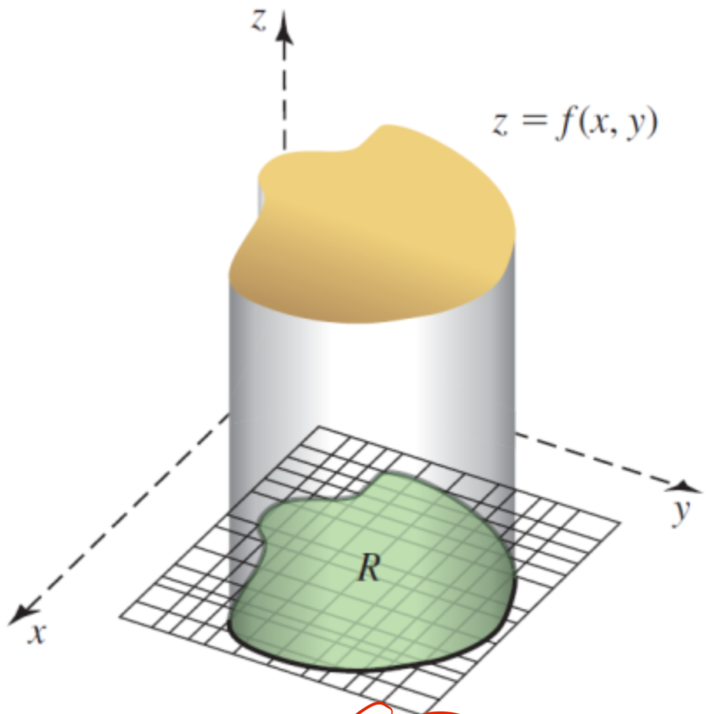
Section 16.2 Double Integrals over General Regions

Figure 16.9



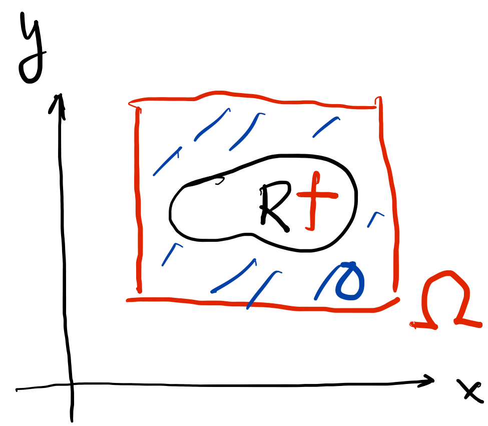
$$\iint_R f(x,y)\,dA = \iint_\Omega \tilde{f}\,dA \qquad \tilde{f} = \begin{cases} f, & \text{in } R \\ 0, & \Omega \setminus R \end{cases}$$

Figure 16.10



Volume of solid $= \iint\limits_R f(x, y)\, dA$

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$



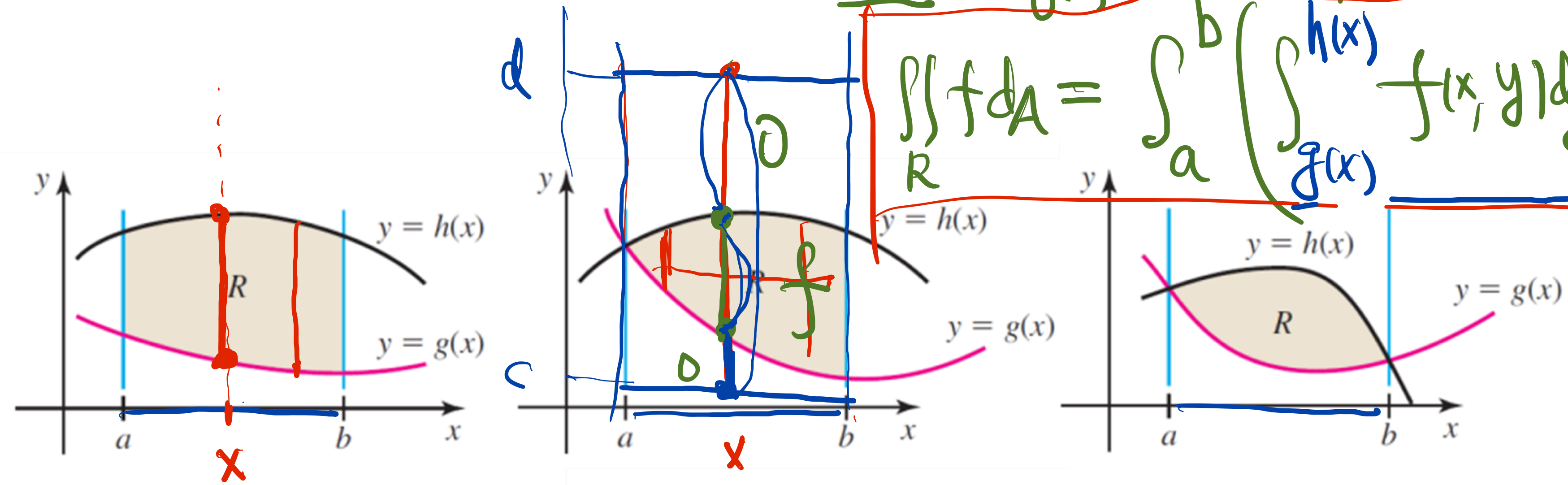
Elementary Regions

$$\iint_R f(x,y)\,dA = \iint_\Omega \tilde{f}\,dA = \int_a^b \left(\int_c^d \tilde{f}\,dy \right) dx$$

$$= \int_a^b \left[\int_c^{g(x)} 0 + \int_{g(x)}^{h(x)} f + \int_{h(x)}^{d} 0 \; dy \right] dx$$

Figure 16.11

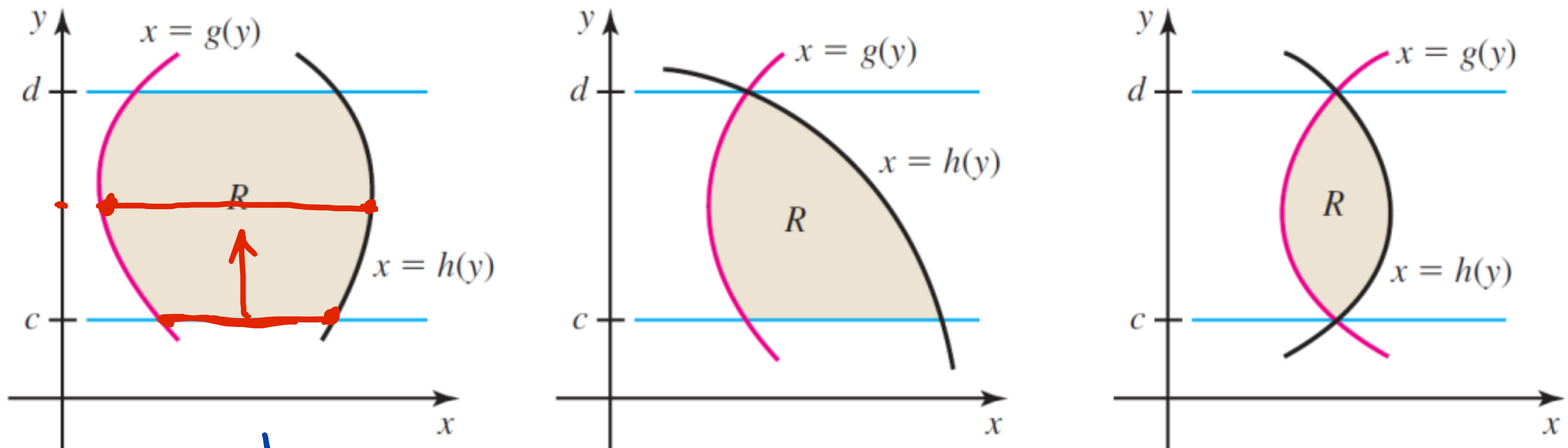
$$\iint_R f\,dA = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$



$$\begin{cases} a \le x \le b \\ g(x) \le y \le h(x) \end{cases}$$

Figure 16.15

$$\iint_R f\,dA = \int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\,dx \right) dy$$



$$R \begin{cases} c \le y \le d \\ g(y) \le x \le h(y) \end{cases}$$

Theorem 14.2 Double Integrals over Nonrectangular Regions

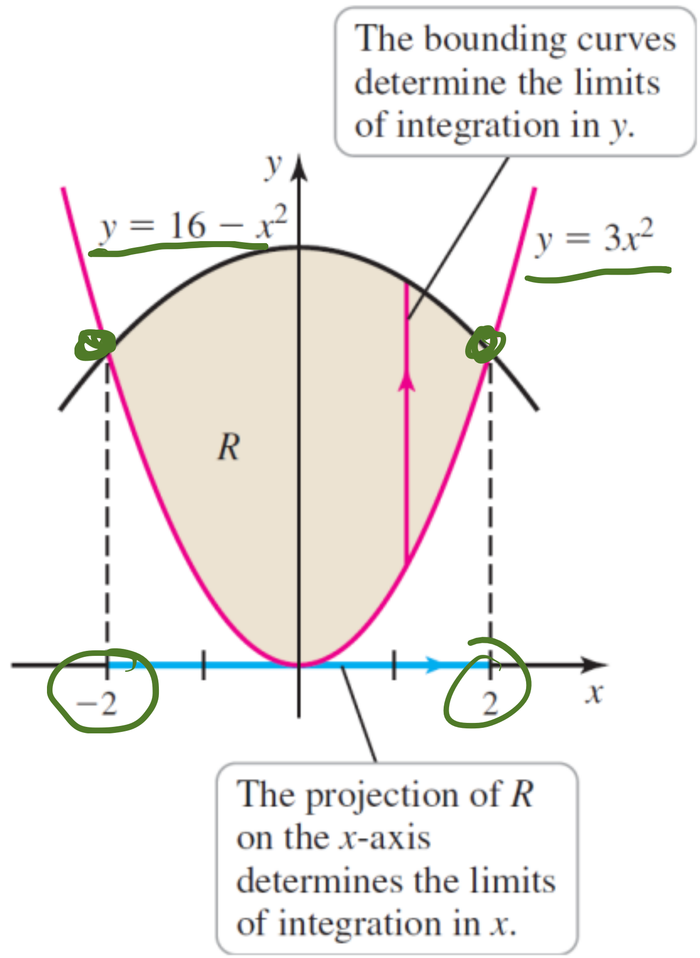
Let *R* be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines *x* = *a* and *x* = *b* (Figure 16.11). If *f* is continuous on *R*, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx.$$

Let *R* be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines *y* = *c* and *y* = *d* (Figure 16.15). If *f* is continuous on *R*, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy.$$

Figure 16.13


The bounding curves determine the limits of integration in y.
y
$y = 16 - x^2$
$y = 3x^2$
R
−2
2
x
The projection of R on the x-axis determines the limits of integration in x.

Example 1 Let R be the region bounded by the parabolas $y=3x^2$ and $y=16-x^2$.

$$\iint_R 2x^2y\,dA$$

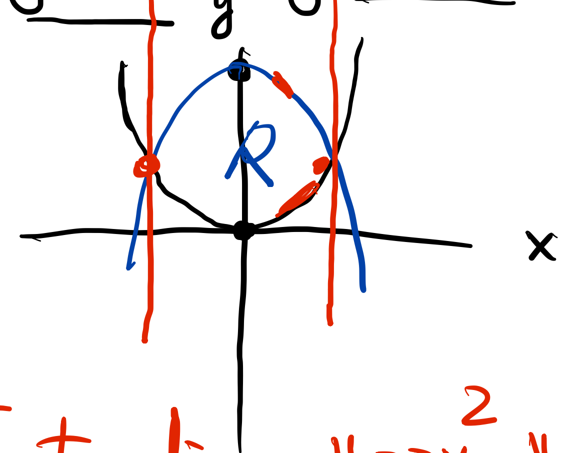


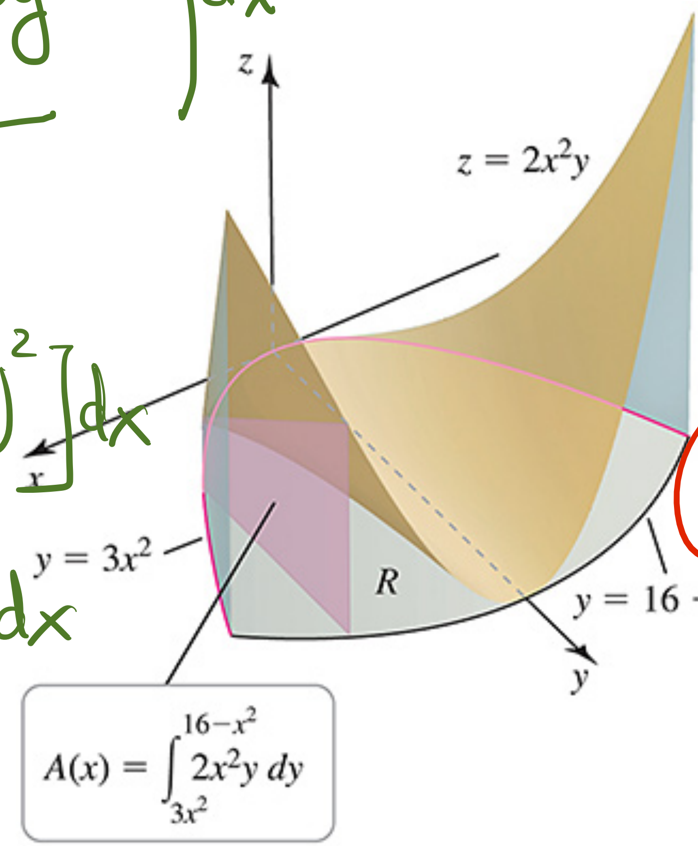
Figure 16.14

$$=\int_{-2}^{2}\left(\int_{3x^2}^{16-x^2}(2x^2y)\,dy\right)dx$$

$$=\int_{-2}^{2} x^2y^2\Big|_{y=3x^2}^{16-x^2}dx$$

$$=\int_{-2}^{2} x^2\left[(16-x^2)^2-(3x^2)^2\right]dx$$

$$=2\int_{0}^{2} x^2\left[\qquad\right]dx$$



intersection $y=3x^2=y=16-x^2$

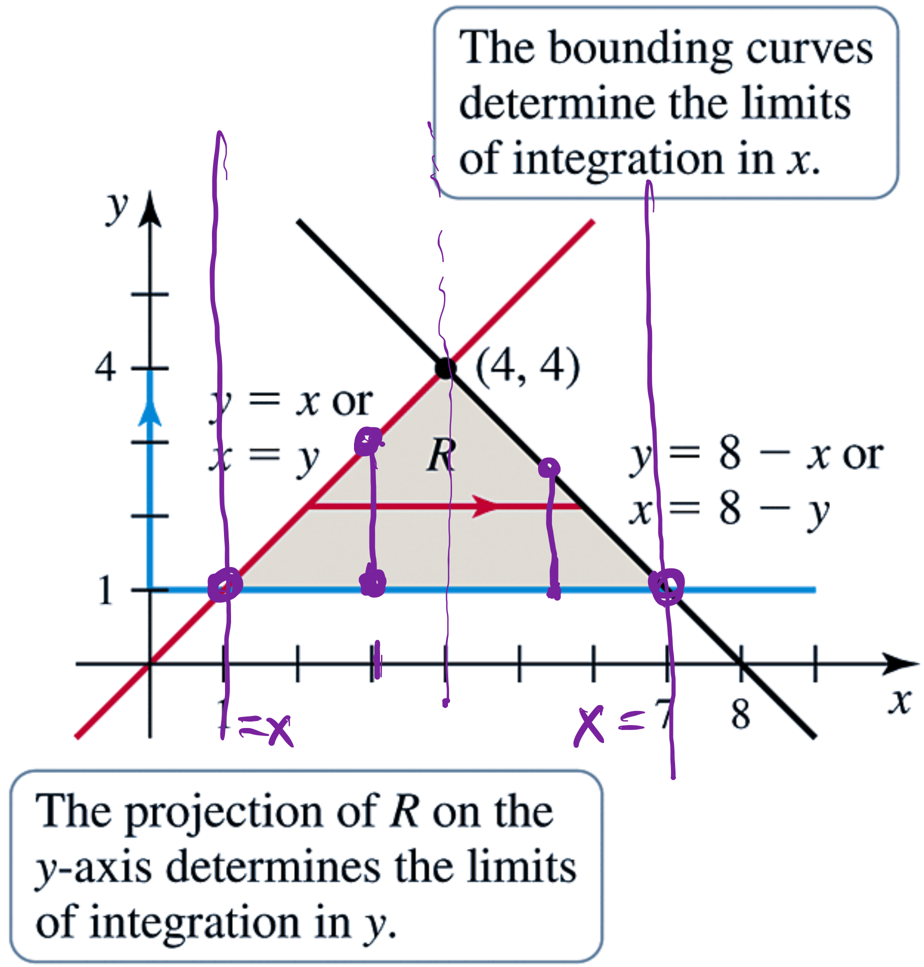
$3x^2=16-x^2 \Rightarrow x^2=4$

$x=\pm 2$

$$R\begin{cases} -2\le x\le 2 \\ 3x^2\le y\le 16-x^2 \end{cases}$$

Example 2 Find the volume of the solid below the surface $f(x,y) = 2 + \frac{1}{y}$ and above the region R in the xy-plane bounded by lines $y = x$, $y = 8-4$, and $y = 1$.

Figure 16.16

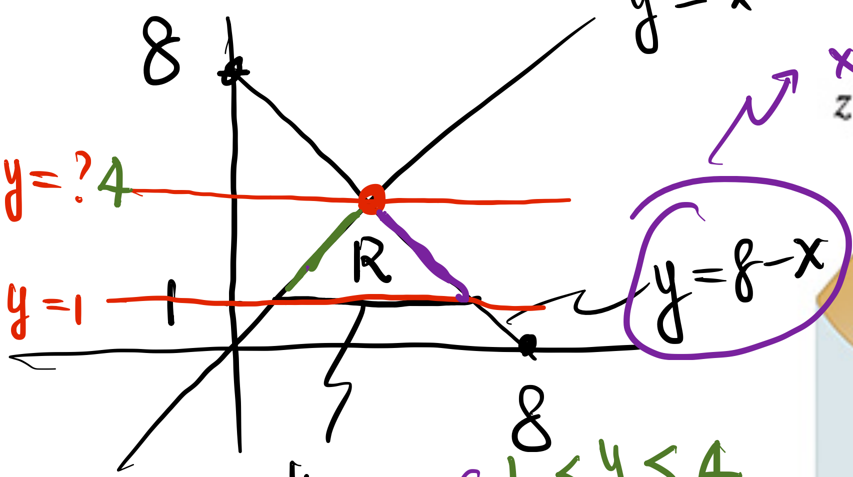


Example 2 Let R be the region in the xy-plane bounded by lines $y=x$, $y=8-x$, and $y=1$.

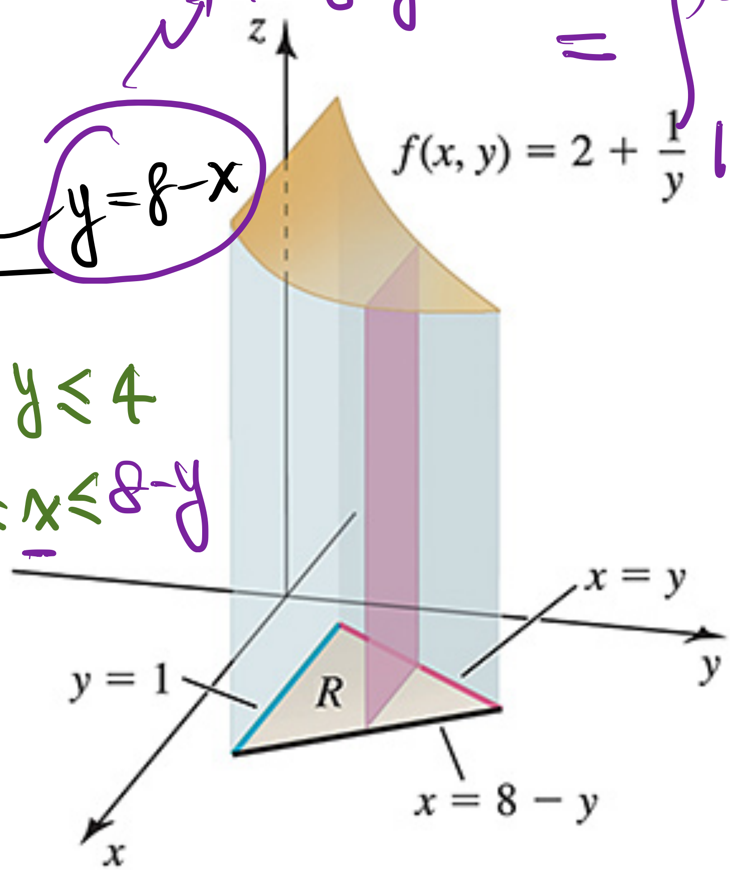
$$\iint_R \left(2+\frac{1}{y}\right) dA = \int_1^4 \left(\int_y^{8-y} \left(2+\frac{1}{y}\right) dx \right) dy$$

$$= \int_1^4 \left(2+\frac{1}{y}\right)(8-2y)\, dy$$

Figure 16.17



$x = 8-y$



$$R \begin{cases} 1 \le y \le 4 \\ y \le x \le 8-y \end{cases}$$

$y=x = y=8-x$

$2x=8,\ x=4$

$y=4$

Figure 16.18

$$\begin{cases} 0 \le y \le 1 \\ y^3 \le x \le \sqrt{y} \end{cases}$$

$y = x^{\frac{1}{3}} = y = x^2$

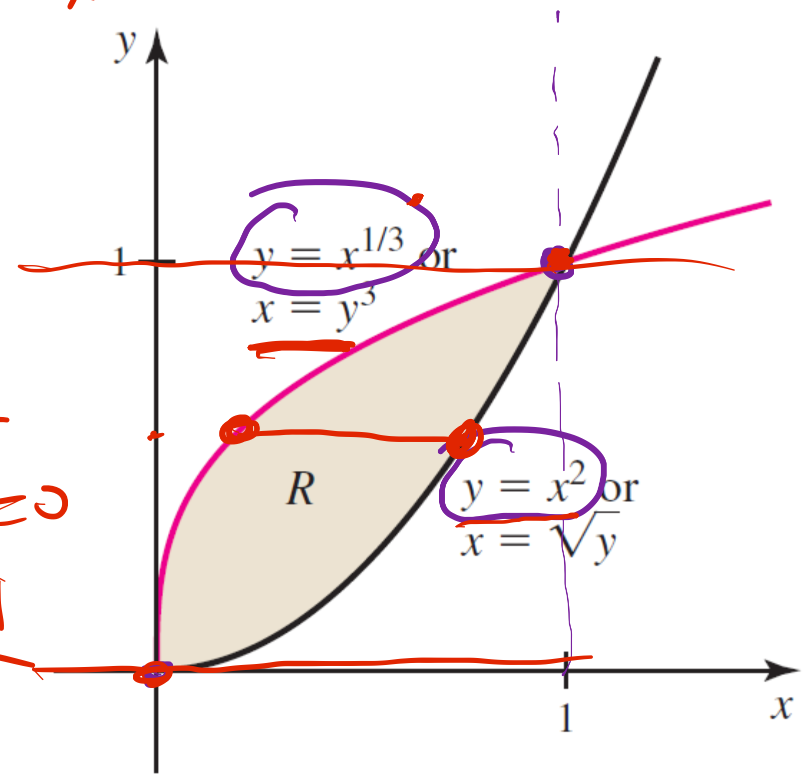
$x^{1/3} = x^2$

$x = x^6$

$0 = x(1 - x^5)$

$\Rightarrow x = 0$ or $1 - x^5 = 0$

$x = 1$



$R \begin{cases} 0 \le x \le 1 \\ x^2 \le y \le x^{\frac{1}{3}} \end{cases}$

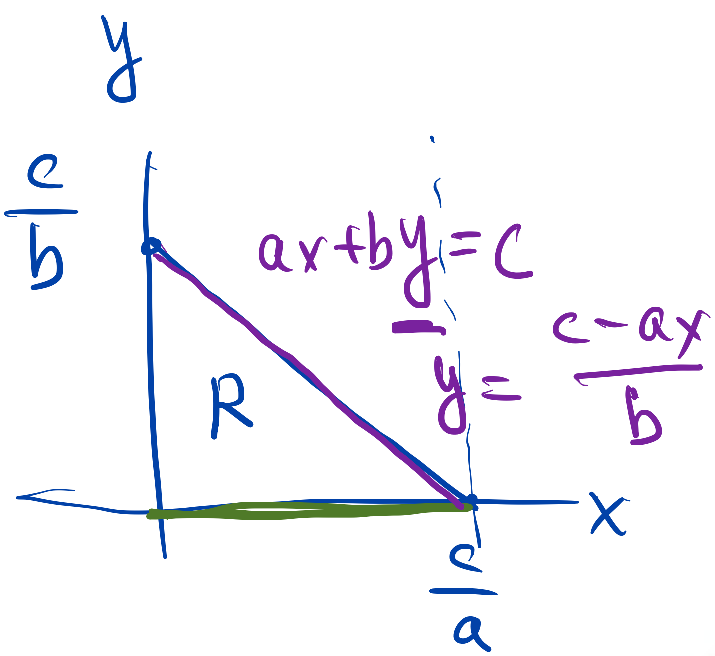
R is bounded above and below, and on the right and left by curves.

Example 3 Find the volume of the tetrahedron in the first octant bounded by the place $z = c - ax - by$ and the coordinates ($x=0$, $y=0$, and $z=0$). Assume $a, b, c > 0$.

Figure 16.19

$$V = \iint_R (c - ax - by)\, dA$$

$$= \int_0^{\frac{c}{a}} \left(\int_0^{\frac{c-ax}{b}} (c - ax - by)\, dy \right) dx$$



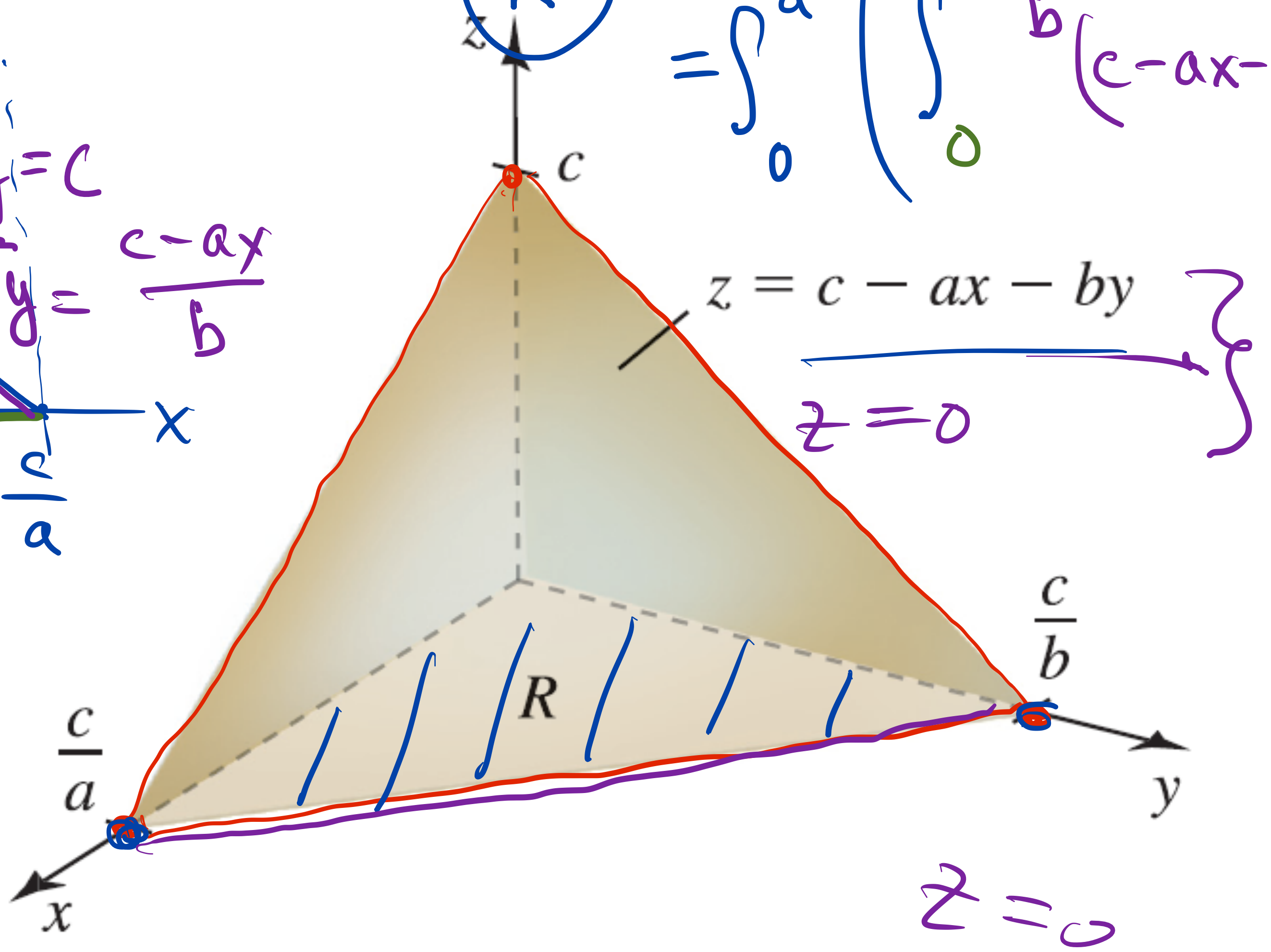
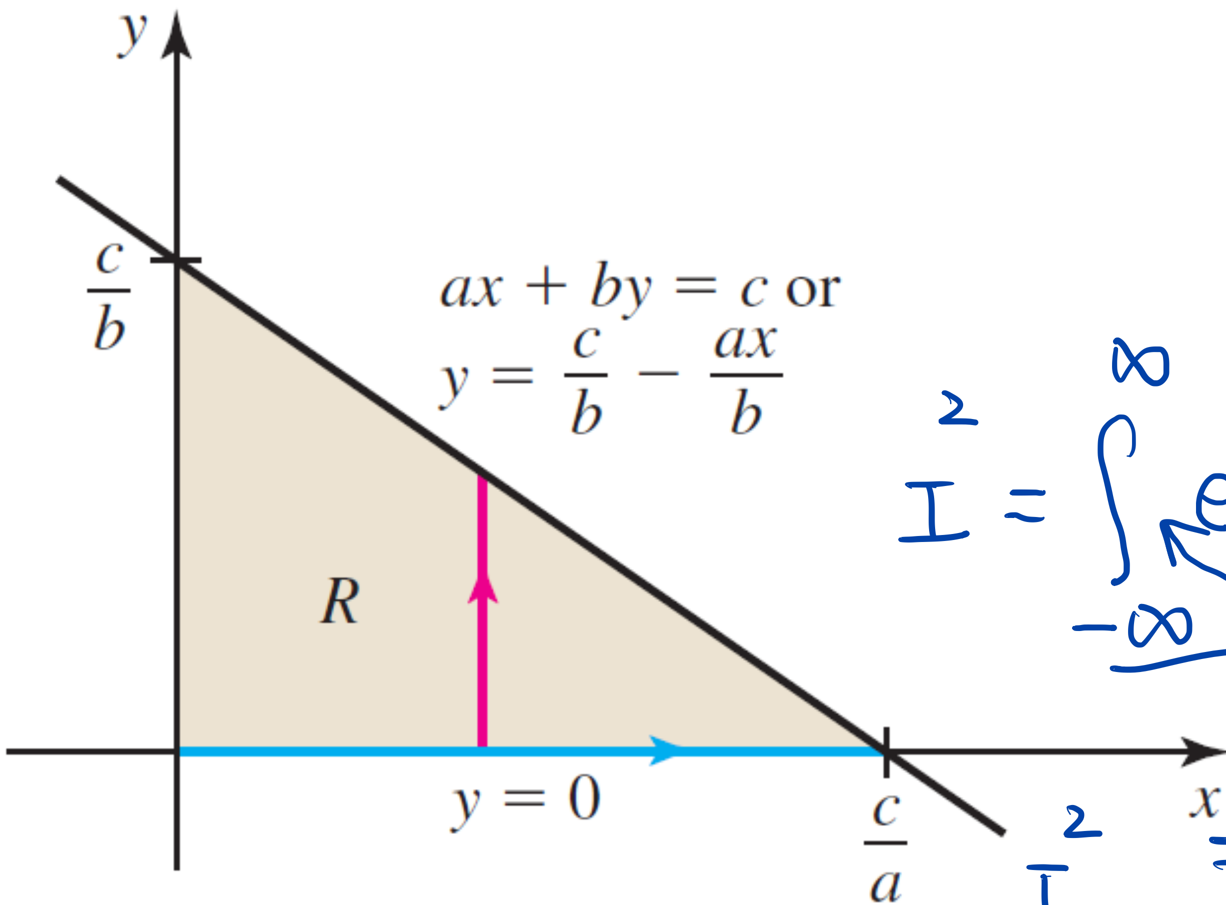


Figure 16.20



$$I^2 = \int_{-\infty}^{\infty} e^{x^2}\,dx \int_{-\infty}^{\infty} e^{y^2}\,dy$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x^2+y^2}\,dA$$

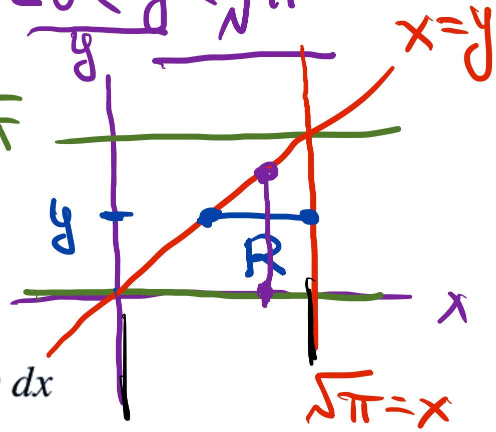
Example 4

$$\int_0^{\sqrt{\pi}}\left(\int_y^{\sqrt{\pi}} \sin x^2\,dx\right)dy = \int_0^{\sqrt{\pi}}\left(\int_0^{x} \sin x^2\,dy\right)dx$$

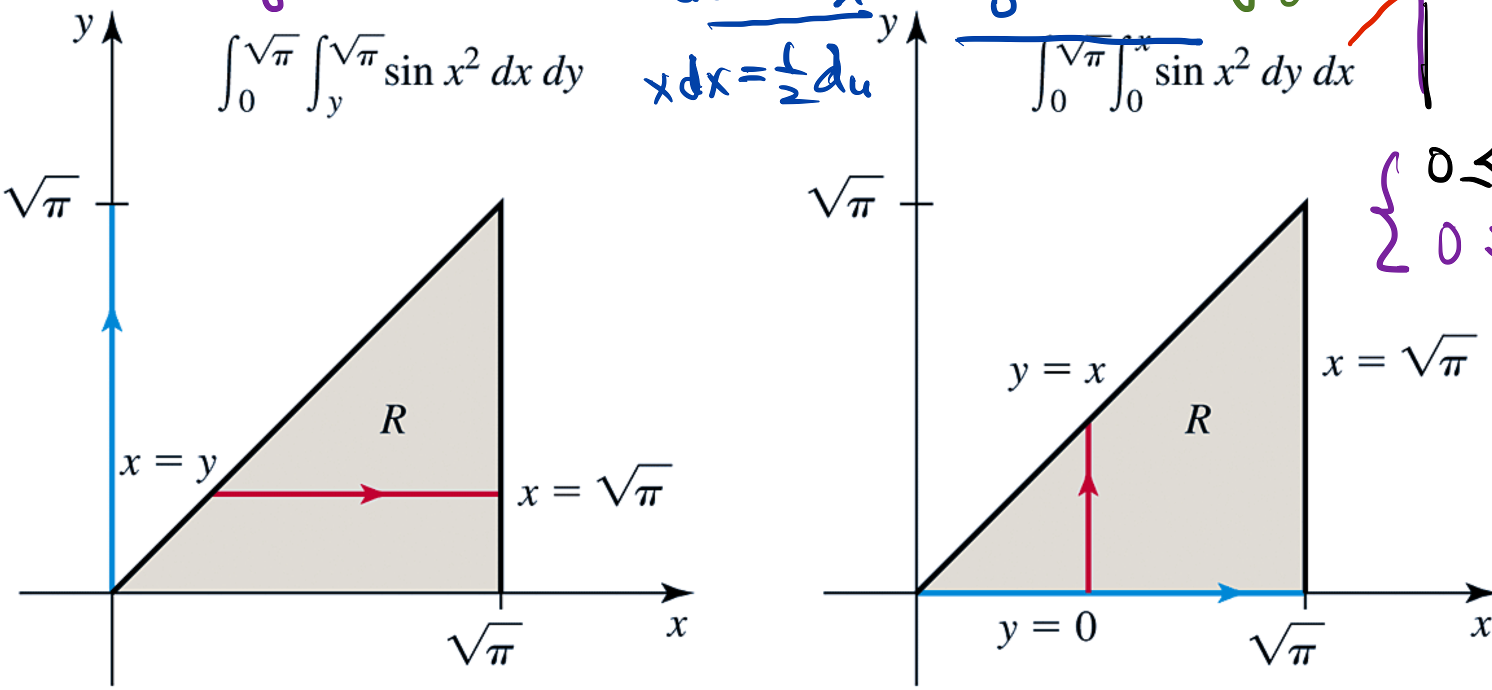
$R: \begin{cases} y \le x \le \sqrt{\pi} \\ 0 \le y \le \sqrt{\pi} \end{cases}$

Figure 16.21 (a & b)

$$= \int_0^{\sqrt{\pi}} x \sin x^2\,dx \qquad \frac{u = x^2 : 0 \to \pi}{du = 2x\,dx} \qquad x\,dx = \tfrac{1}{2}du \qquad \int_0^{\pi} \tfrac{1}{2}\sin u\,du$$



$\begin{cases} 0 \le x \le \sqrt{\pi} \\ 0 \le y \le x \end{cases}$

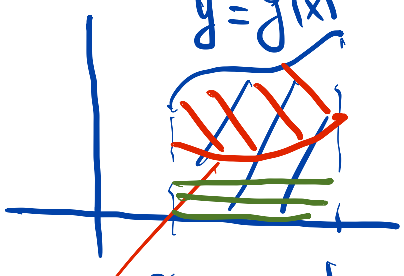


Integrating first with respect to x does not work. Instead...

... we integrate first with respect to y.

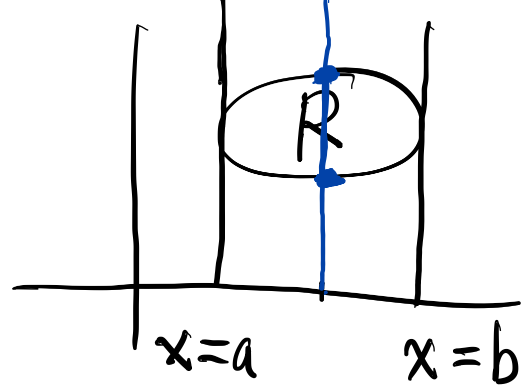
• Regions Between Two Surfaces

Figure 16.22



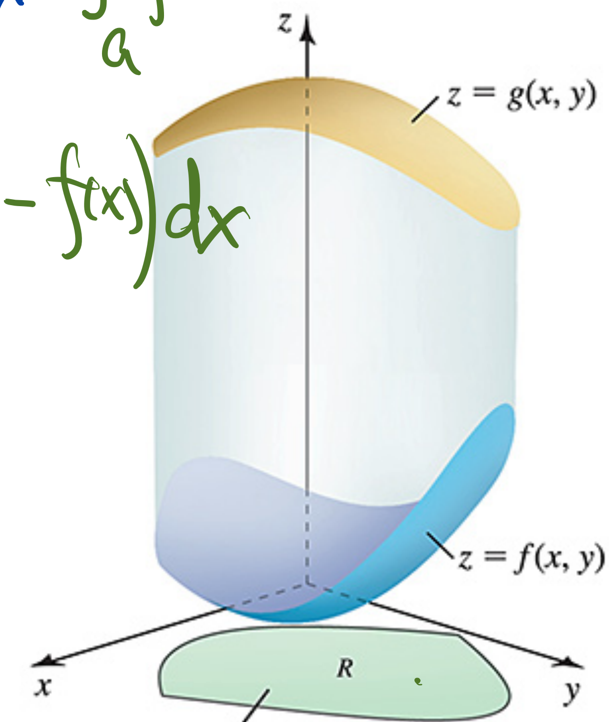
$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$

$$= \int_a^b \big(g(x) - f(x)\big)\,dx$$



$$\begin{cases} a \le x \le b \\ g(x) \le y \le h(x) \end{cases}$$

$$V = \iint_R f(x,y)\,dA$$



$$\text{Volume} = \iint_R (g(x, y) - f(x, y))\,dA$$

Example 5 Find the volume of the solid bounded by the parabolic cylinder $z = 1 + x^2$ and the planes $z = 5 - y$ and $y = 0$.

$$V = \iint_R \left[(5-y) - (1+x^2)\right] dA$$

$$= \int_{-2}^{2} \int_0^{4-x^2} \left[\qquad \right] dA$$

Figure 16.23

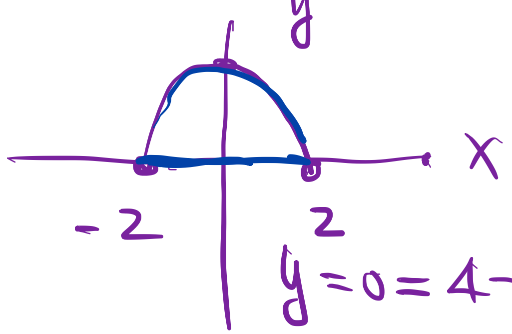
intersection

$$\begin{cases} z = 1 + x^2 \\ z = 5 - y \end{cases}$$

$$1 + x^2 = 5 - y$$

$$y = 4 - x^2$$

$$y = 0 = 4 - x^2 \Rightarrow x = \pm 2$$



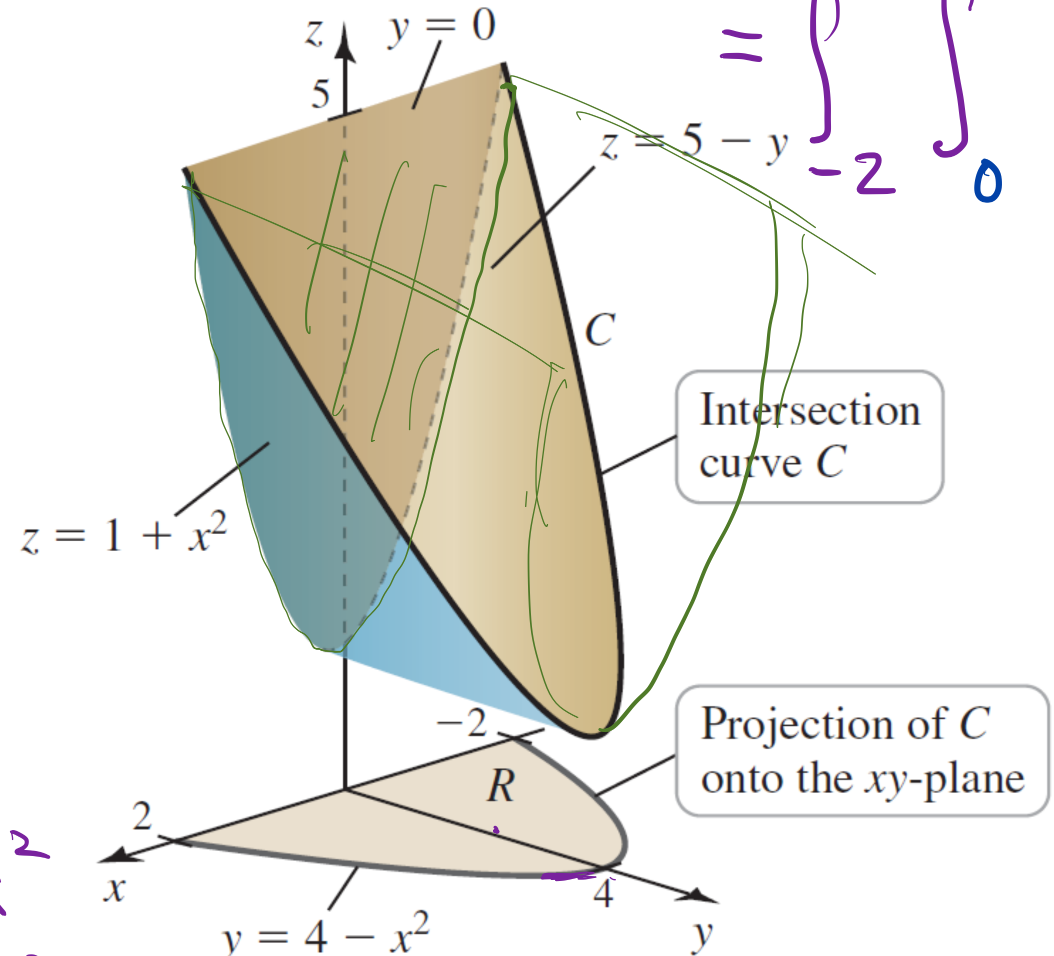
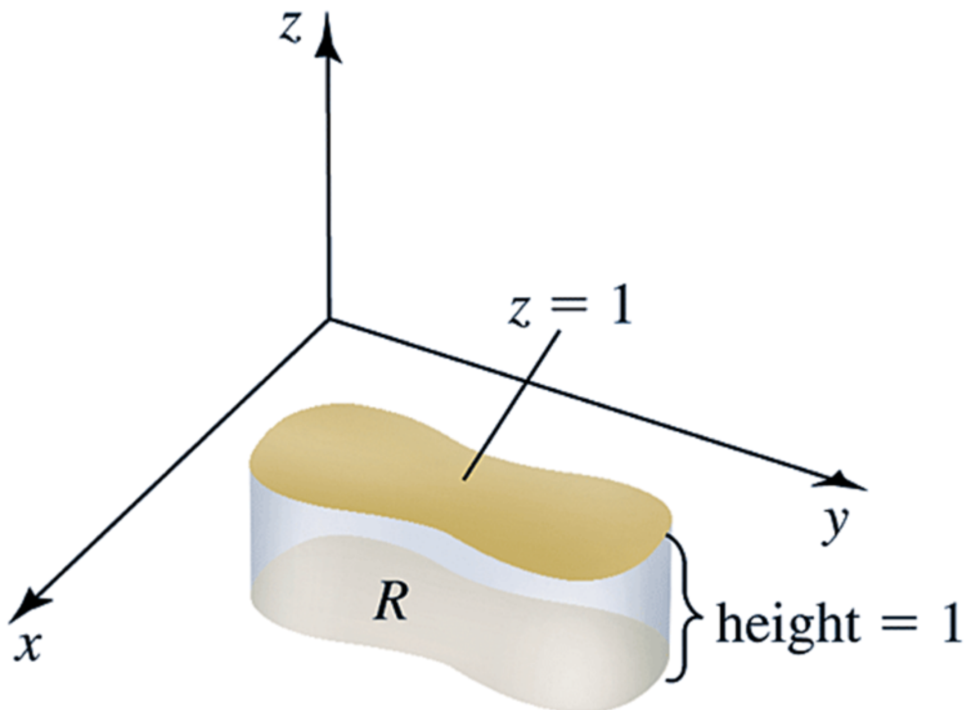


Figure 16.25



Volume of solid = (area of R) × (height)

$$= \text{area of } R = \iint_R 1\, dA$$

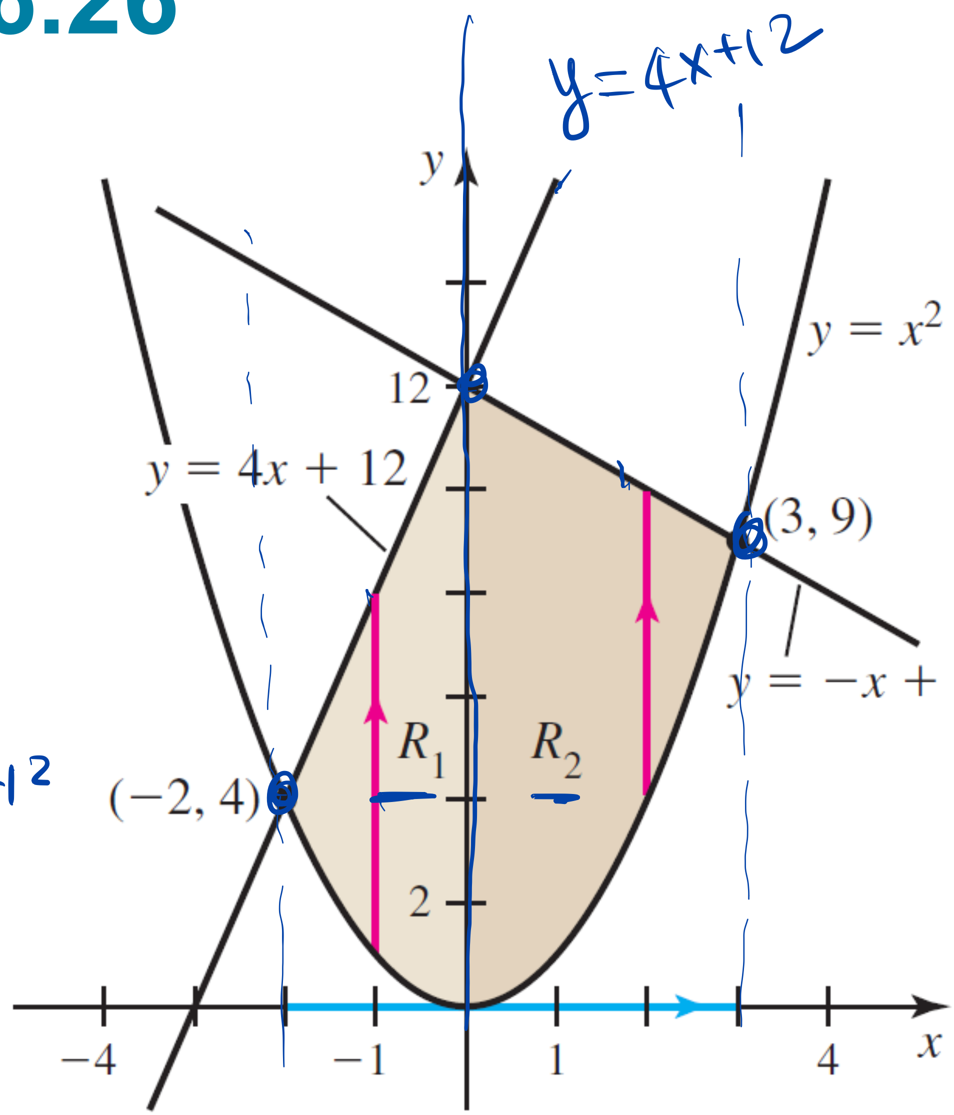
Area of Regions by Double Integrals

Let R be the region in the xy-plane. Then

$$\text{area of } R = \iint\limits_R dA.$$

Example 6 Find the area of the region R bounded by $y = x^2$, $y = -x + 12$, and $y = 4x + 12$.

Figure 16.26



$$\begin{cases} -2 \le x \le 0 \\ x^2 \le y \le 4x+12 \end{cases}$$

$$\begin{cases} 0 \le x \le 3 \\ x^2 \le y \le -x+12 \end{cases}$$

Section 16.3 Double Integrals in Polar Coordinates

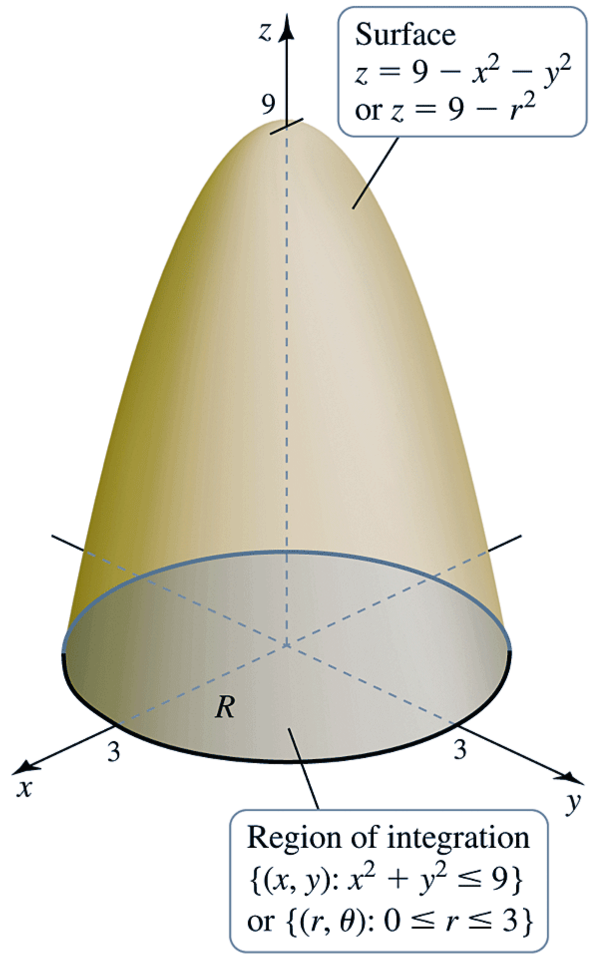
$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, r\,d\theta\,dr = 2\pi\int_0^{\infty} r\,e^{-r^2}\,dr$$

$$0 \le \theta \le 2\pi$$

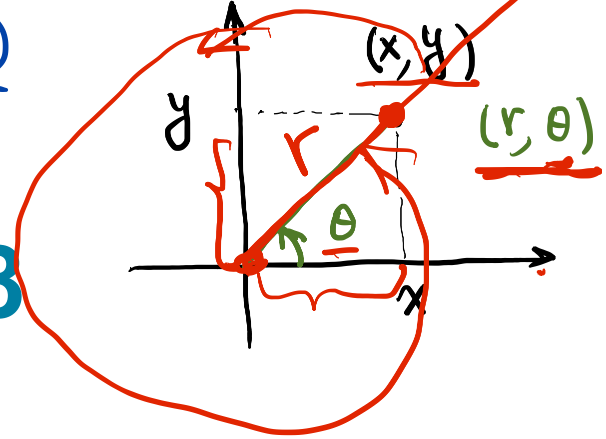
$$V = \iint_R \left(9 - (x^2 + y^2)\right) dA$$

Figure 16.27



• Polar Coordinate (§12.2)

Figure 16.28



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\frac{y}{x}$$

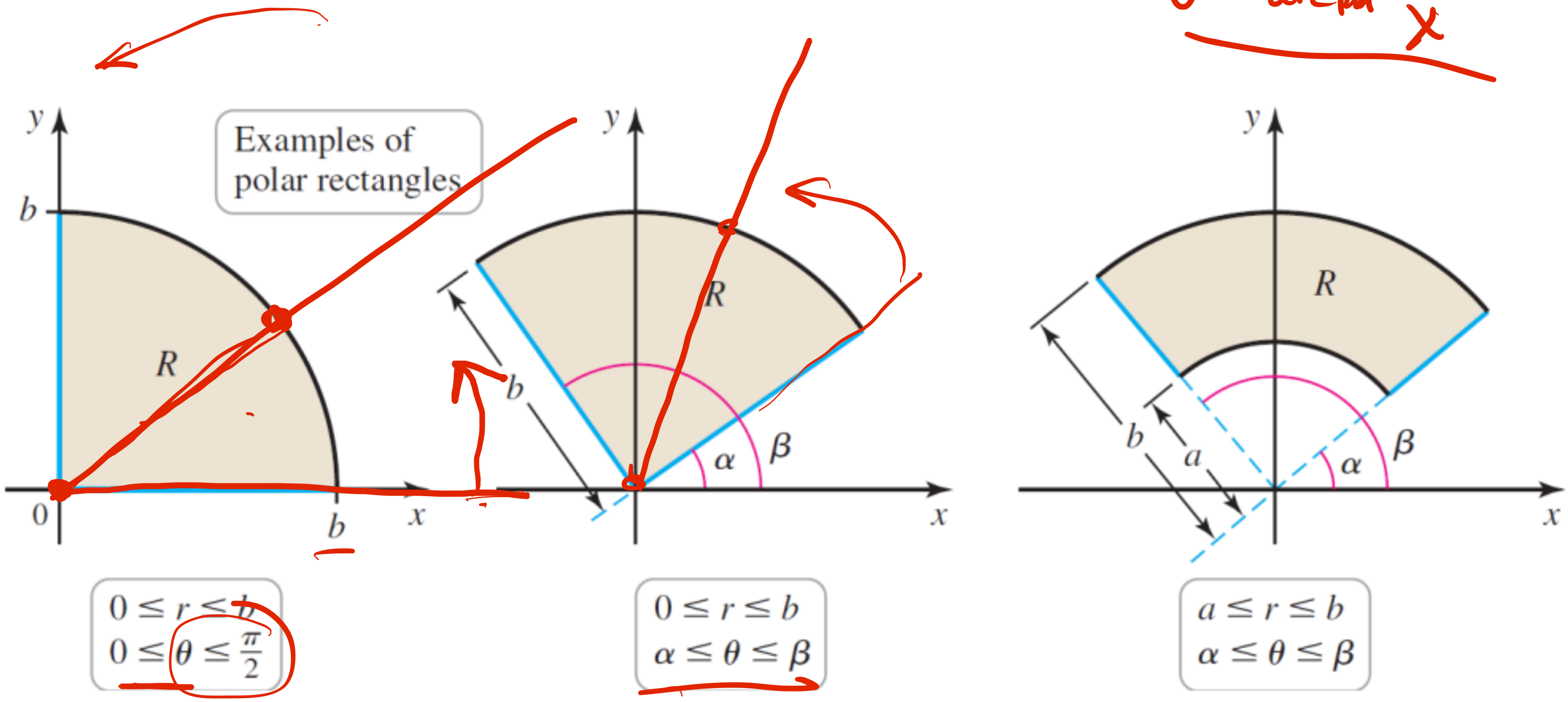
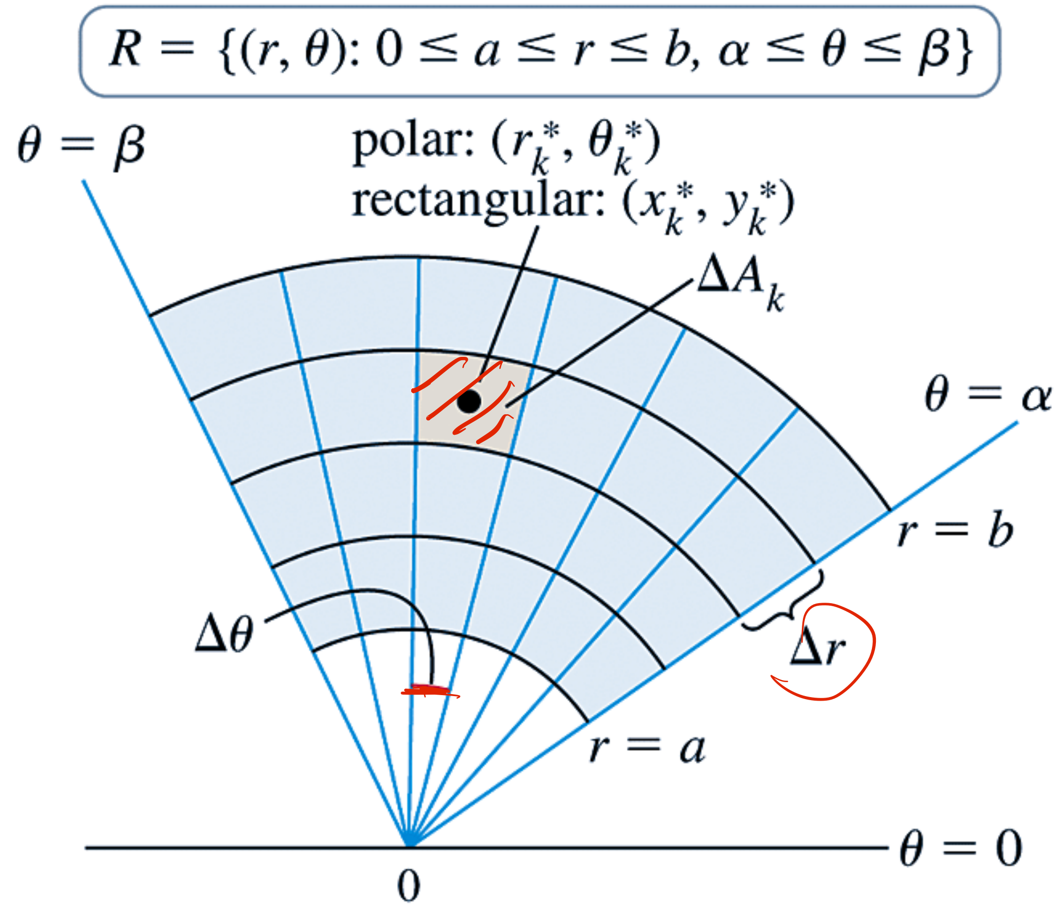


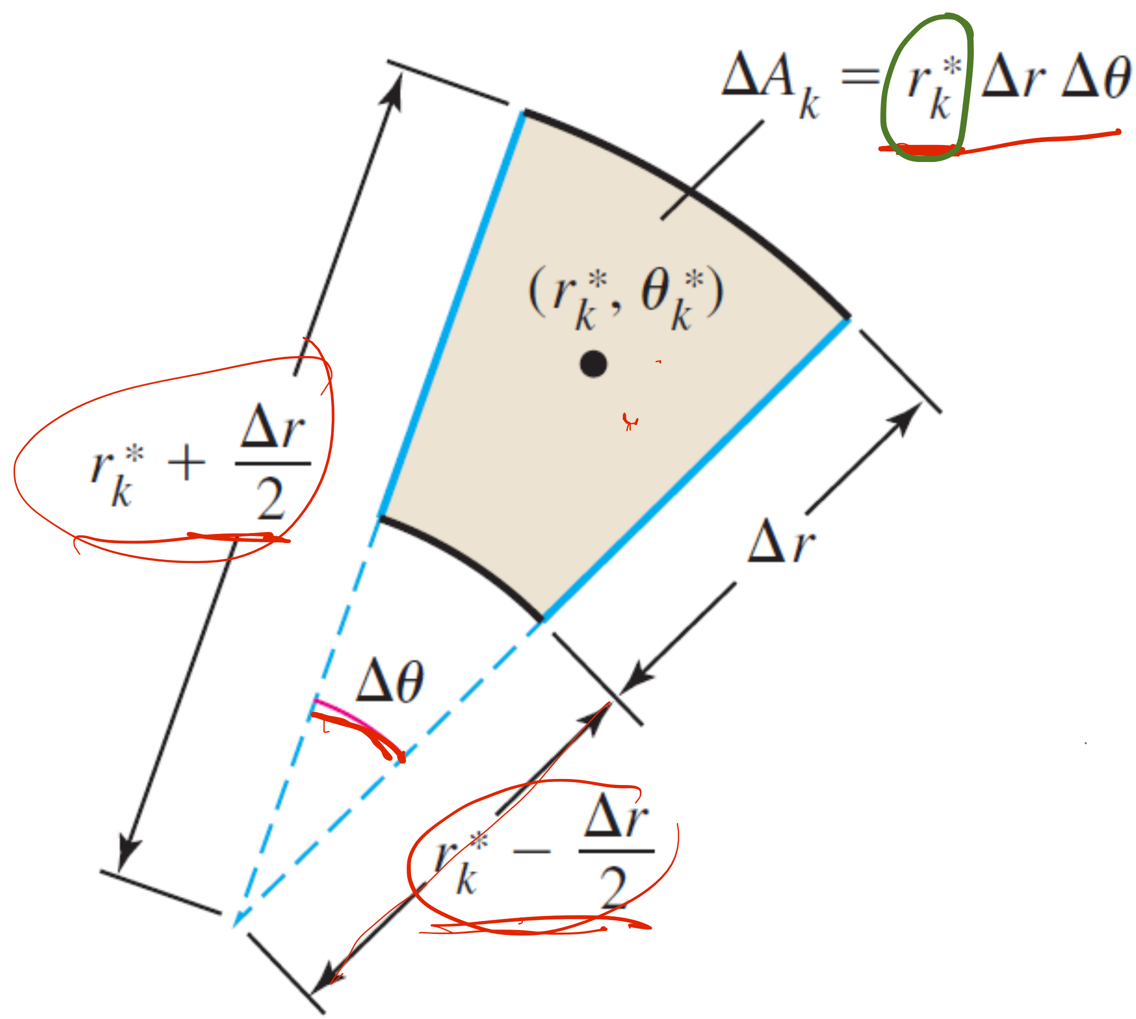
Figure 16.29



$R = \{(r, \theta): 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$
$\theta = \beta$
polar: (r_k^*, θ_k^*)
rectangular: (x_k^*, y_k^*)
ΔA_k
$\theta = \alpha$
$r = b$
$\Delta\theta$
Δr
$r = a$
$\theta = 0$
0

area of sector $= \frac{1}{2} r^2 \theta$

Figure 16.30



Theorem 16.3 Change of Variables for Double Integrals over Polar Rectangular Regions

Let *f* be continuous on the region *R* in the *xy*-plane expressed in a polar coordinates as $R = \{(r,\theta): 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$. Then *f* is integrable over *R*, and the double integral of *f* over *R* is

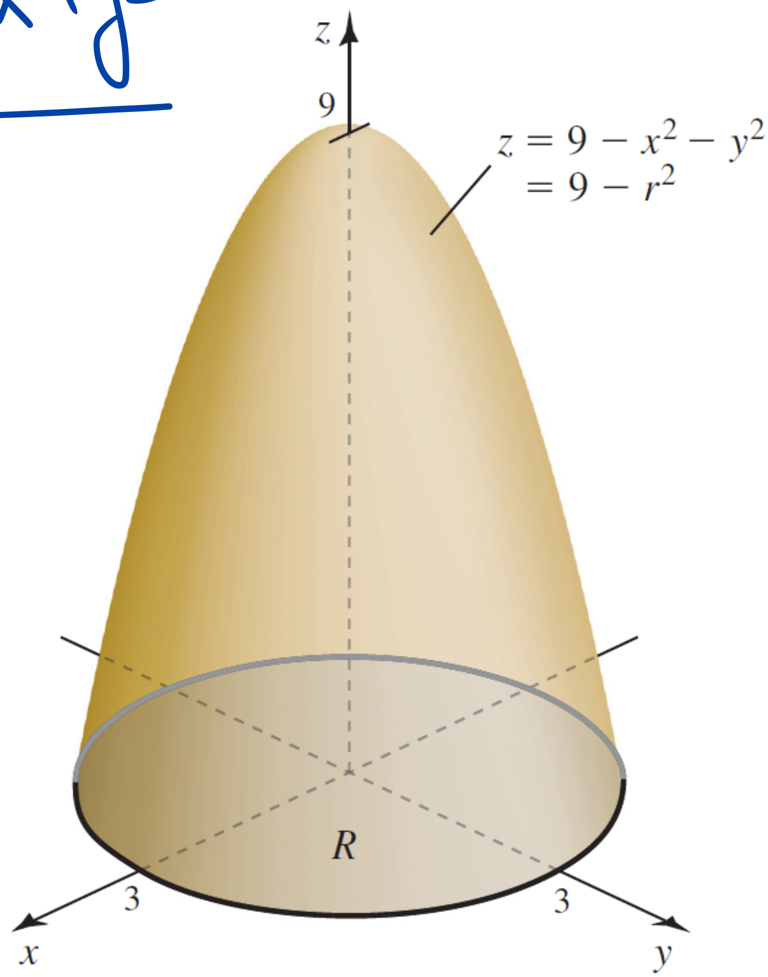
$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

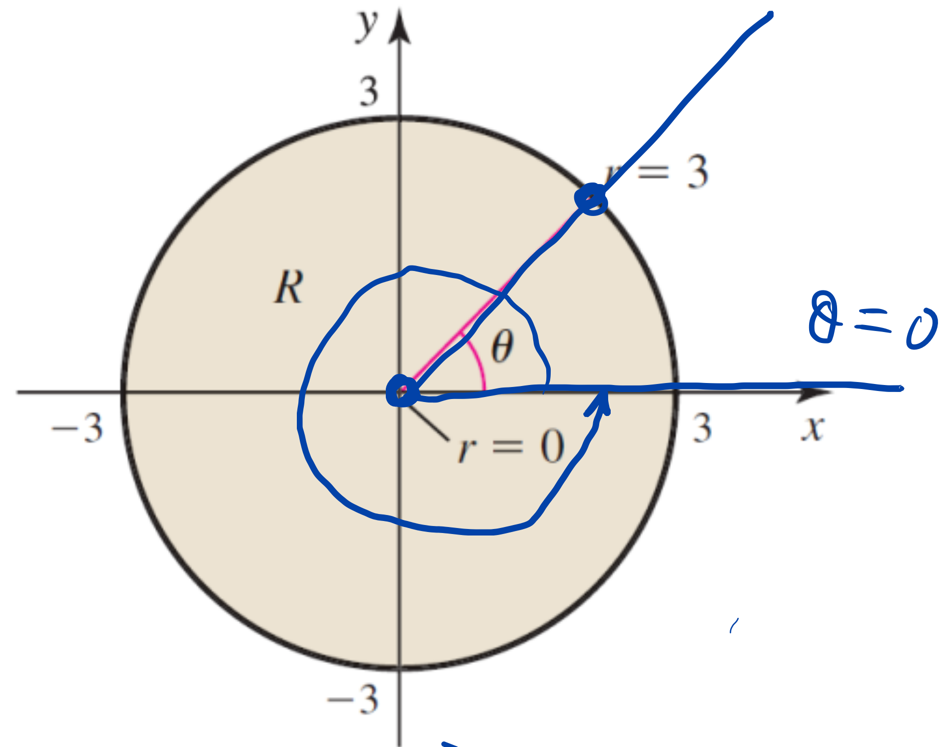
Example 1 $V = \iint_R (9 - x^2 - y^2)\, dA = \int_0^3 \int_0^{2\pi} (9 - r^2)\, r\, d\theta\, dr$

$= 2\pi \int_0^3 (9r - r^3)\, dr$

Figure 16.31

$r = \sqrt{x^2 + y^2}$





$R = \{(r, \theta): 0 \le r \le 3, 0 \le \theta \le 2\pi\}$

Example 2 Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the cone $z = 2 - \sqrt{x^2+y^2}$.

$$V = \iint_R \left[\left(2 - \sqrt{x^2+y^2}\right) - \left(x^2+y^2\right) \right] dA$$

$$= \int_0^1 \int_0^{2\pi} \left[2 - r - r^2\right] r \, d\theta \, dr$$

Figure 16.32

$$x^2 + y^2 = r^2$$

intersection

$$\begin{cases} z = 2 - \sqrt{x^2+y^2} \\ z = x^2 + y^2 \end{cases}$$

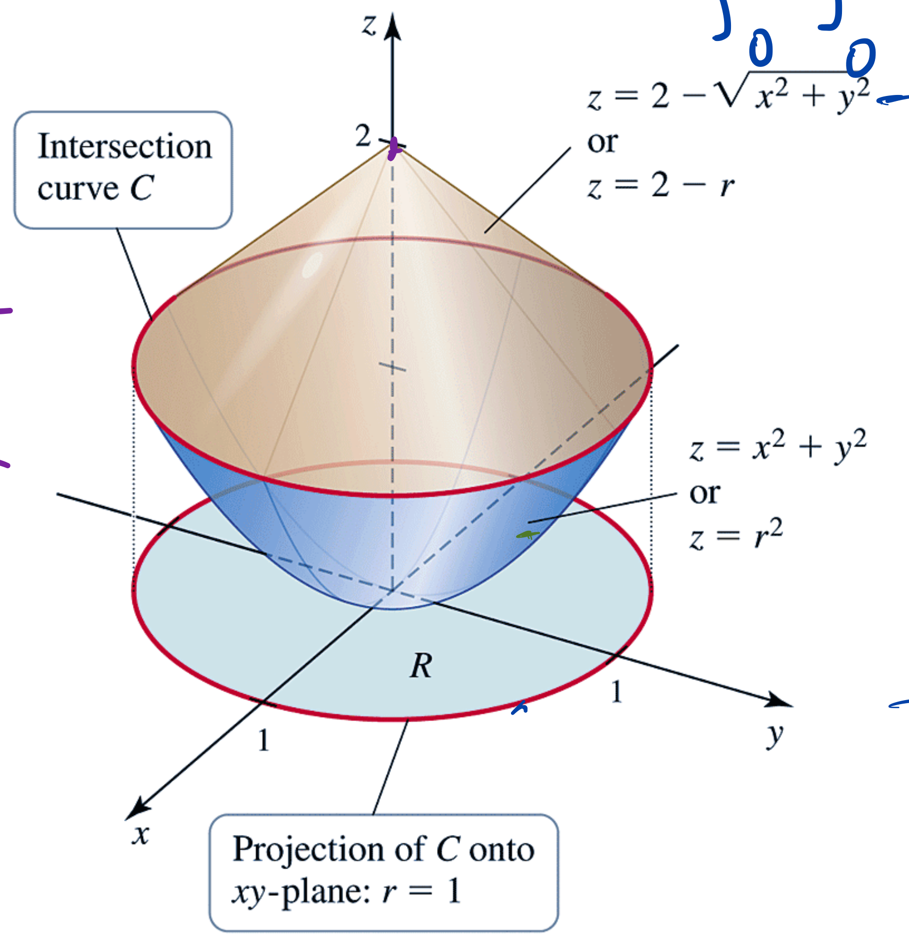
$$2 - \sqrt{x^2+y^2} = x^2 + y^2$$

$$2 - r = r^2$$

$$0 = r^2 + r - 2$$

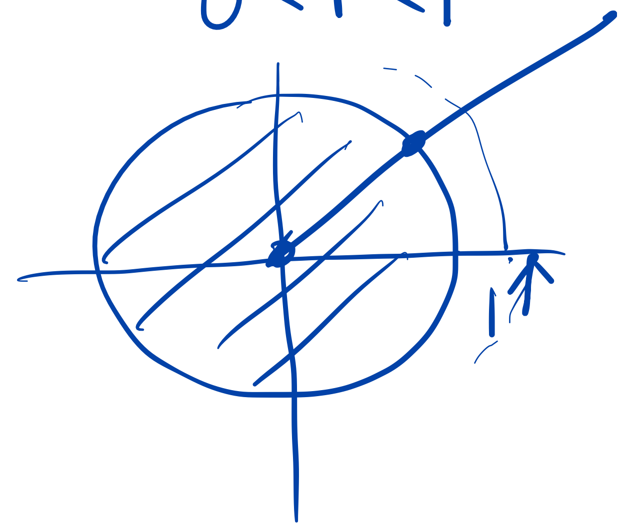
$$= (r+2)(r-1)$$

$\Rightarrow$ ~~$r = -2$~~ or $r = 1$



$0 \le \theta \le 2\pi$

$0 \le r \le 1$



Example 3 Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \{(r,\theta): 2 \le r \le 4,\ 0 \le \theta \le 2\pi\}$.

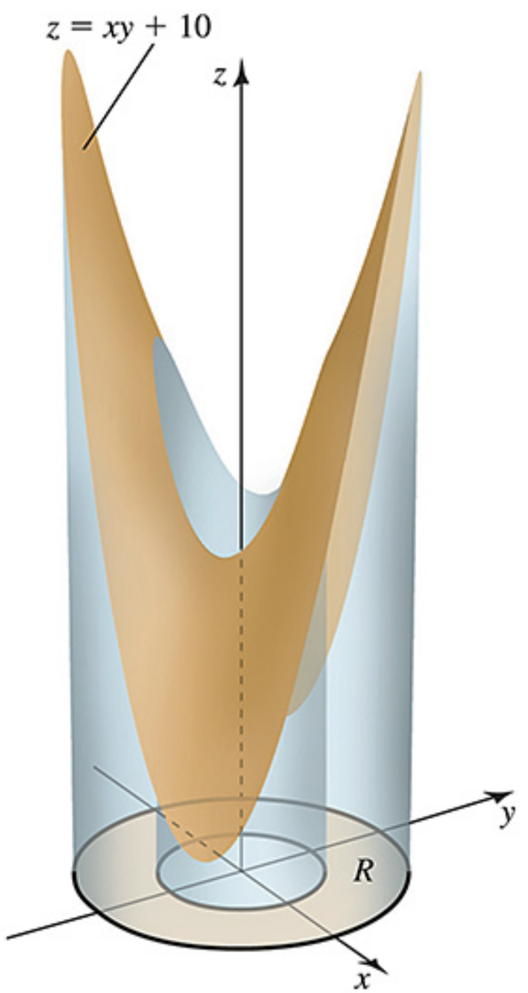
Figure 16.33

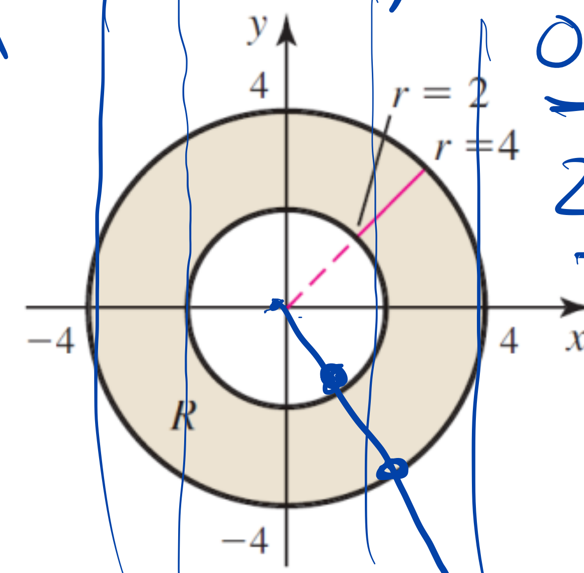
$$V = \iint_R (xy+10)\,dA$$

$$= \int_2^4 \int_0^{2\pi} \left(r^2 \cos\theta \sin\theta + 10 \right) r \, d\theta \, dr$$

$0 \le \theta \le 2\pi$

$2 \le r \le 4$

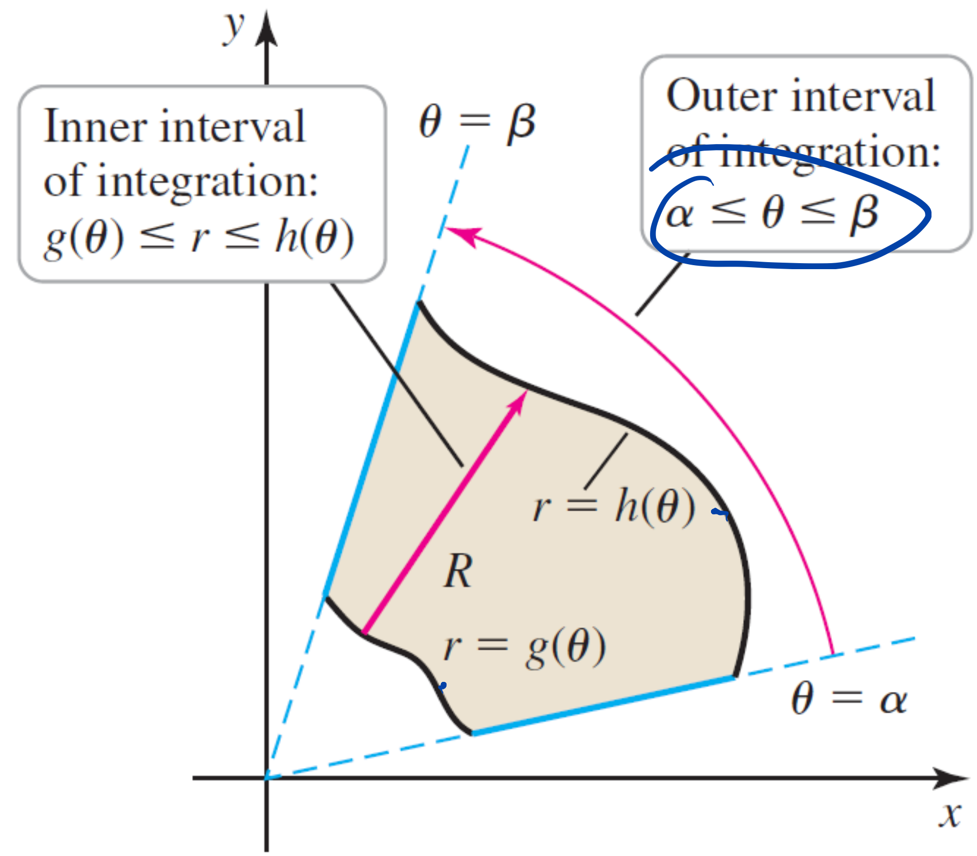




$R = \{(r, \theta): 2 \le r \le 4, 0 \le \theta \le 2\pi\}$

• More General Polar Regions

Figure 16.34



$R = \{(r, \theta)\colon 0 \le g(\theta) \le r \le h(\theta),\ \alpha \le \theta \le \beta\}$

Theorem 16.4 Change of Variables for Double Integrals over More General Polar Regions

Let *f* be continuous on the region *R* in the *xy*-plane expressed in polar coordinates as

$$R = \{(r,\theta) : 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\},$$

where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Example 4 Write an iterated integral in polar coordinates for $\iint_R g(r,\theta)\,dA$ for the following regions R in xy-plane.

Figure 16.35

(a) The region outside the circle $r=2$ (with radius 2 centered at (0,0)) and inside the circle $r=4\cos\theta$ (with radius 2 centered at (2,0))

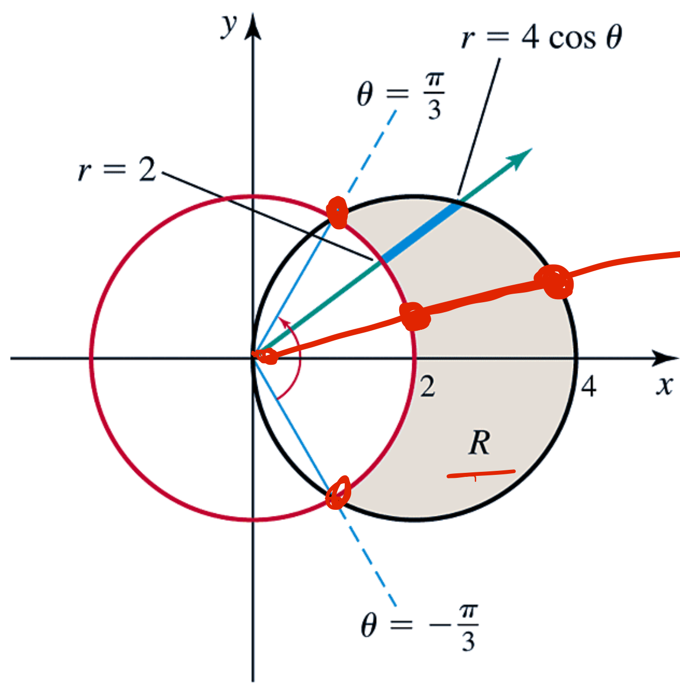
intersection

$$\begin{cases} r=2 \\ r=4\cos\theta \end{cases}$$

$2=4\cos\theta$

$\cos\theta=\frac{1}{2}$, $\theta=\pm\frac{\pi}{3}$

Radial lines enter the region R at $r = 2$ and exit the region at $r = 4\cos\theta$.



The inner and outer boundaries of R are traversed as θ varies from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$.

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{2}^{4\cos\theta} g(r,\theta)\, r\, dr\, d\theta$$

$$\begin{cases} -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \\ 2 \le r \le 4\cos\theta \end{cases}$$

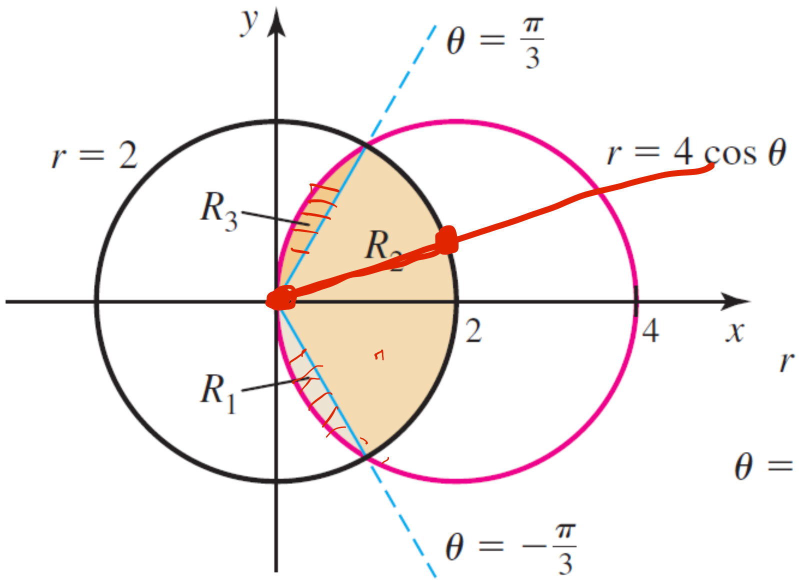
(b) The region inside both circles of Part (a).

Figure 16.36 (a & b)

$R_2 \begin{cases} -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \\ 0 \le r \le 2 \end{cases}$

$R_3 \begin{cases} \frac{\pi}{3} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 4\cos\theta \end{cases}$

In R_1, radial lines begin at the origin and exit at $r = 4\cos\theta$.



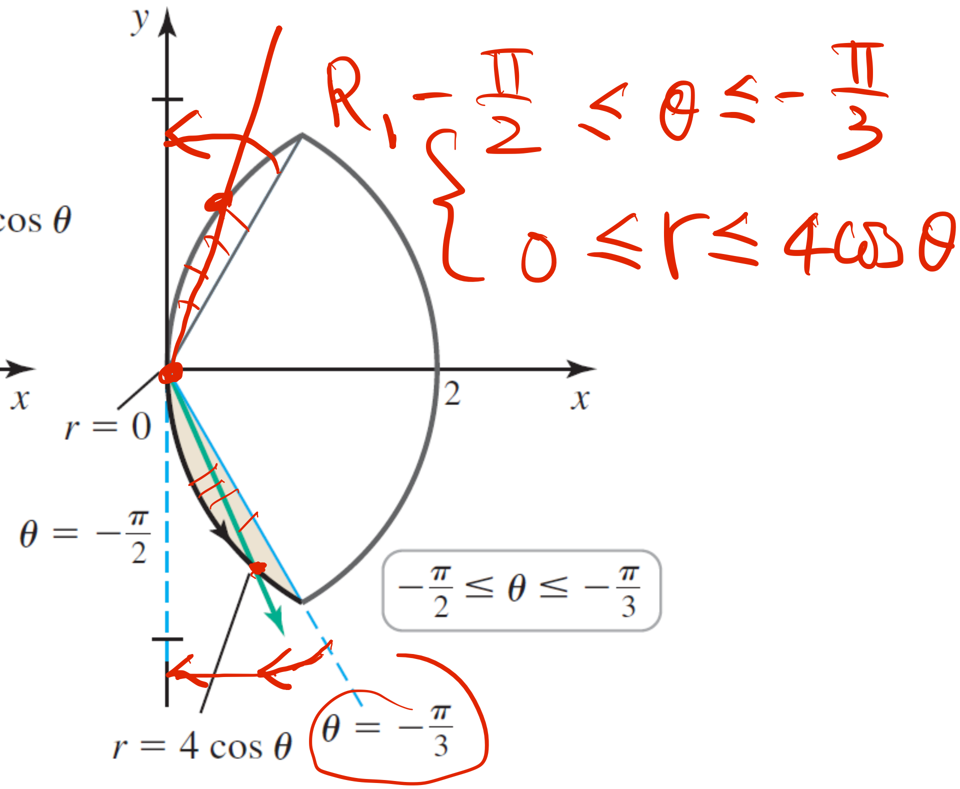
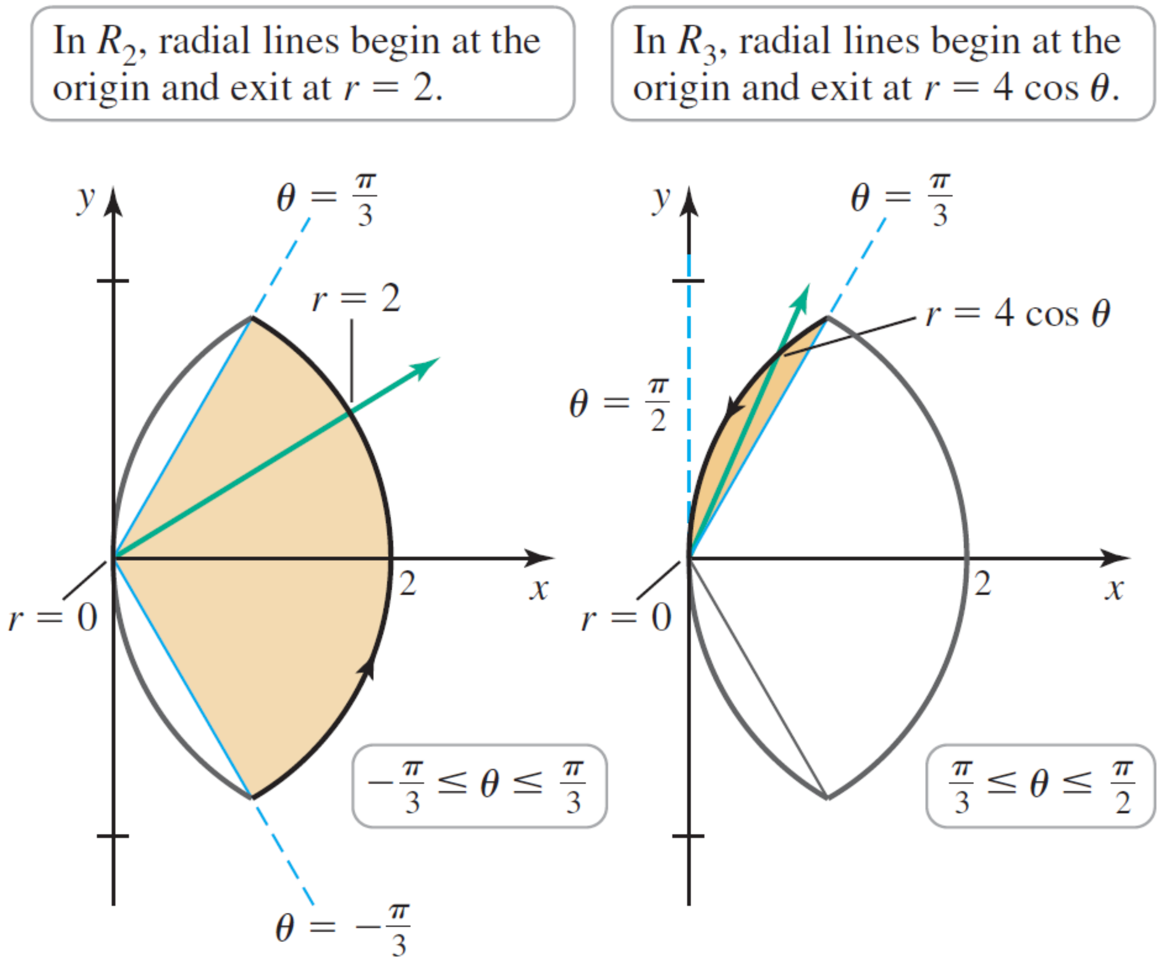


Figure 16.36 (c & d)



Area of Polar Regions

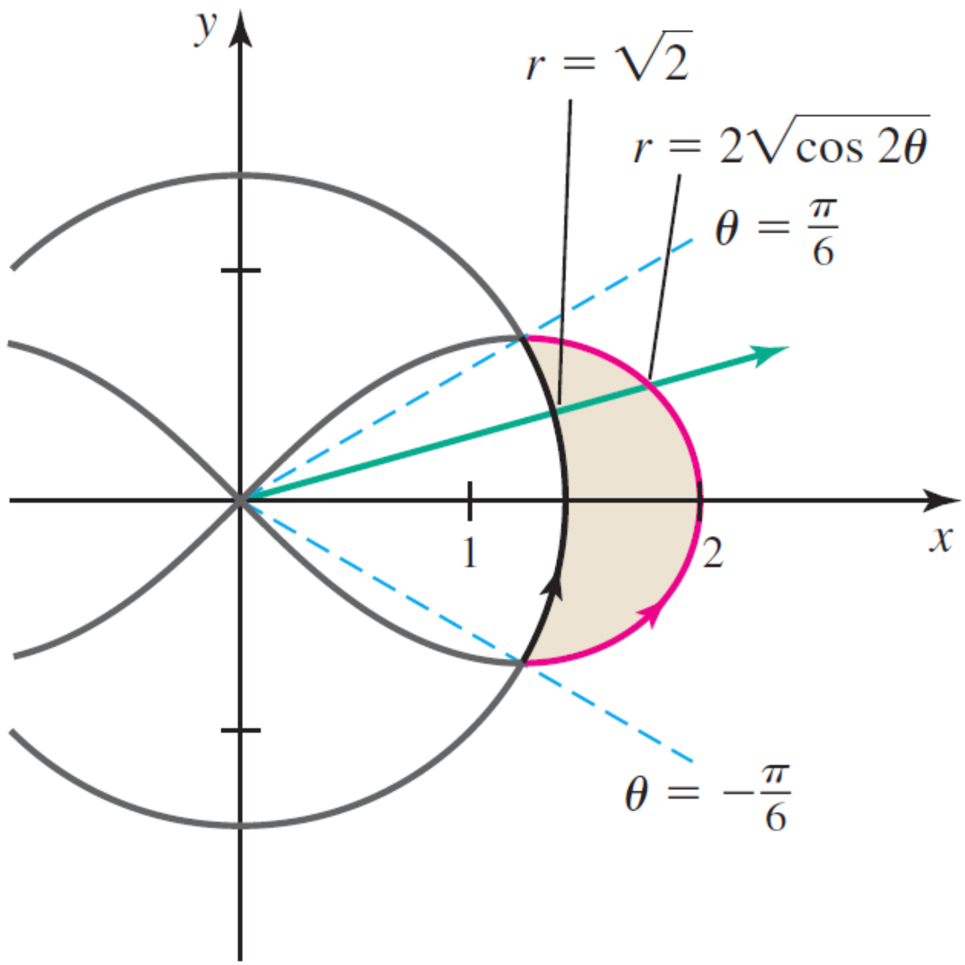
The area of the region $R = \{(r, \theta) : 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\}$,

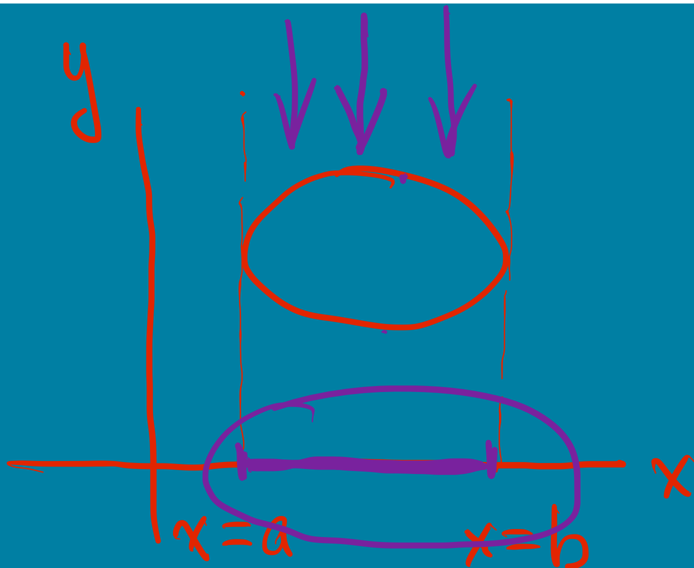
where $0 < \beta - \alpha \le 2\pi$, is

$$A = \iint_R dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} r \, dr \, d\theta.$$

Example 5 Compute the area of the region in the first and fourth quadrants outside the circle $r=\sqrt{2}$ and inside the lemniscate $r^2=4\cos 2\theta$.

Figure 16.37





$$\iint_R f(x,y)\,dA = \int_a^b dx \int_{g(x)}^{h(x)} f\,dy$$

Section 16.4 Triple Integrals

$$\iiint_D f(x,y,z)\,dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f\,dz \right) dA$$

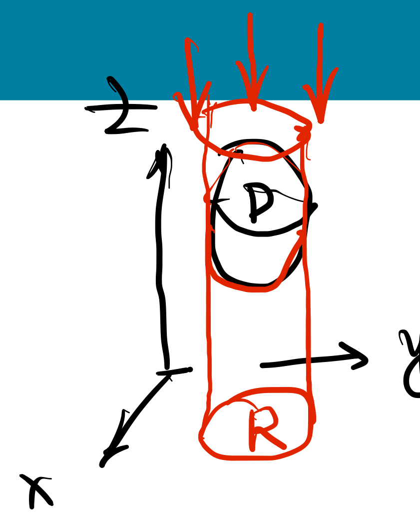


Figure 16.38

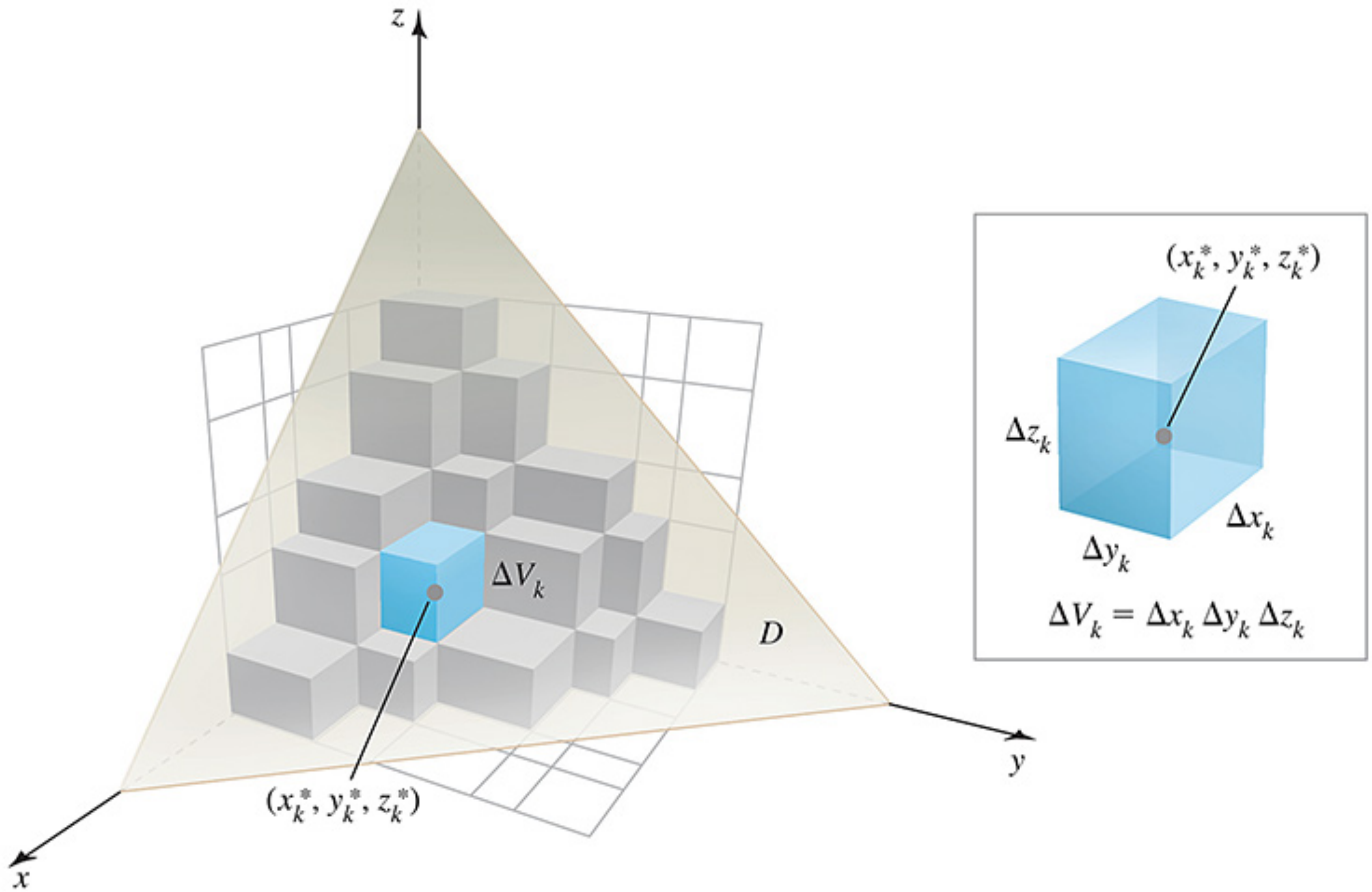
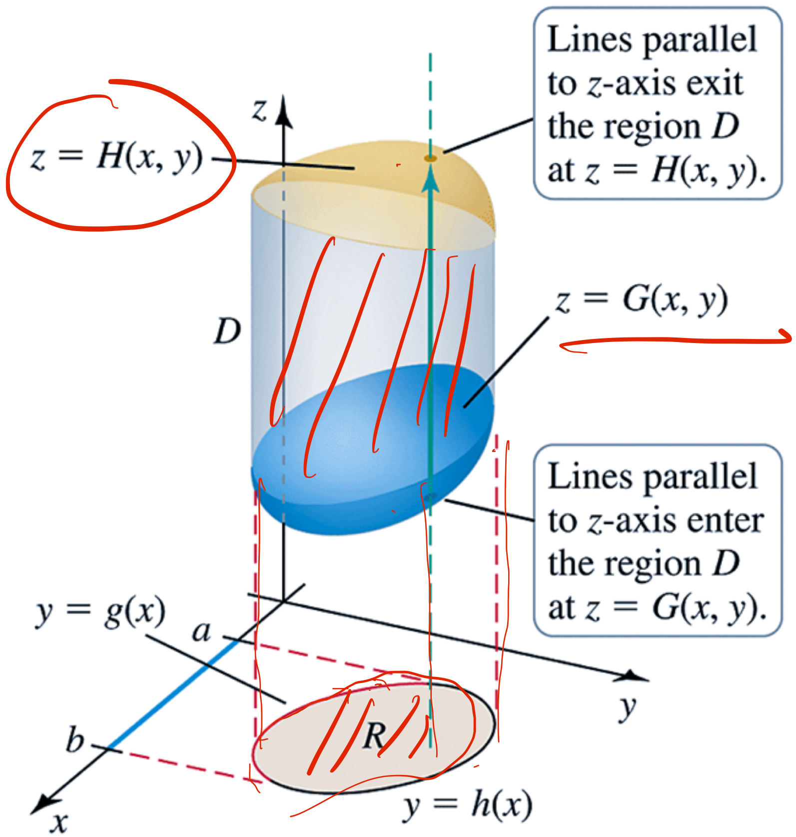

z
ΔV_k
D
(x_k^*, y_k^*, z_k^*)
x
y
(x_k^*, y_k^*, z_k^*)
Δz_k
Δx_k
Δy_k
$\Delta V_k = \Delta x_k \, \Delta y_k \, \Delta z_k$

Figure 16.39



$$\iiint_D f(x, y, z)\, dV = \iint_R \left(\int_{G(x, y)}^{H(x, y)} f(x, y, z)\, dz \right) dA$$

Figure 16.40

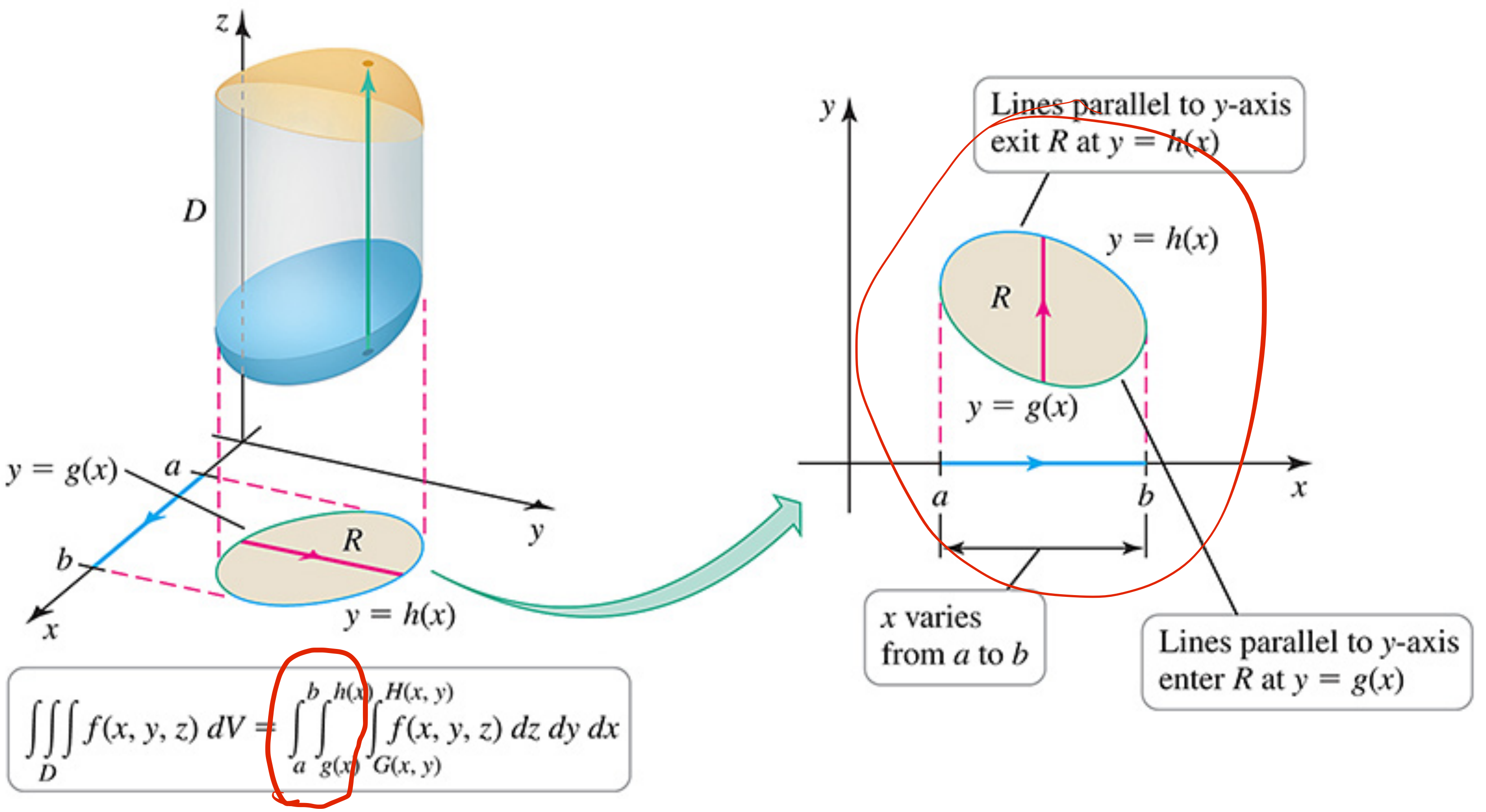

z
D
y = g(x)
a
b
x
R
y
y = h(x)
$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x, y)}^{H(x, y)} f(x, y, z)\, dz\, dy\, dx$$
Lines parallel to y-axis exit R at y = h(x)
y
y = h(x)
R
y = g(x)
a
b
x
x varies from a to b
Lines parallel to y-axis enter R at y = g(x)

Table 16.2

Integral	Variable	Interval
Inner	z	$G(x, y) \le z \le H(x, y)$
Middle	y	$g(x) \le y \le h(x)$
Outer	x	$a \le x \le b$

Theorem 16.5 Triple Integrals

Let *f* be continuous over the region

$$D = \{(x, y, z) : a \le x \le b, g(x) \le y \le h(x), G(x, y) \le z \le H(x, y)\},$$

where *g*, *h*, *G*, and *H* are continuous functions. Then *f* is integrable over *D* and the triple integral is evaluated as the iterated integral

$$\iiint\limits_D f(x, y, z)\,dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\,dz\,dy\,dx.$$

Example 1 Find the mass of the box. The density is $f(x,y,z) = 2 - z$.

Figure 16.41

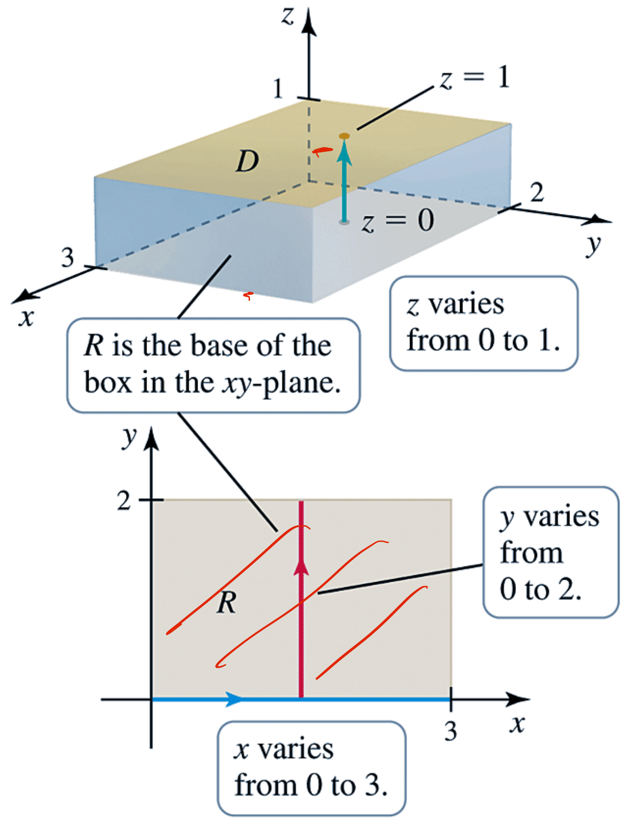
$$\text{mass} = \iiint_D \text{density}\, dV = \int_0^3 \int_0^2 \int_0^1 (2 - z)\, dz\, dy\, dx$$

$$= \int_0^3 dx \int_0^2 dy \int_0^1 dz\, (2 - z)$$

$0 \le z \le 1$

$R: \begin{cases} 0 \le x \le 3 \\ 0 \le y \le 2 \end{cases}$

2



$$M = \int_0^3 \int_0^2 \int_0^1 (2 - z)\, dz\, dy\, dx$$

Table 16.3

Integral	Variable	Interval
Inner	z	$0 \le z \le 1$
Middle	y	$0 \le y \le 2$
Outer	x	$0 \le x \le 3$

Example 2 Find the volume of the prism D in the first octant bounded by the planes $y = 4 - 2x$ and $z = 6$.

$z = 0$, xy-plane
$y = 0$, xz-plane
$x = 0$, yz-plane

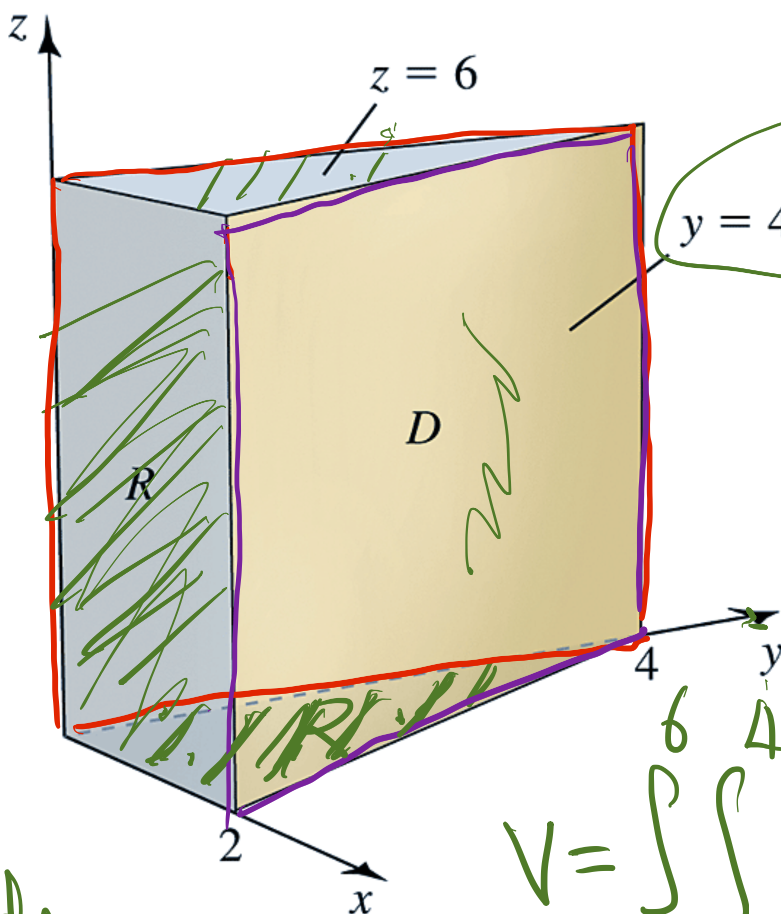
Figure 16.42

$$V = \iiint_D 1 \, dV$$

$$= \iint_R \left(\int_0^6 dz \right) dA$$

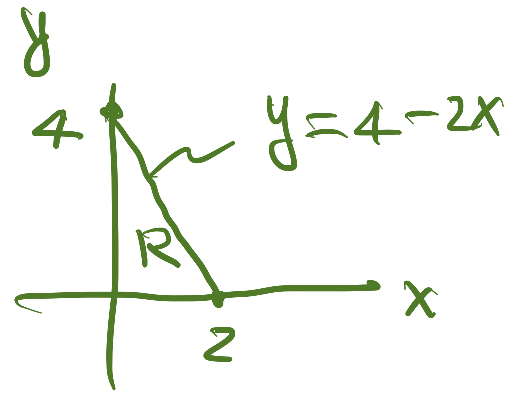
$$= 6 \iint_R dA$$

$$= 6 \cdot \frac{1}{2} \cdot 2 \cdot 4$$



$0 \le z \le 6$

R:

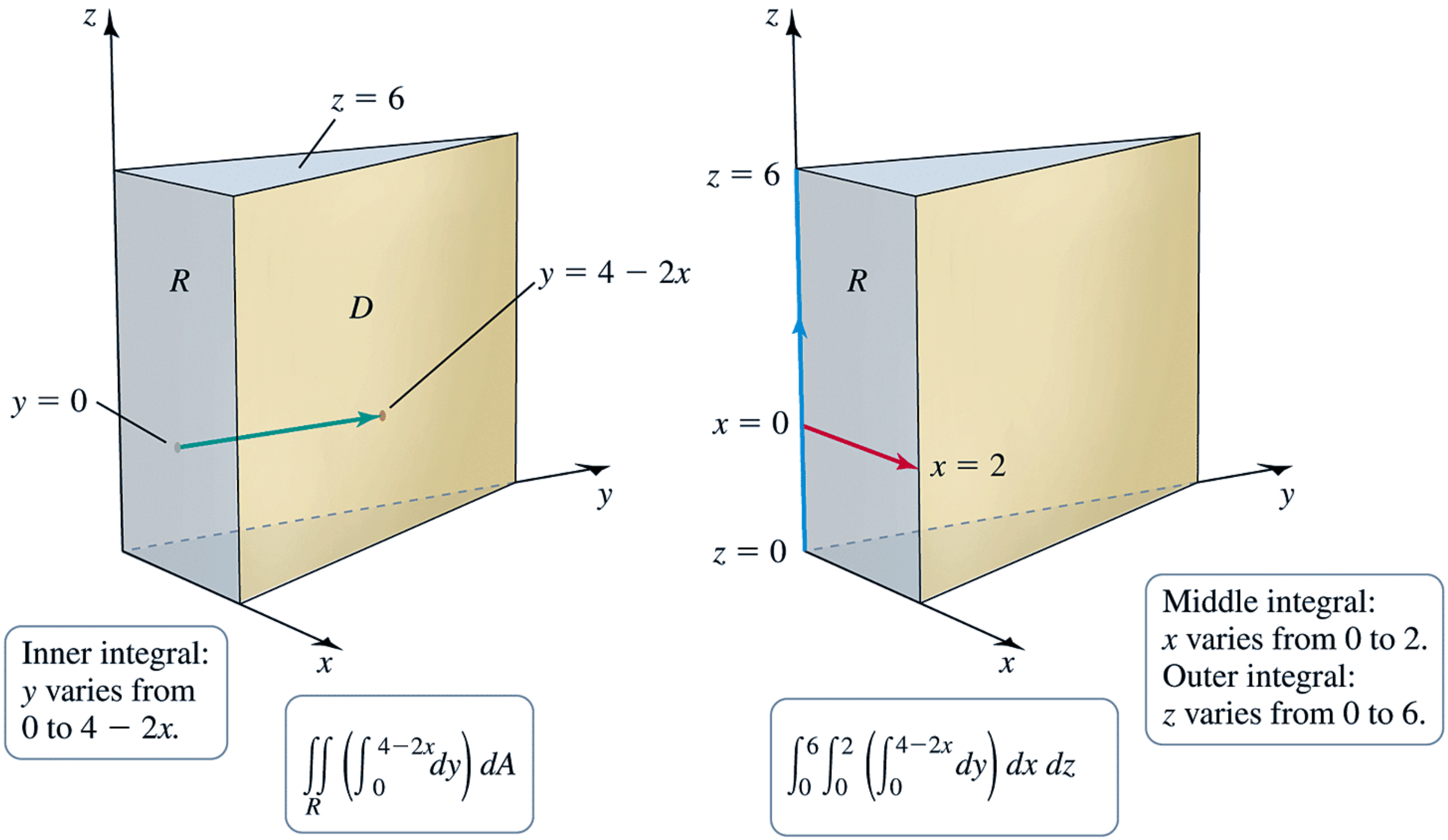


R: $\begin{cases} 0 \le x \le 2 \\ 0 \le z \le 6 \end{cases}$

$$V = \iint_R \left(\int_0^{4-2x} dy \right) dA$$

$$V = \int_0^6 \int_0^4 \int_0^{2 - \frac{1}{2}y} dx$$

Figure 16.43 (a & b)



Example 3 Find the volume of the solid D bounded by the paraboloids $y = x^2 + 3z^2 + 1$ and $y = 5 - 3x^2 - z^2$.

Figure 16.44 (a & b)

$$V = \iiint_D 1\, dV = \iint_R \left(\int_{x^2+3z^2+1}^{5-3x^2-z^2} 1\, dy \right) dA$$

intersection

$$\begin{cases} y = x^2 + 3z^2 + 1 \\ y = 5 - 3x^2 - z^2 \end{cases}$$

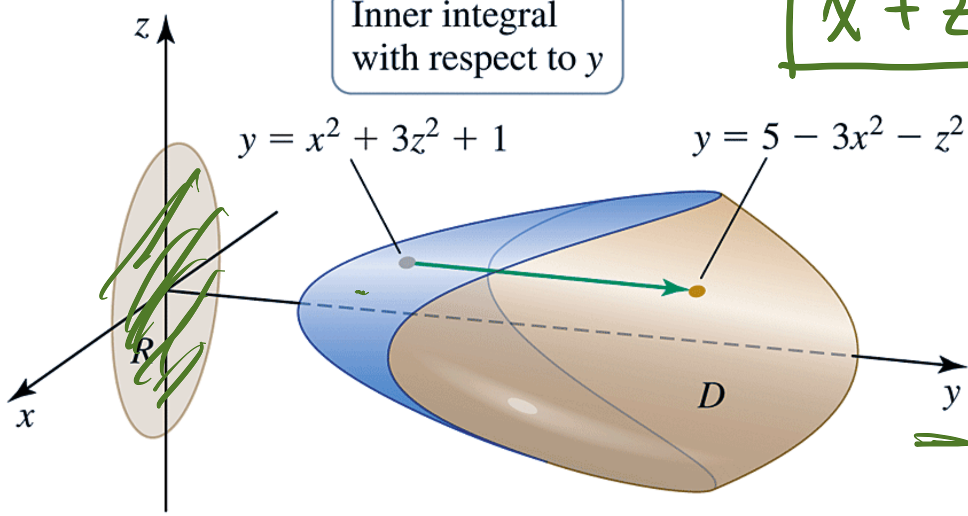
$4x^2 + 4z^2 = 4$

$x^2 + z^2 = 1$

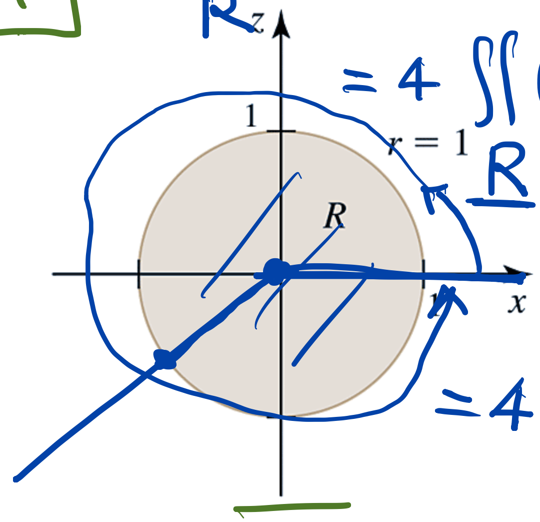
$$= \iint_R \left(4 - 4x^2 - 4z^2\right) dA$$

$$= 4 \iint_R (1 - x^2 - z^2)\, dA$$

$$= 4 \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$



$$\text{Volume} = \iint_R \left(\int_{x^2+3z^2+1}^{5-3x^2-z^2} dy \right) dA$$



$$R = \{(r, \theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$$

$$\begin{cases} x = r\cos\theta \\ z = r\sin\theta \end{cases}$$

Example 4 $I=\int_0^{\pi^{1/4}}\int_0^z\int_y^z 12y^2z^3\sin x^4\,dx\,dy\,dz$.

(a) Sketch the region of D.

(b) Evaluate the integral by changing the order of integration.

$x^3\sin x^4$ $\quad u=x^4$

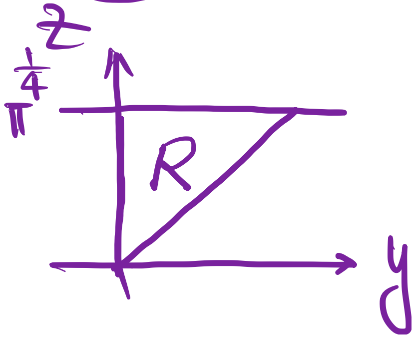
Figure 16.45

$$I=\iint_R\left(\int_0^x 12y^2z^3\sin x^4\,dy\right)dA=\iint_R 4z^3\sin x^4\cdot x^3\,dA$$

$y\le x\le z$

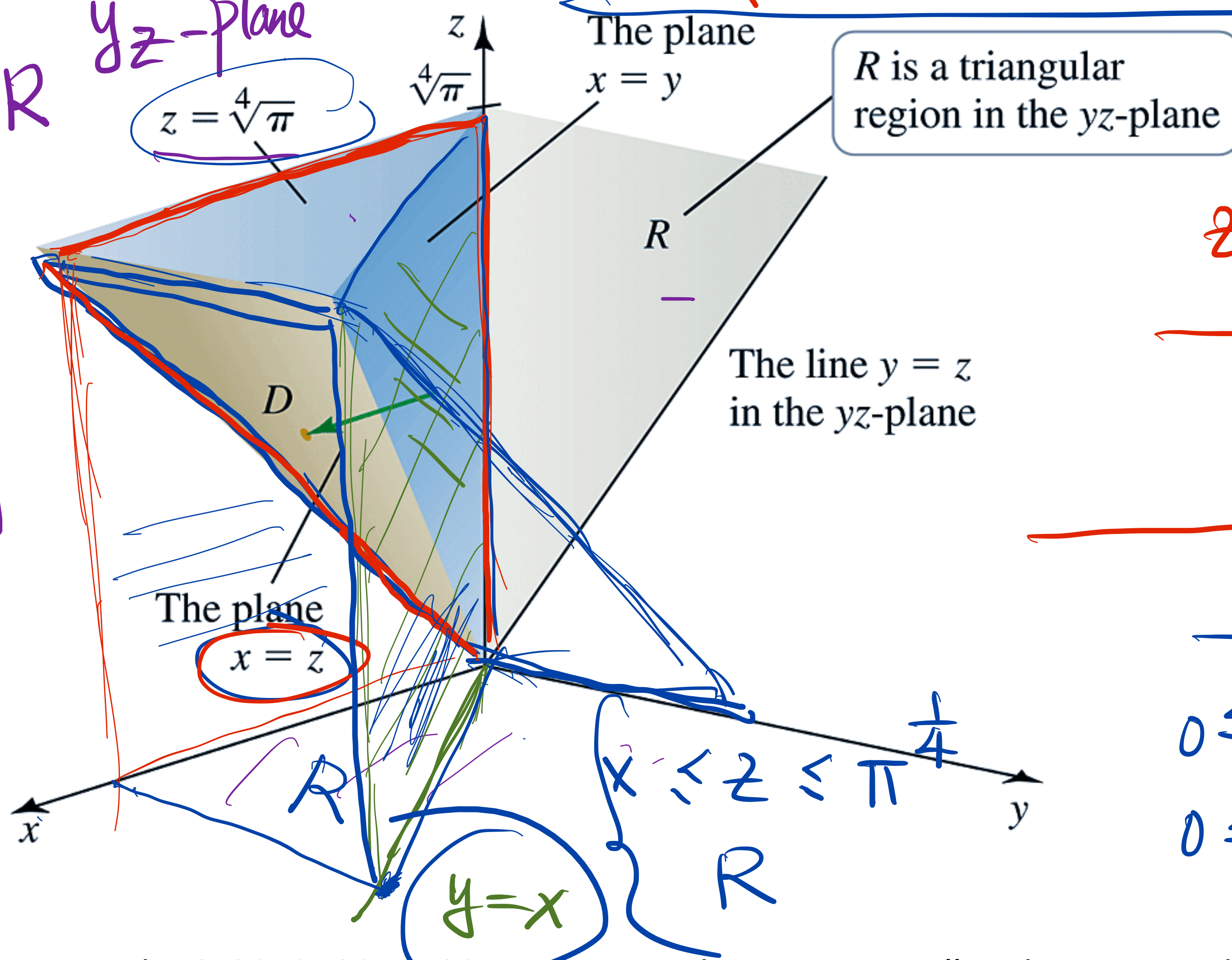
$R:\ 0\le y\le z,\quad 0\le z\le \pi^{1/4}$

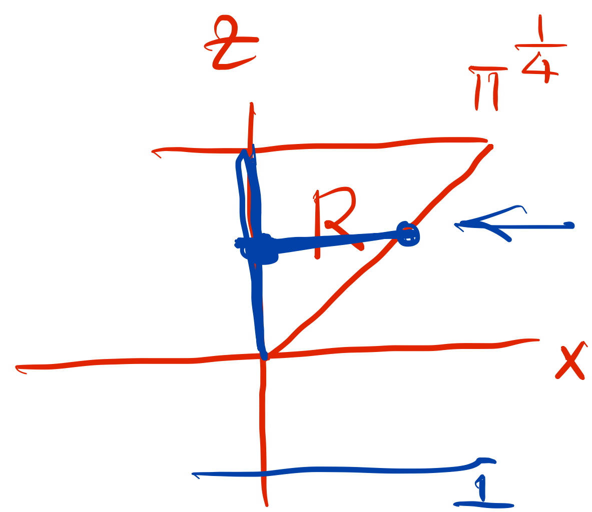
yz-plane



$y=x$

$x=z$

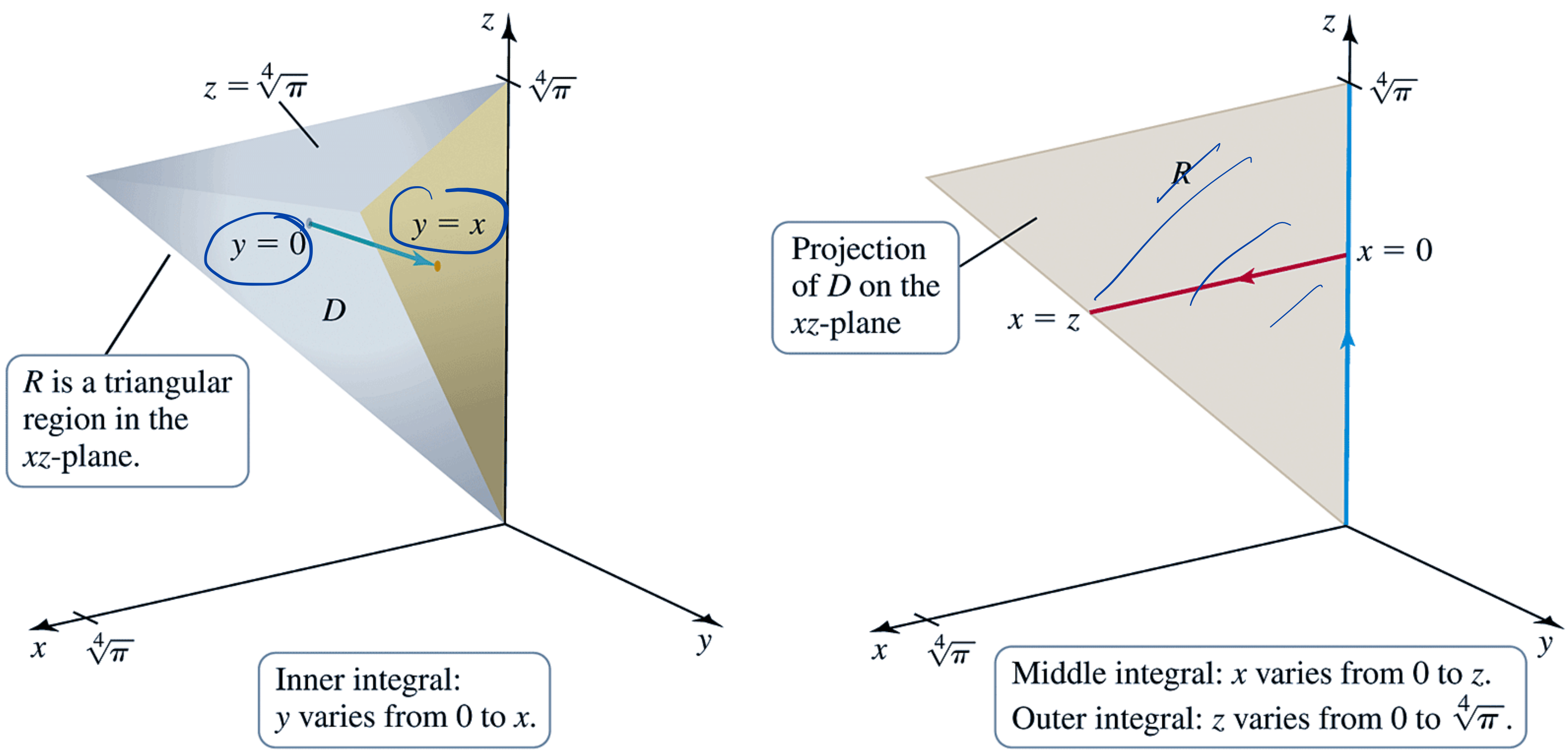




$0\le z\le\pi^{1/4}$

$0\le x\le z$

Figure 16.46 (a & b)



Definition Average Value of a Function of Three Variables

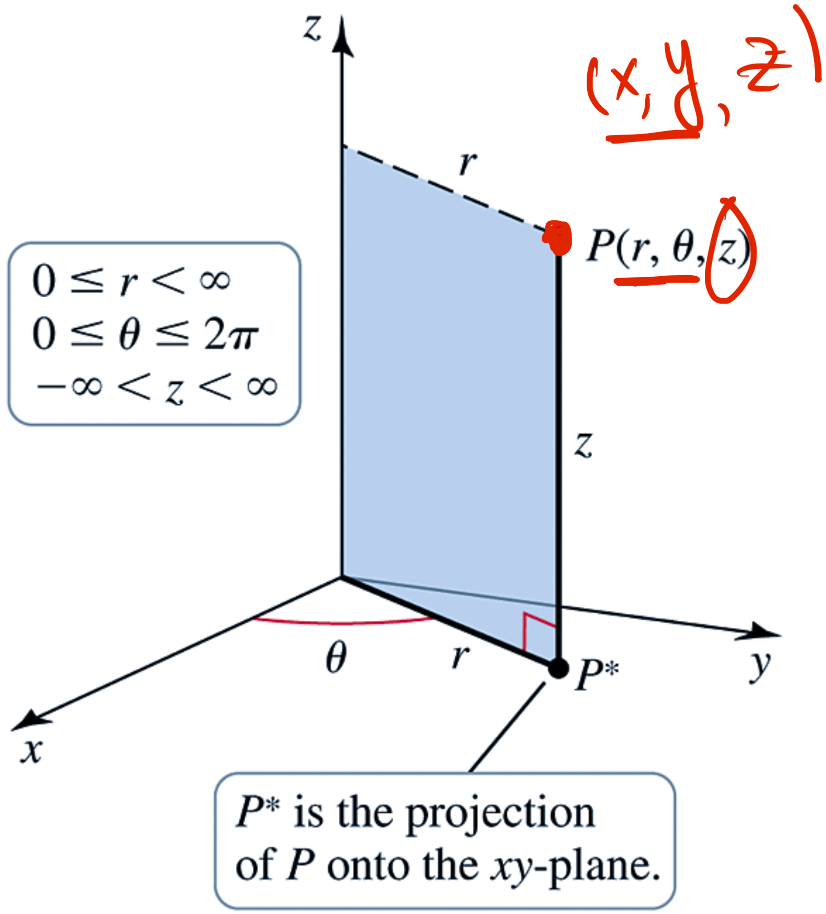
If *f* is continuous on a region *D* of $\circ^{3}$, then the **average value** of *f* over *D* is

$$\overline{f} = \frac{1}{\text{Volume of } D} \iiint_D f(x, y, z)\; dV.$$

$$\iiint_D f\,dV = \iint_R \left(\int_{G(x,y)}^{H(x,y)} f\,dz \right) dA$$

Section 16.5 Triple Integrals in Cylindrical and Spherical Coordinates

Figure 16.47



z
(x, y, z)
r
P(r, θ, z)
0 ≤ r < ∞
0 ≤ θ ≤ 2π
−∞ < z < ∞
z
θ
r
P*
y
x
P* is the projection
of P onto the xy-plane.

Table 16.4 (1 of 3)

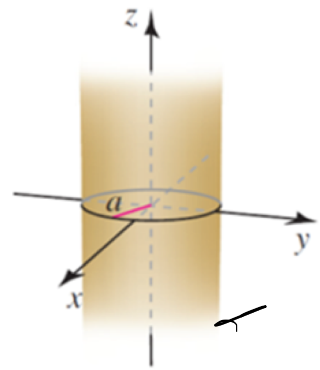
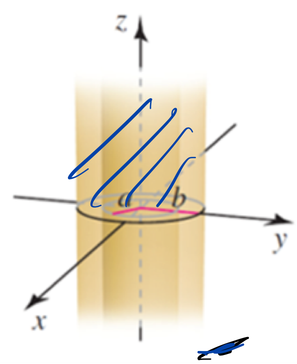
Name	Description	Example
Cylinder	$\{(r,\theta,z): r = a\}, a > 0$	
Cylindrical Shell	$\{(r,\theta,z): 0 < a \le r \le b\}$	

Table 16.4 (2 of 3)

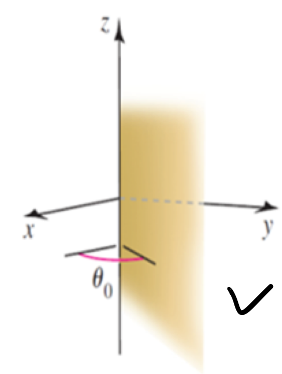
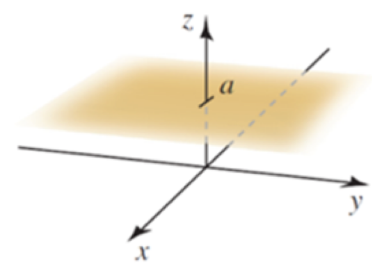
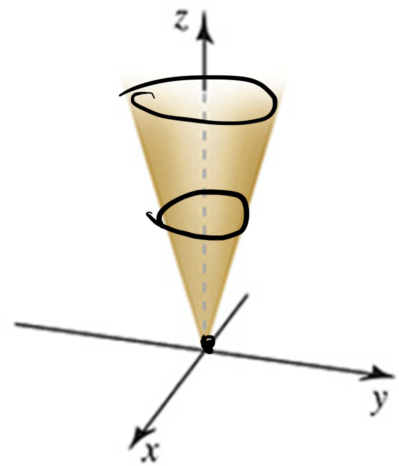
Name	Description	Example
Vertical half plane	$\{(r,\theta,z): \theta = \theta_0\}$	
Horizontal plane	$\{(r,\theta,z): z = a\}$	

Table 16.4 (3 of 3)

Name	Description	Example
Cone	$\{(r, \theta, z) : z = ar\}, a \neq 0$	

Example 1 Identify and sketch the following sets in cylindrical coordinates.

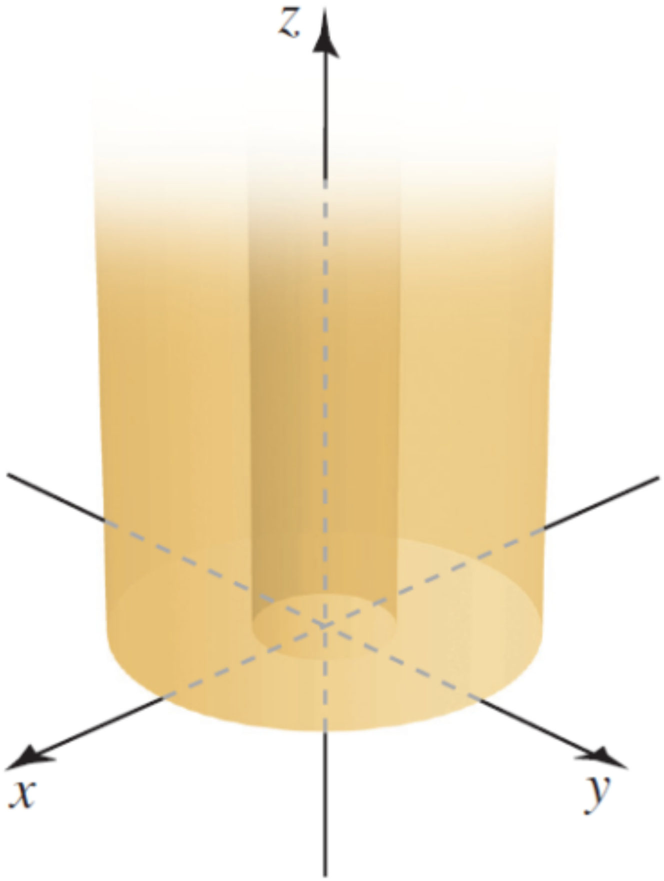
(a) $Q = \{(r,\theta,z) : 1 \le r \le 3,\ z \ge 0\}$.

(b) $S = \{(r,\theta,z) : z = 1 - r,\ 0 \le r \le 1\}$.

Figure 16.48 (a & b)

$z = k = 1 - r$ $\quad r = 1 - k$

$k \le 1$



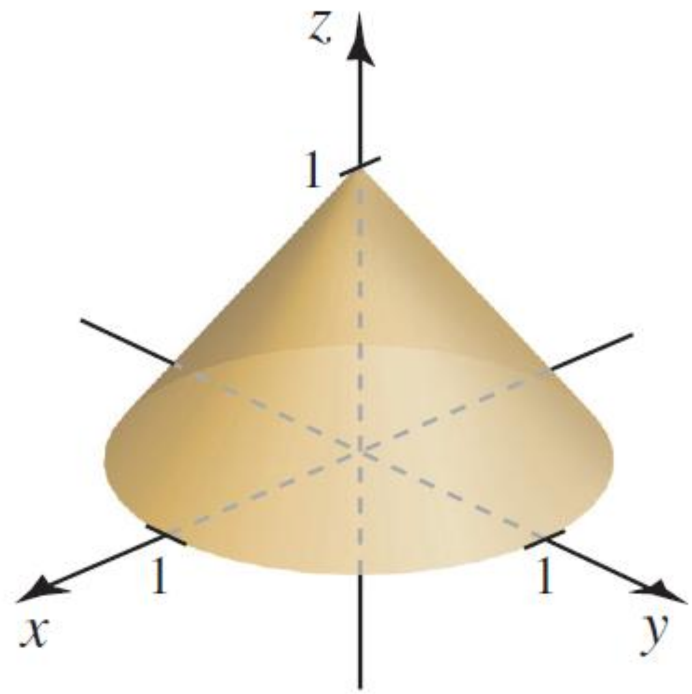
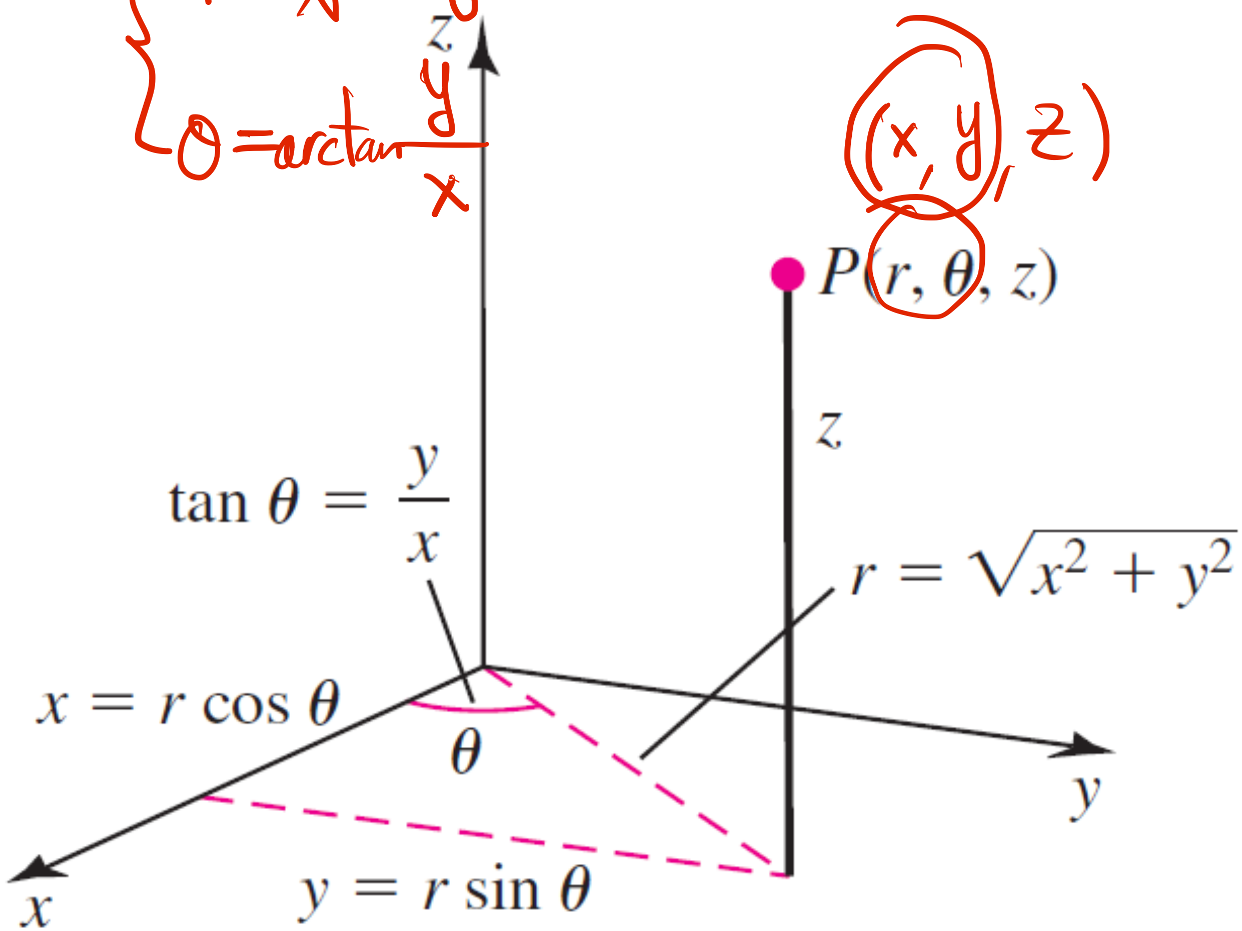


Figure 16.49

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{cases}$$



Transformations Between Cylindrical and Rectangular Coordinates

Rectangular → Cylindrical

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

$$z = z$$

Cylindrical → Rectangular

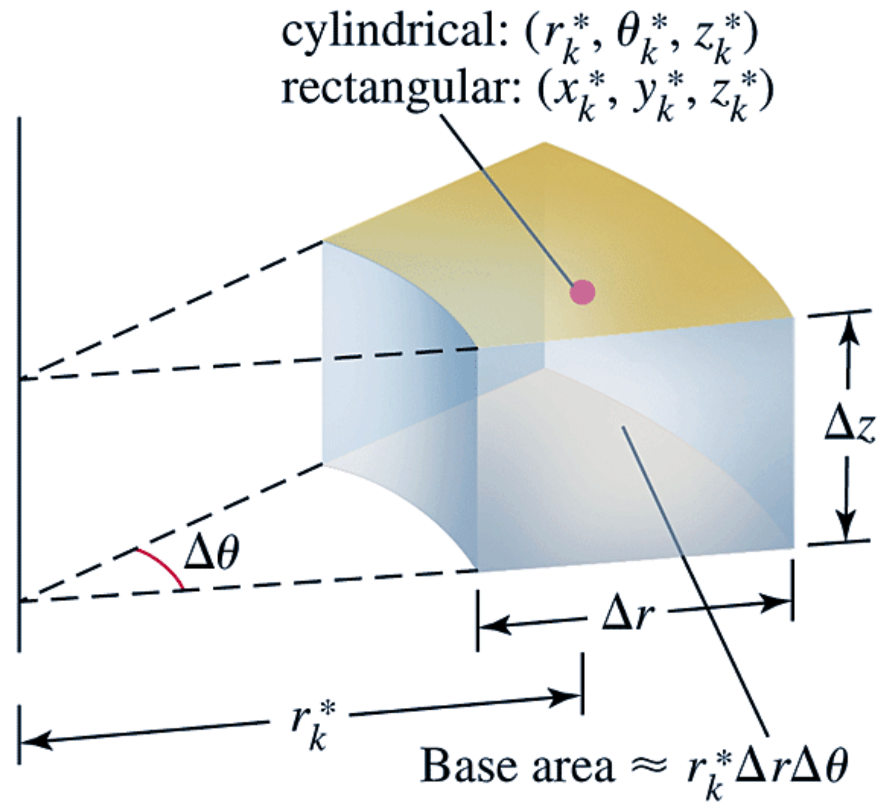
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

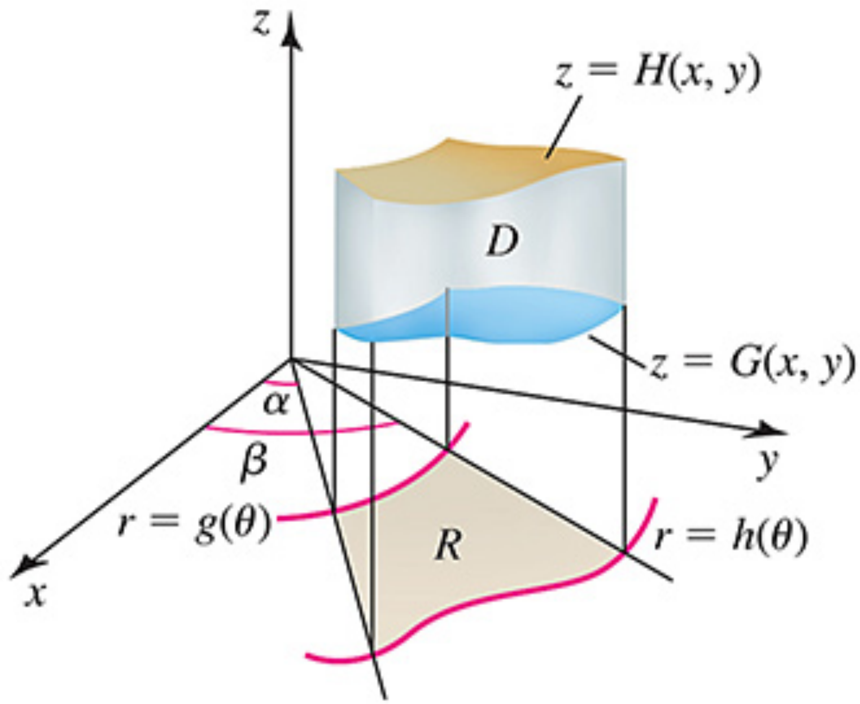
Figure 16.50

$$\iiint_D f(x,y,z)\,dV = \iint_R \left(\int_{G(x,y)}^{H(x,y)} f\,dz\right) dA$$



Approximate volume $\Delta V_k = r_k^*\Delta r\Delta\theta\Delta z$

Figure 16.51



Theorem 16.6 Change of Variables for Triple Integrals in Cylindrical Coordinates

Let *f* be continuous over the region *D*, expressed in cylindrical coordinates as

$$D = \{(r, \theta, z) : 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta, G(x, y) \le z \le H(x, y)\}.$$

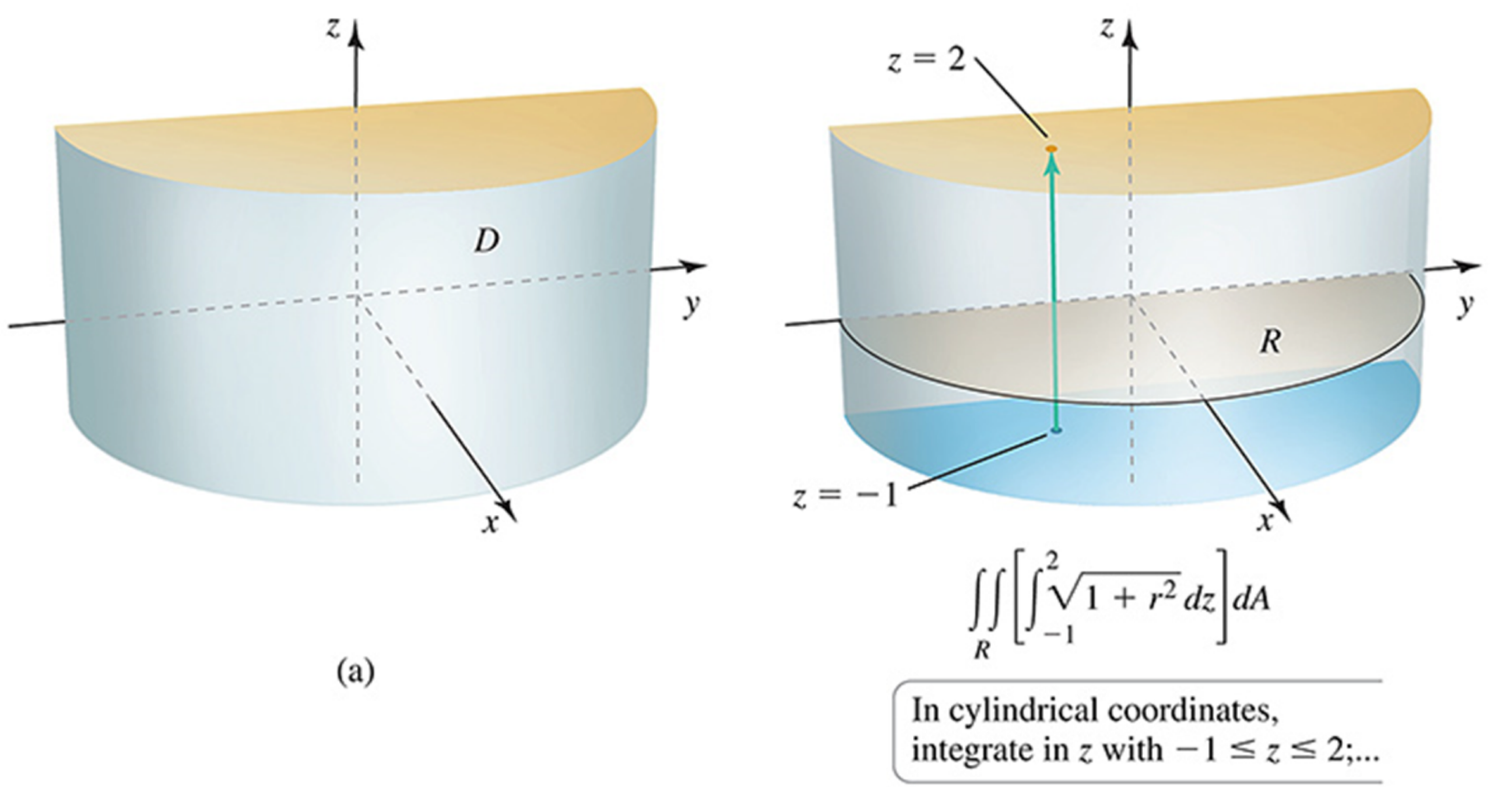
Then *f* is integrable over *D*, and the triple integral of *f* over *D* is

$$\iiint_D f(x, y, z)\,dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \left(\int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz \right) r\,dr\,d\theta.$$

R

dA

Figure 16.52 (a & b)



Example 2 (switching coordinate system)

$$I=\int_0^{2\sqrt{2}}\int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}}\left(\int_{-1}^{2}\sqrt{1+x^2+y^2}\,dz\right)dy\,dx$$

$$=\int_0^{2\sqrt{2}}\int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} 3\sqrt{1+x^2+y^2}\,dy\,dx$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\sqrt{2}} 3\sqrt{1+r^2}\,r\,dr\,d\theta$$

$u=1+r^2$

$du=2r\,dr$

$$\pi\int_1^9 \frac{3\sqrt{u}}{2}\,du$$

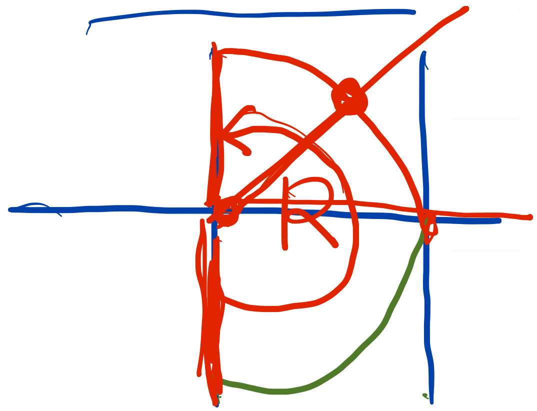
$$=\frac{3}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_1^9=27-1$$

Figure 16.52 (c)

$-\sqrt{8-x^2}\le y\le\sqrt{8-x^2}$

$0\le x\le 2\sqrt{2}$

} R



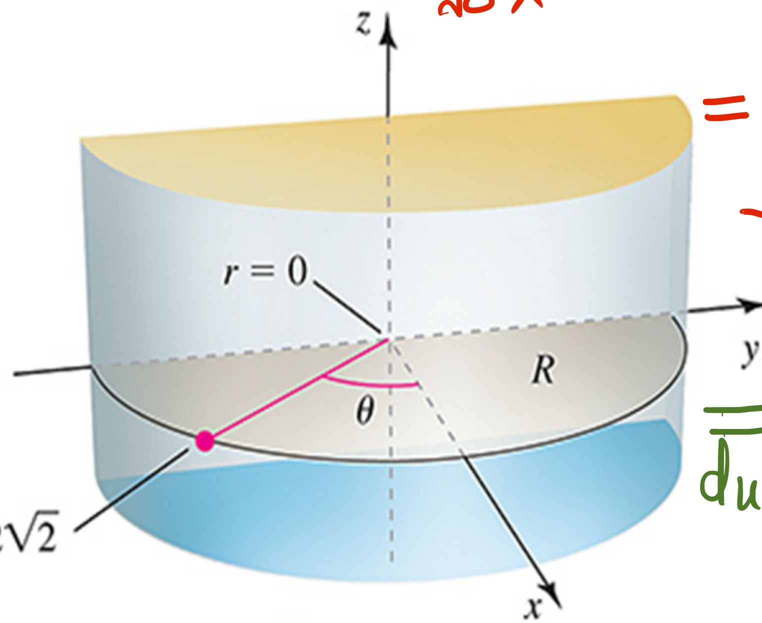
$x=0$ $\quad$ $x=2\sqrt{2}$

$y=\pm\sqrt{8-x^2}\Rightarrow y^2=8-x^2$

$x^2+y^2=8$

$0\le r\le\sqrt{8}$

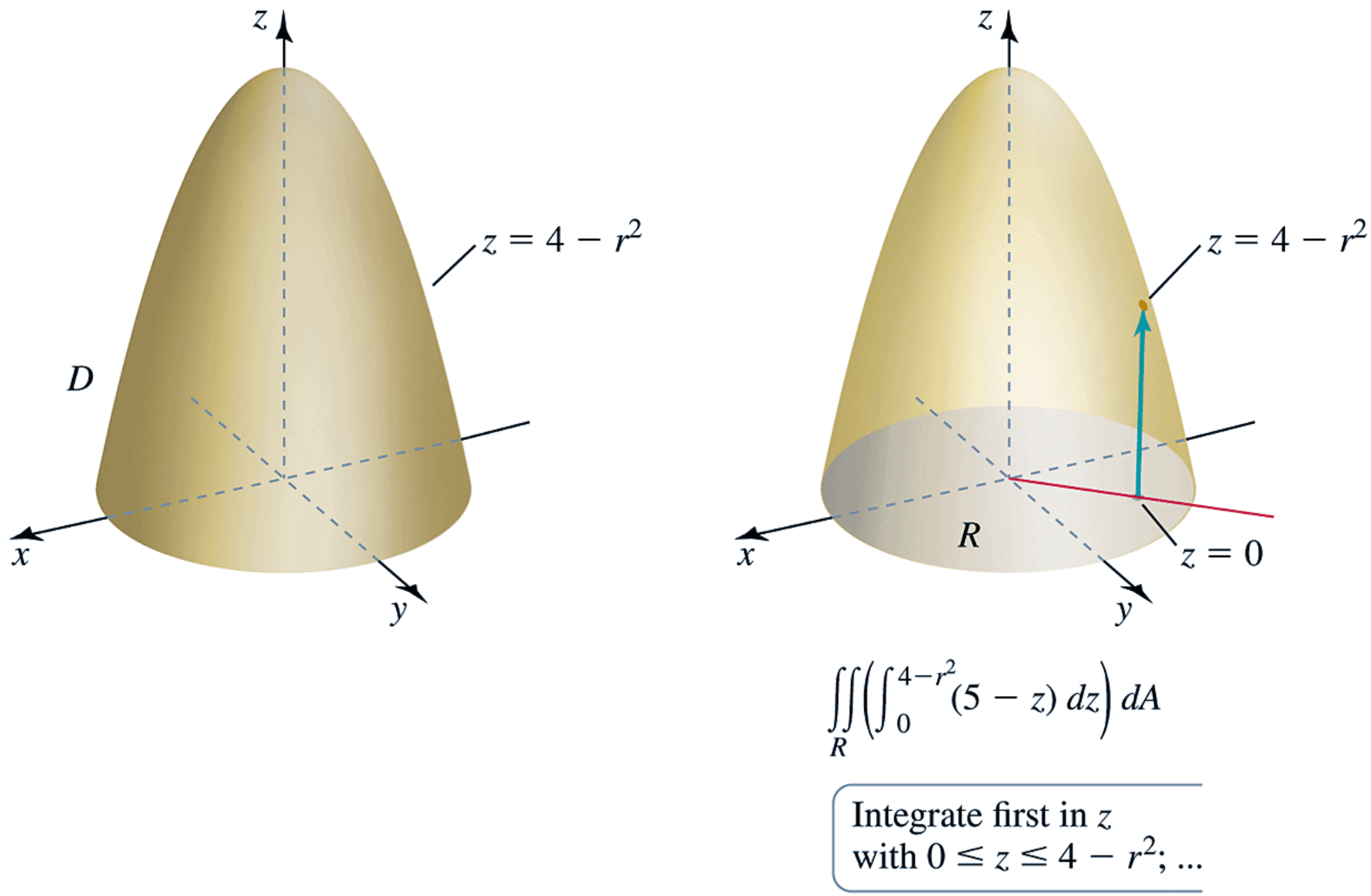
$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\sqrt{2}}\int_{-1}^{2}\sqrt{1+r^2}\,dz\,r\,dr\,d\theta$$

... then integrate over R with $0\le r\le 2\sqrt{2},\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.

Figure 16.53 (a & b)



Example 3 Find the mass of the solid D bounded by $z = 1 - r^2$ and $z = 0$. The density is $f(r,\theta,z) = 5 - z$.

Figure 16.53 (c)

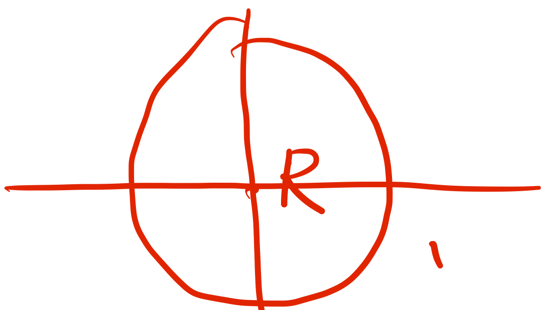
$$M = \iiint_D (5-z)\,dV = \iint_R \left(\int_0^{1-r^2} (5-z)\,dz \right) dA$$

$$= \iint_R \left[5z - \frac{1}{2}z^2 \right]_0^{1-r^2} dA$$

$$= \iint_R \left[5(1-r^2) - \frac{1}{2}(1-r^2)^2 \right] dA$$

intersection R

$$\begin{cases} z = 1 - r^2 \\ z = 0 \end{cases} \Rightarrow 1 - r^2 = 0 \Rightarrow r^2 = 1 \Rightarrow r = 1$$



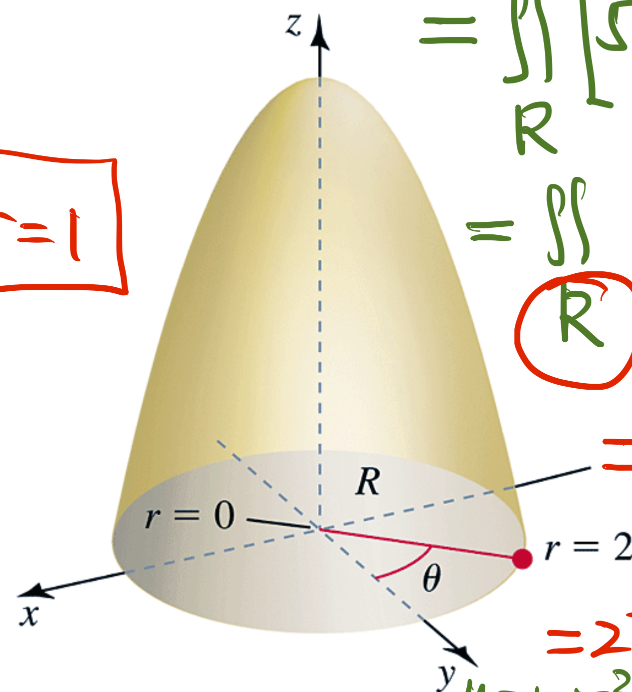
$$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{cases}$$

$$= \int_0^{2\pi}\int_0^1 [\quad]\, r\,dr\,d\theta$$

$$= 2\pi \int_0^1 \left[5(1-r^2) - \frac{1}{2}(1-r^2)^2 \right] r\,dr$$

$u = 1 - r^2$, $du = -2r\,dr$

$$2\pi \int_1^0 \left[5u - \frac{1}{2}u^2 \right] \cdot \left(-\frac{1}{2}du \right)$$



$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} (5 - z)\,dz\,r\,dr\,d\theta$$

... then integrate over R with $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

Example 4 Find the volume of the solid D between the cone $z=\sqrt{x^2+y^2}$ and the inverted paraboloid $z=12-x^2-y^2$.

$$V=\iiint_D dV=\iint_R\left(\int_{\sqrt{x^2+y^2}}^{12-x^2-y^2} dz\right)dA$$

Figure 16.54 (a, b & c)

R=? intersection

$$\sqrt{x^2+y^2}=12-x^2-y^2$$

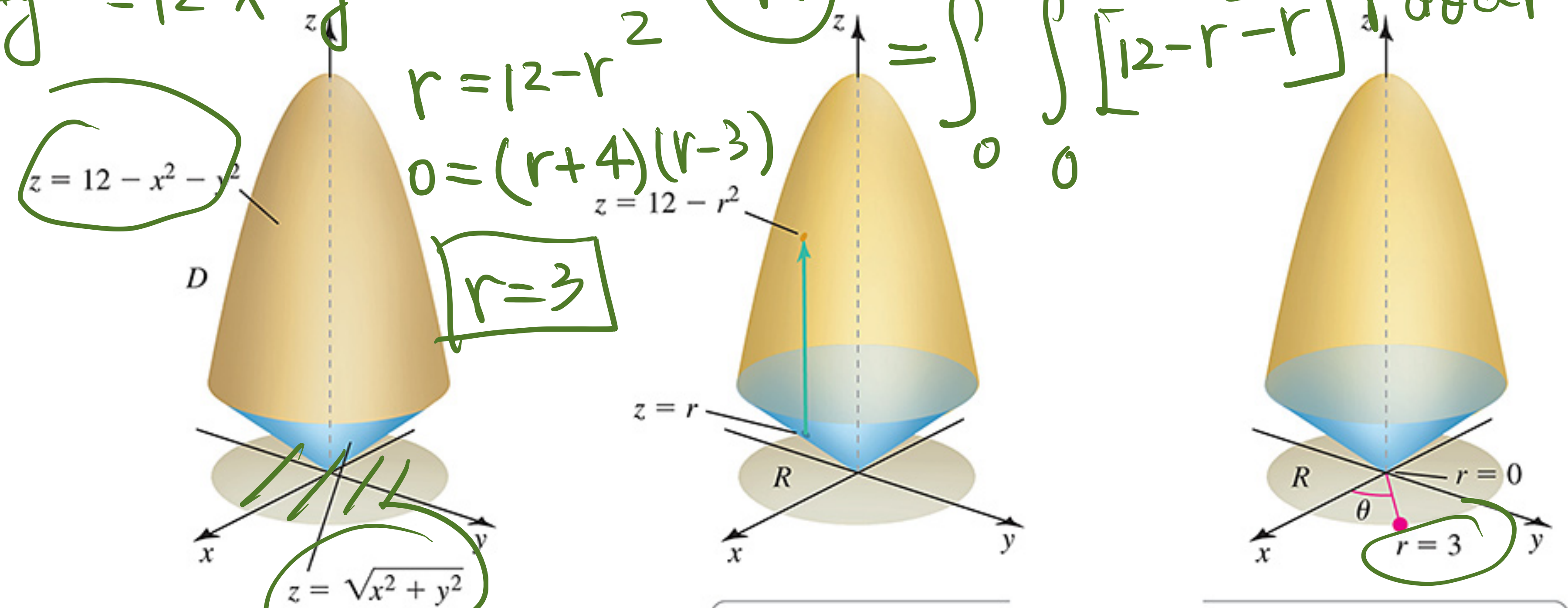
$$r=12-r^2$$

$$0=(r+4)(r-3)$$

$$r=3$$

$$=\iint_R\left[12-x^2-y^2-\sqrt{x^2+y^2}\right]dA$$

$$=\int_0^3\int_0^{2\pi}\left[12-r^2-r\right]r\,d\theta\,dr$$



Integrate first in z with $r \le z \le 12 - r^2$;...

... then integrate over R with $0 \le r \le 3$, $0 \le \theta \le 2\pi$.

Spherical Coordinates

Figure 16.55

$$\begin{cases} -\infty \le x \le +\infty \\ -\infty \le y \le +\infty \\ -\infty \le z \le +\infty \end{cases}$$

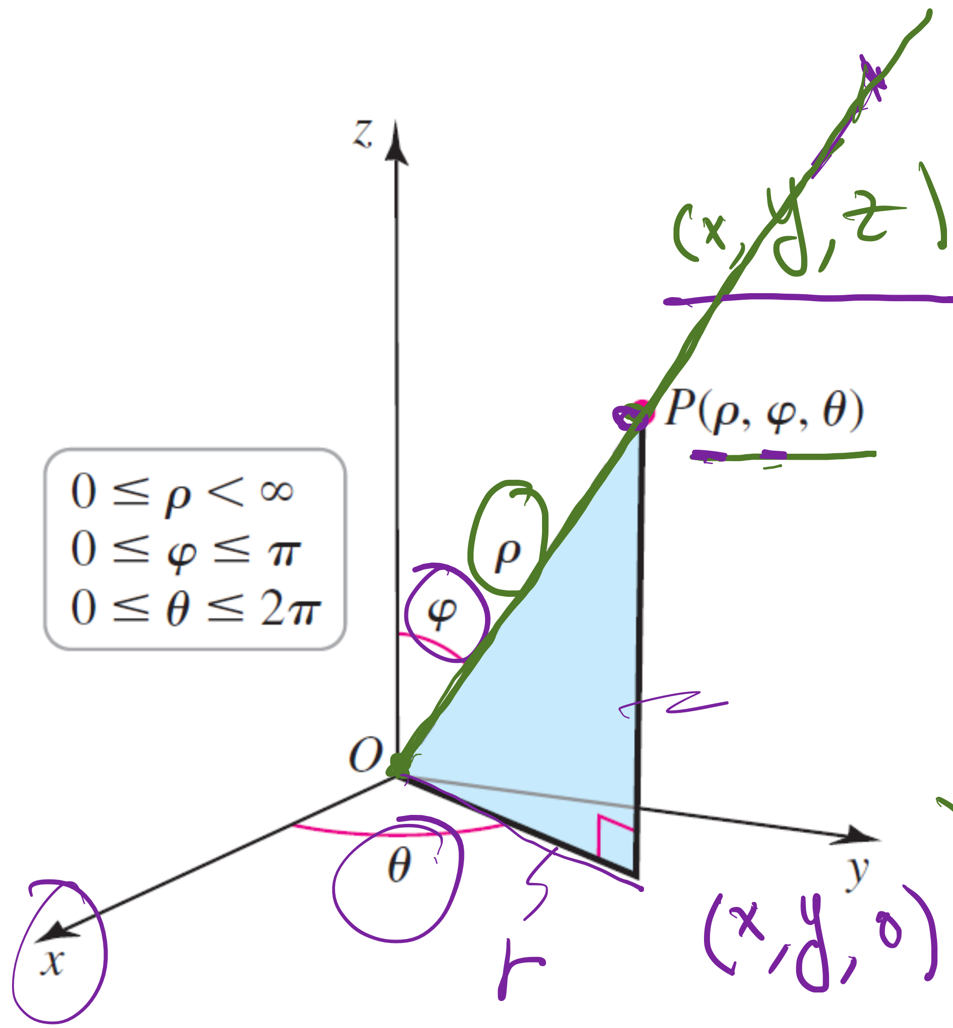
$0 \le \rho \le +\infty$

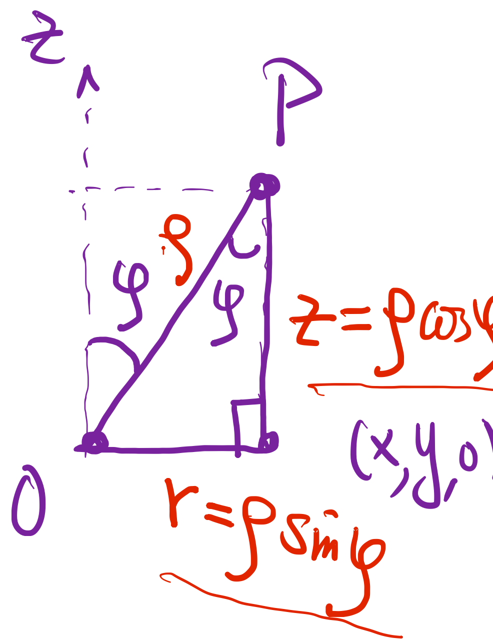
$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$$\rho = \sqrt{x^2+y^2+z^2}$$

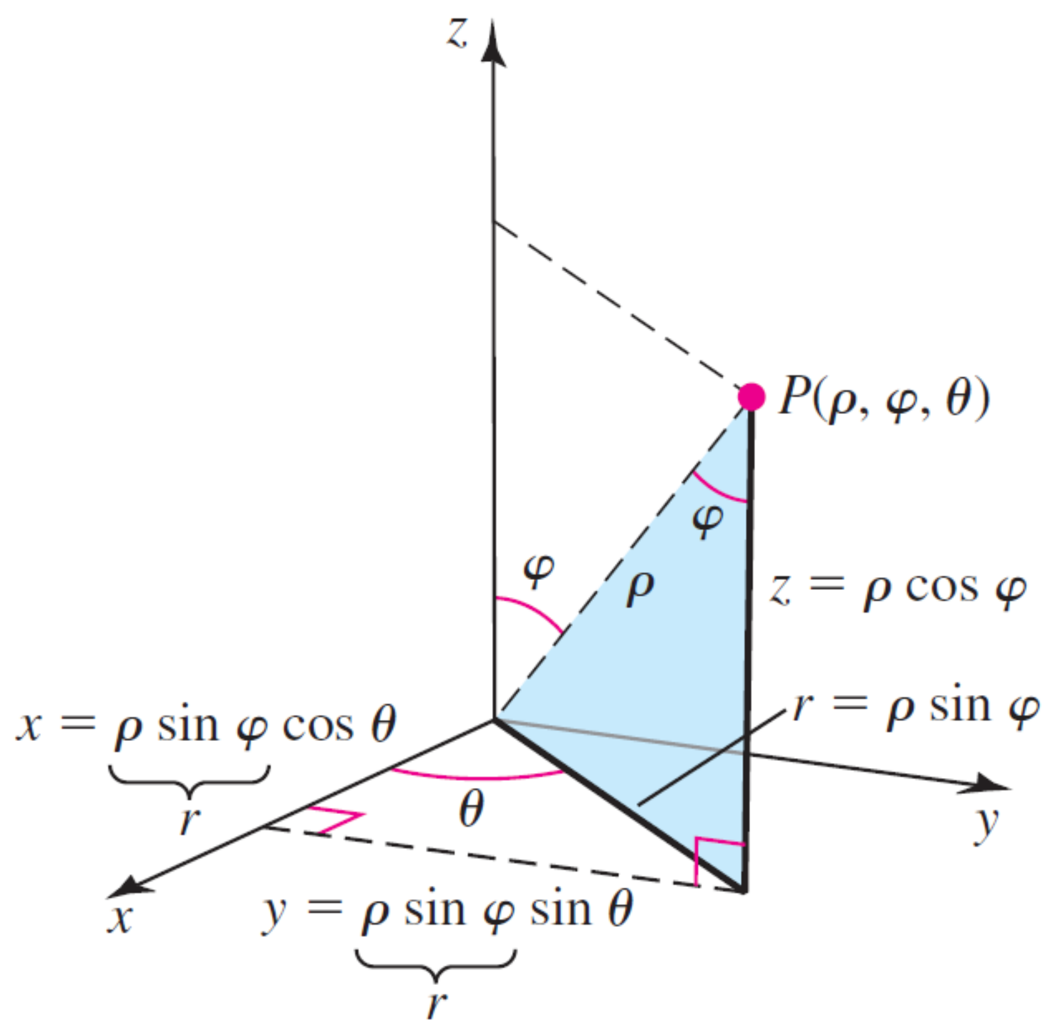
$$\begin{cases} \theta = \arctan\left(\dfrac{y}{x}\right) \\ \varphi = \arctan\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \end{cases}$$





$$\begin{cases} x = r\cos\theta \\ x = \rho\sin\varphi\cos\theta \\ y = r\sin\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

Figure 16.56


z
$P(\rho, \varphi, \theta)$
φ
φ
ρ
$z = \rho \cos \varphi$
$r = \rho \sin \varphi$
$x = \rho \sin \varphi \cos \theta$
r
θ
y
x
$y = \rho \sin \varphi \sin \theta$
r

Transformations Between Spherical and Rectangular Coordinates

Rectangular → Spherical

$$\rho^2 = x^2 + y^2 + z^2$$

Use trigonometry to find φ and θ

Spherical → Rectangular

$$x = \rho \sin\varphi \cos\theta$$

$$x = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi$$

Example 5 Express the following sets in rectangular coordinates and identify the set.

(a) $\{(\rho,\varphi,\theta): \rho = 2a\cos\varphi,\ 0\le\varphi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi\}$. (b) $\{(\rho,\varphi,\theta): \rho = 4\sec\varphi,\ 0\le\varphi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi\}$

$= \frac{4}{\cos\varphi}$

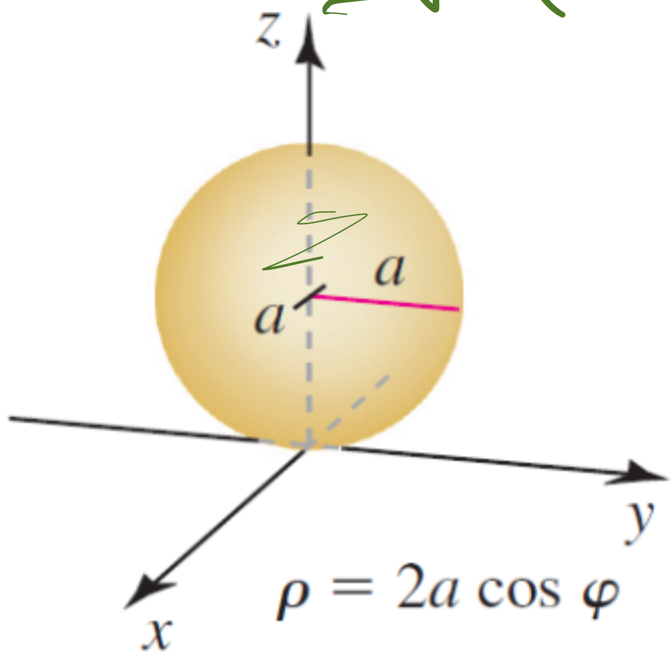
Figure 16.57 (a & b)

$\rho^2 = 2a\rho\cos\varphi = 2az$

$\rho^2 = x^2+y^2+z^2$

$x^2+y^2+(z-a)^2 = a^2$

$z = \rho\cos\varphi = 4 \Rightarrow z = 4$



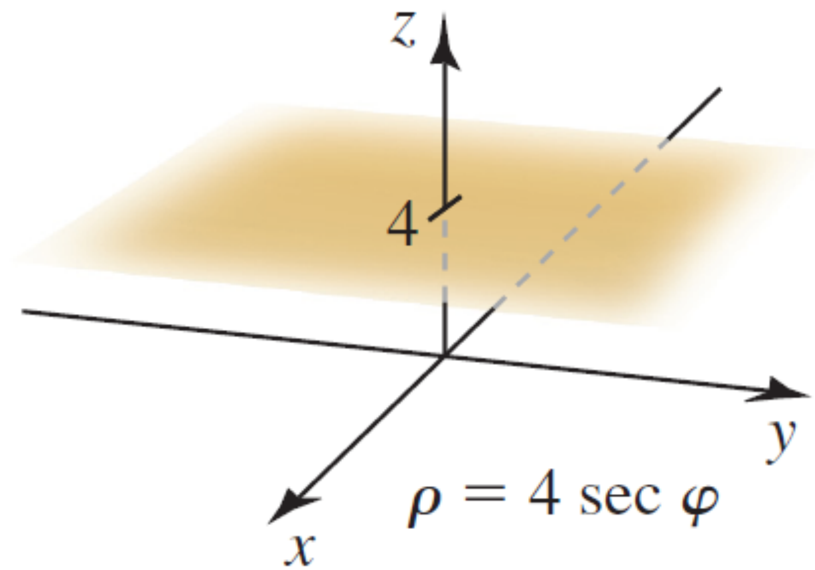


Table 16.5 (1 of 3)

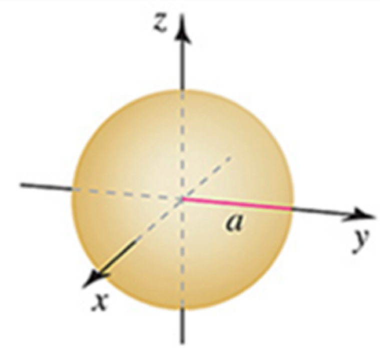
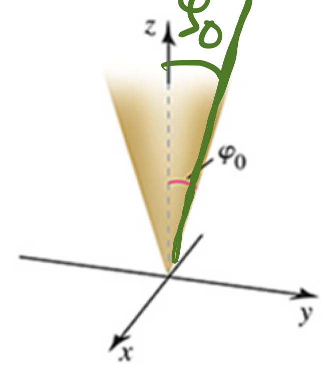
Name	Description	Example
Sphere, radius *a*, center (0, 0, 0)	$\{(\rho, \varphi, \theta) : \rho = a\}, a > 0$	
Cone	$\{(\rho, \varphi, \theta) : \varphi = \varphi_0\}, \varphi_0 \neq 0, \frac{\pi}{2}, \pi$	

Table 16.5 (2 of 3)

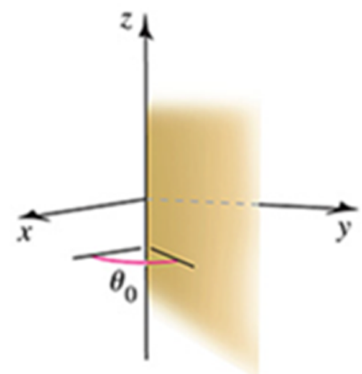
Name	Description	Example
Vertical half plane	$\{(\rho, \varphi, \theta) : \theta = \theta_0\}$	

Table 16.5 (3 of 3)

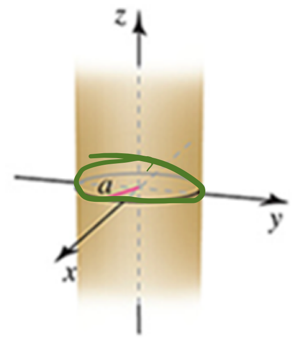
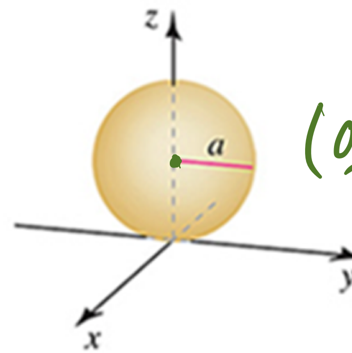
Name	Description	Example
Cylinder, radius $a > 0$	$\{(\rho,\varphi,\theta) : \rho = a\csc\varphi, 0 < \varphi < \pi\}$ $= a\frac{1}{\sin\varphi}$ $a = \rho\sin\varphi = r$	
Sphere, radius $a > 0$, center (0, 0, a)	$\{(\rho,\varphi,\theta) : \rho = 2a\cos\varphi, 0 \le \varphi \le \frac{\pi}{2}\}$	 (0,0,a)

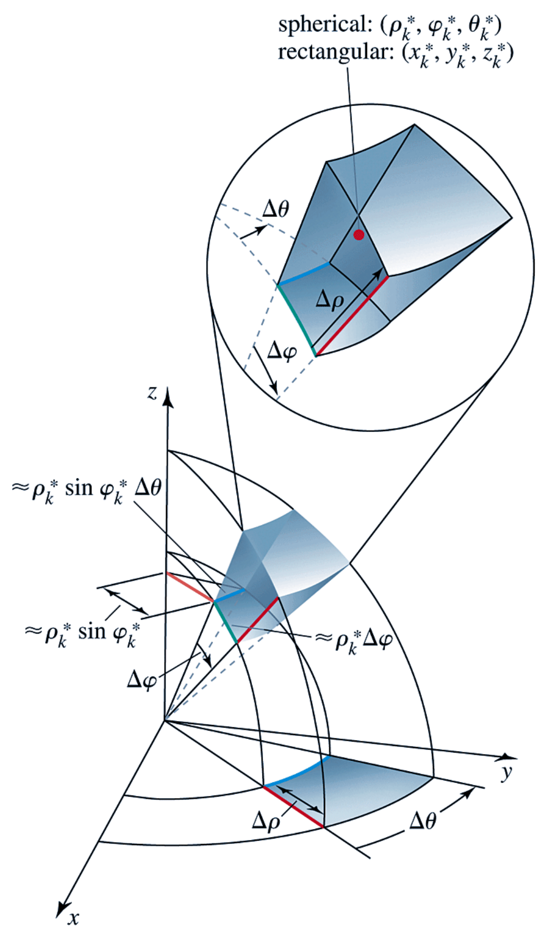
Figure 16.58

$$\iiint_D f(x,y,z)\,dV = \iiint_\Omega f$$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$$\left| \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} \right| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \vdots & & \end{vmatrix}$$



Approximate volume = $\Delta V_k = \rho_k^{*2} \sin\varphi_k^* \Delta\rho\Delta\varphi\Delta\theta$

Figure 16.59

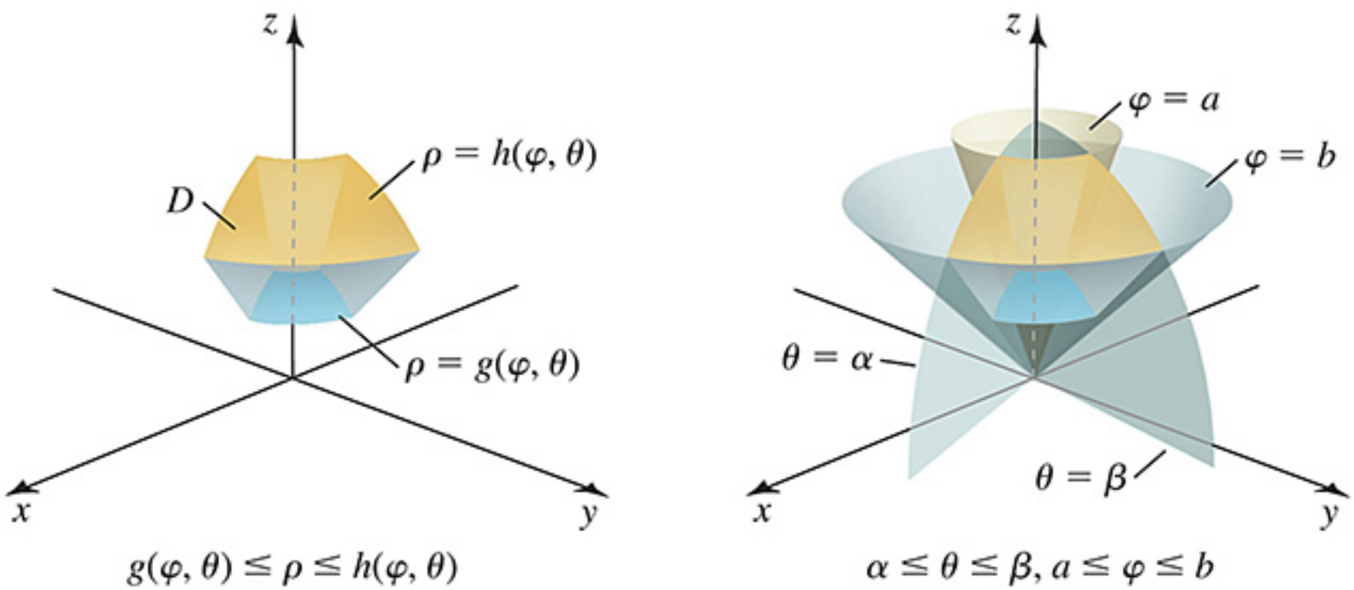
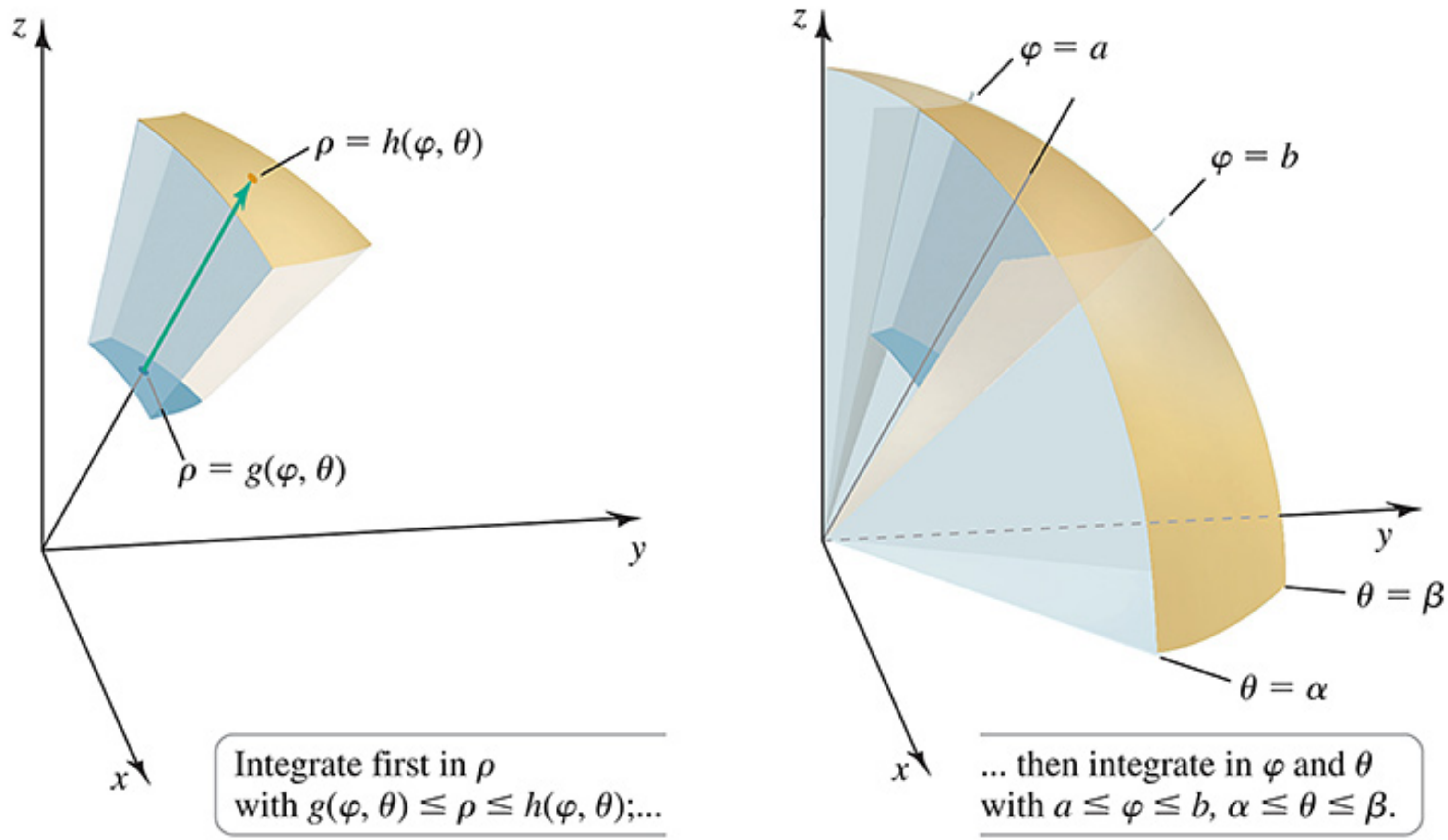

z
D
$\rho = h(\varphi, \theta)$
$\rho = g(\varphi, \theta)$
x
y
$g(\varphi, \theta) \le \rho \le h(\varphi, \theta)$
z
$\varphi = a$
$\varphi = b$
$\theta = \alpha$
$\theta = \beta$
x
y
$\alpha \le \theta \le \beta, a \le \varphi \le b$

Figure 16.60


z
$\rho = h(\varphi, \theta)$
$\rho = g(\varphi, \theta)$
y
x
Integrate first in ρ
with $g(\varphi, \theta) \le \rho \le h(\varphi, \theta)$;...
z
$\varphi = a$
$\varphi = b$
y
$\theta = \beta$
$\theta = \alpha$
x
... then integrate in φ and θ
with $a \le \varphi \le b$, $\alpha \le \theta \le \beta$.

Theorem 16.7 Change of Variable for Triple Integrals in Spherical Coordinates

Let *f* be continuous over the region *D*, expressed in spherical coordinates as

$$D = \{(\rho, \varphi, \theta) : 0 \le g(\varphi, \theta) \le \rho \le h(\varphi, \theta), a \le \varphi \le b, \alpha \le \theta \le \beta\}.$$

Then *f* is integrable over *D*, and the triple integral of *f* over *D* is

$$\iiint_D f(x, y, z)\,dV =$$

$$\int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)\,\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

dV = $\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

dA = r dr dθ

Example 6 $I = \iiint_D (x^2+y^2+z^2)^{-3/2}\, dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin.

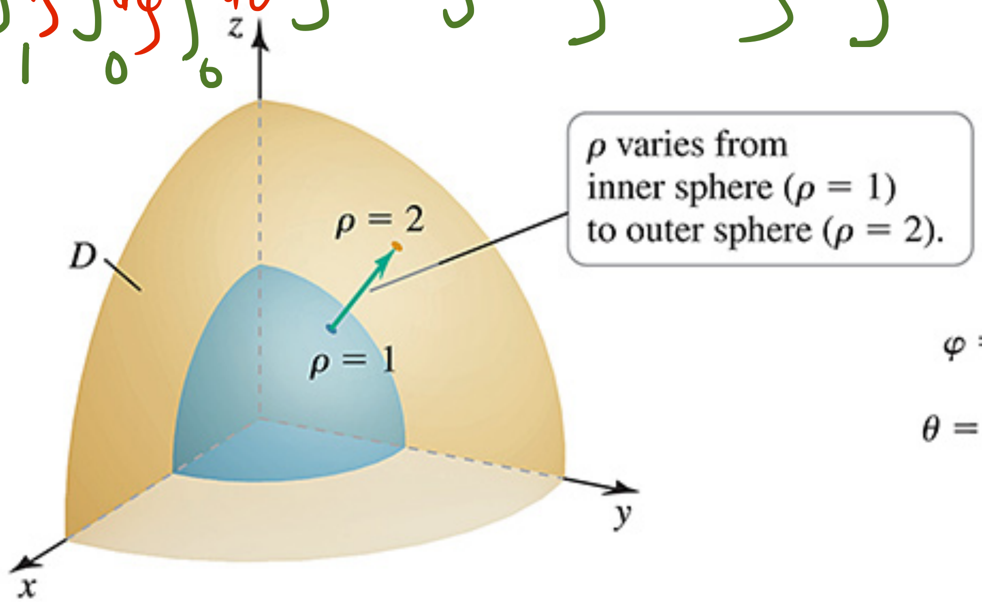
$1 \le \rho \le 2$

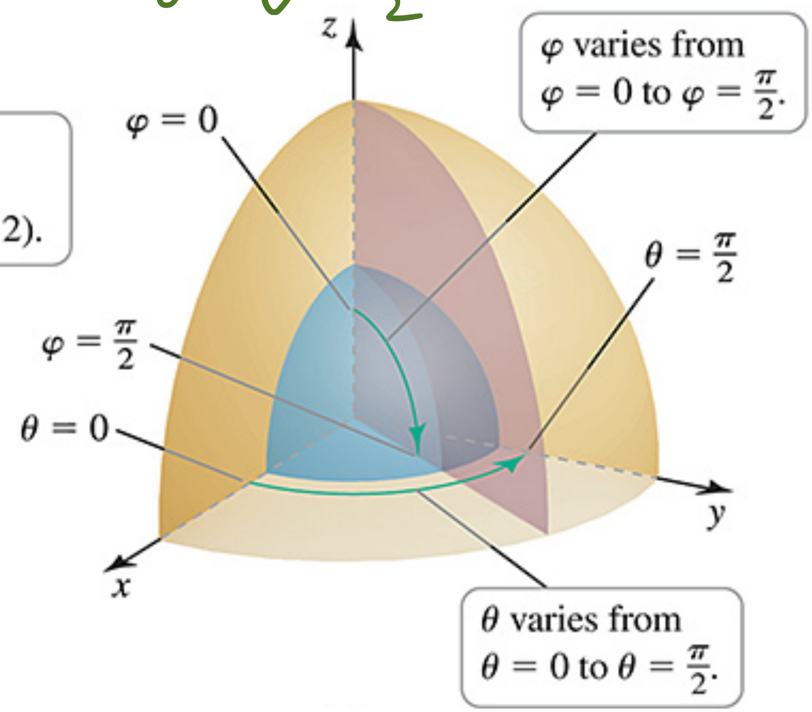
$0 \le \varphi \le \frac{\pi}{2}$

$0 \le \theta \le \frac{\pi}{2}$

Figure 16.61 (a & b)

$I = \int_1^2 d\rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta\, \rho^{-3} \cdot \rho^2 \sin\varphi \, d\theta\, d\varphi\, d\rho$

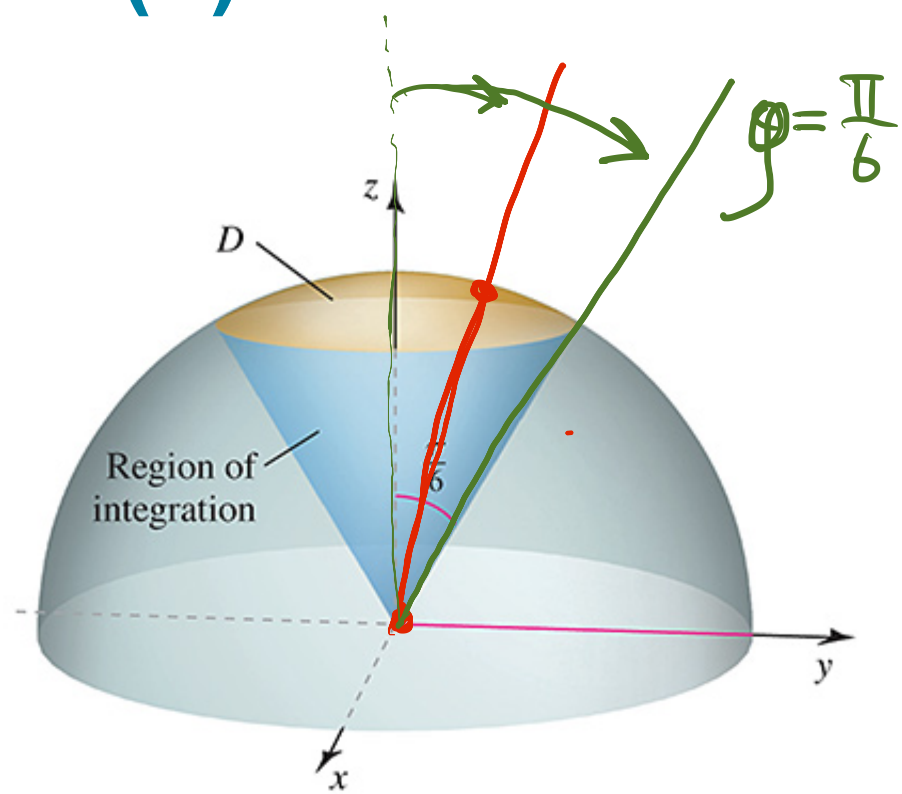




Example 7 Find the volume of the solid region D that lies inside the cone $\varphi = \pi/6$ and inside the sphere $\rho = 4$.

$$V = \iiint_D dV = \int_0^4 d\rho \int_0^{\frac{\pi}{6}} d\varphi \int_0^{2\pi} d\theta \; \rho^2 \sin\varphi$$

Figure 16.62 (a)

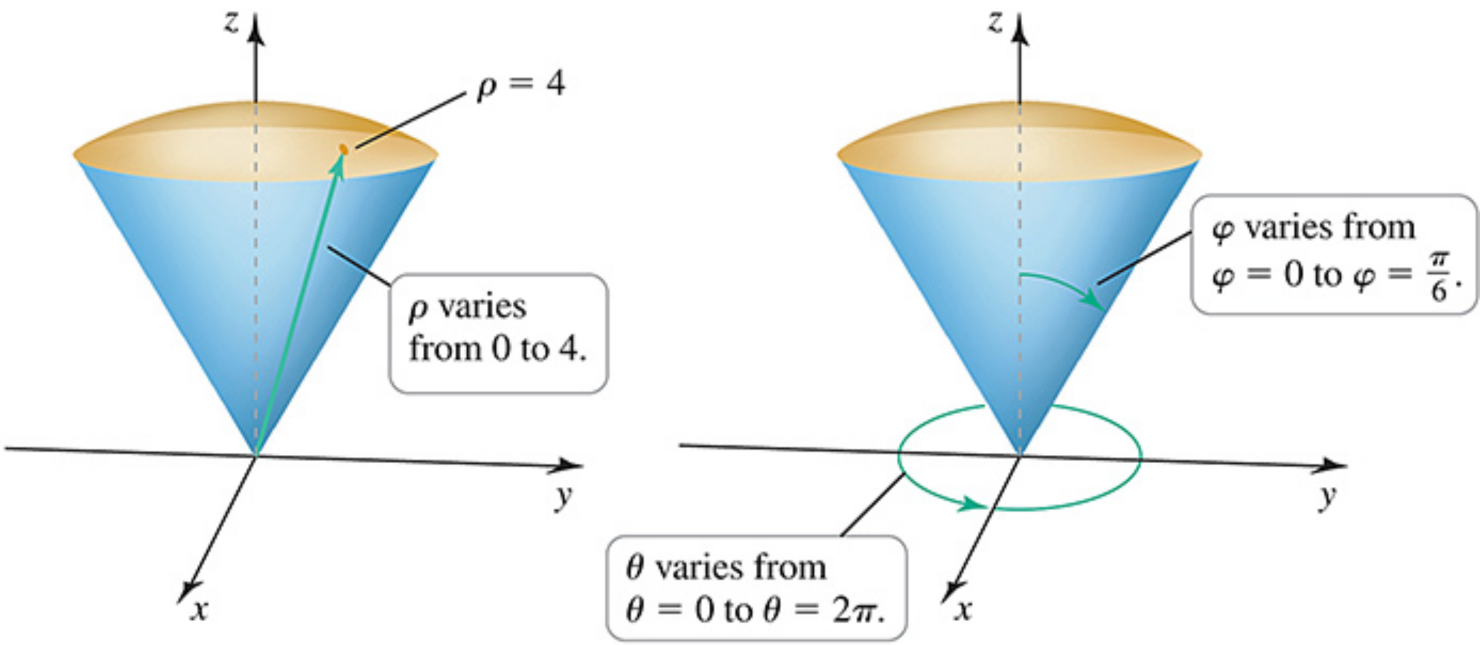


$0 \le \rho \le 4$

$0 \le \varphi \le \frac{\pi}{6}$

$0 \le \theta \le 2\pi$

Figure 16.62 (b & c)



Section 16.6 Integrals for Mass Calculations

Figure 16.63

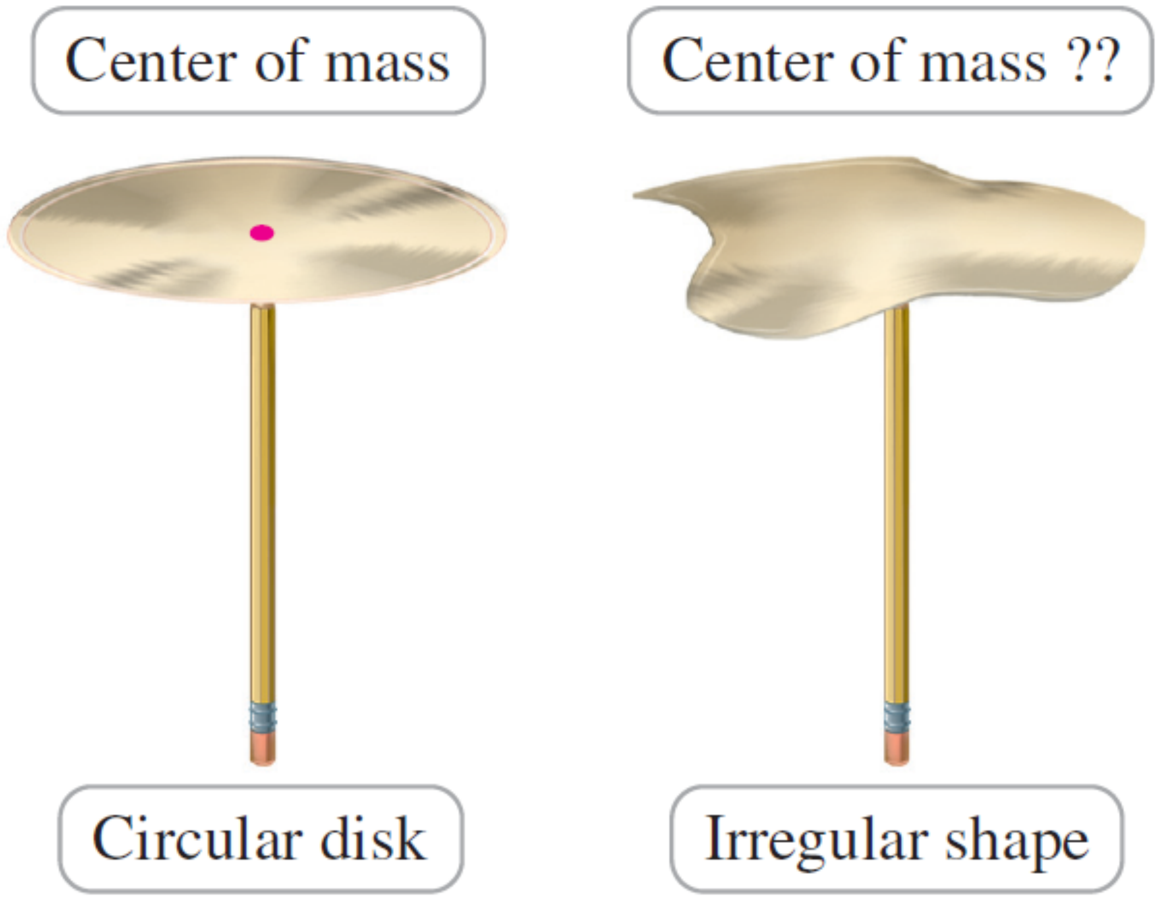
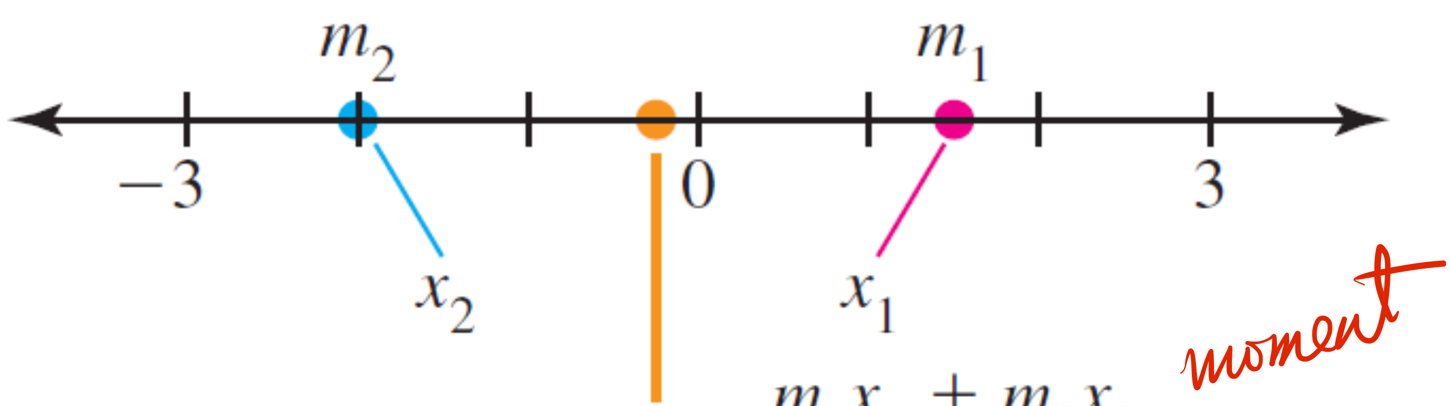

Center of mass
Center of mass ??
Circular disk
Irregular shape

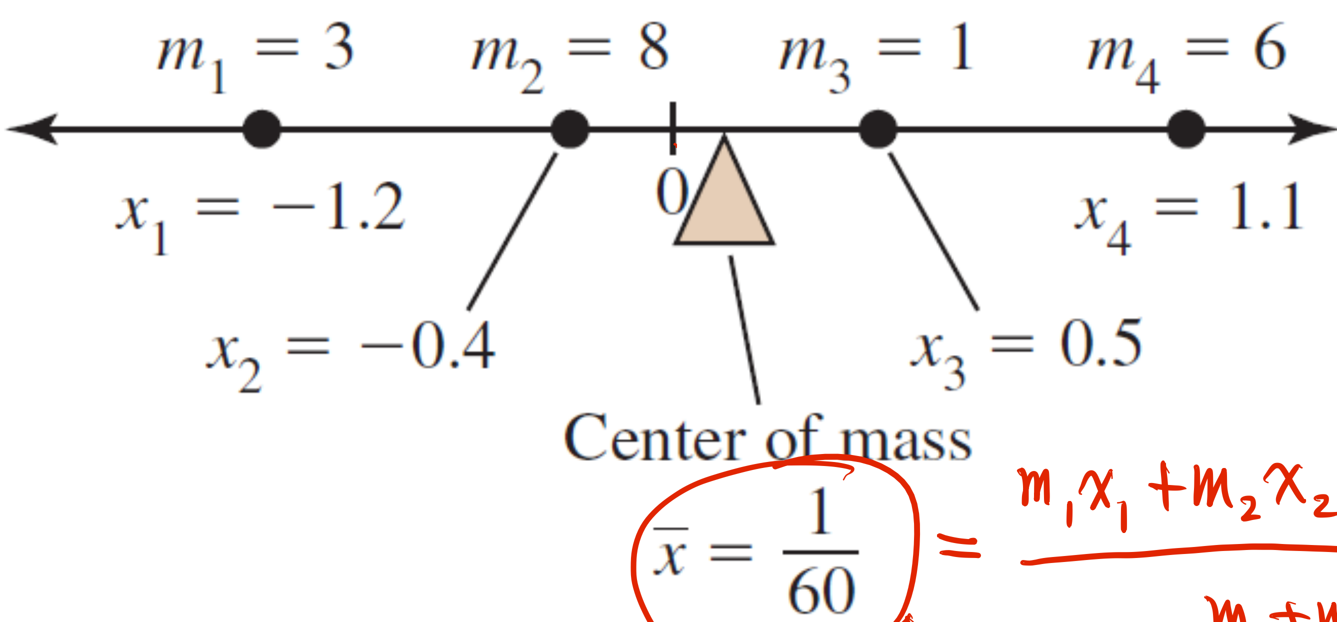
Figure 16.65



$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Example 1 Find the point at which the system shown in Fig. 16.67 balances.

Figure 16.67



$$= \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{\sum_{i=1}^{4} m_i x_i}{\sum_{i=1}^{4} m_i} \longrightarrow \frac{\int_a^b x\,\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$$

Definition Center of Mass in One Dimension

Let ρ be an integrable density function on the interval $[a,b]$ (which represents a thin rod or wire). The **center of mass** is located at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)dx \text{ and } m = \int_a^b \rho(x)dx.$$

Example 2 Suppose a thin 2-m bar is made of an alloy whose density in kg/m is $\rho(x) = 1 + x^2$, where $0 \le x \le 2$. Find the center of the bar.

$$m = \int_0^2 (1+x^2)\,dx$$

$$\bar{x} = \frac{M}{m}$$

$$M = \int_0^2 x(1+x^2)\,dx$$

Figure 16.70

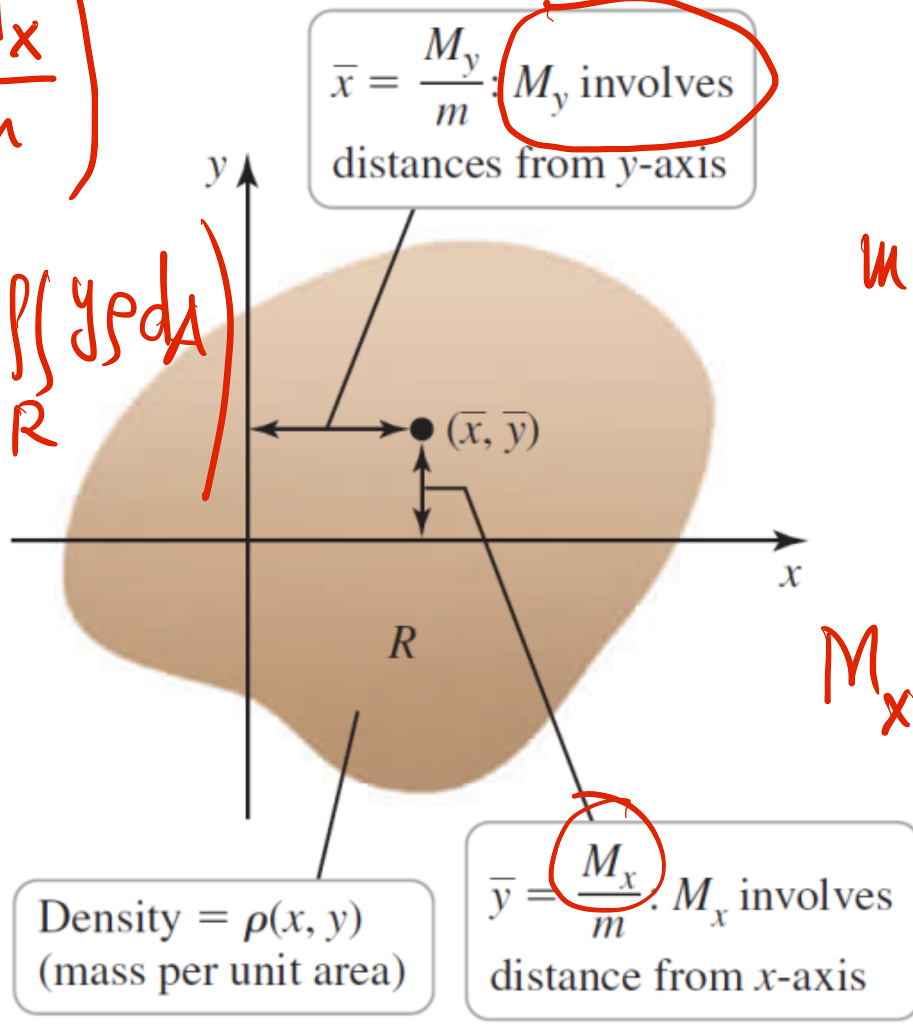
$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

$$= \frac{1}{\iint_R \rho \, dA}\left(\iint_R x \rho \, dA, \iint_R y \rho \, dA\right)$$

$$M_y = \iint_R x \, \rho(x,y) \, dA$$

$$m = \iint_R \rho \, dA$$

$$M_x = \iint_R y \rho \, dA$$



Definition Center of Mass in Two Dimensions

Let ρ be an integrable area density function defined over a closed bounded region *R* in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by *R* are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,y)\ dA \text{ and } \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)\ dA,$$

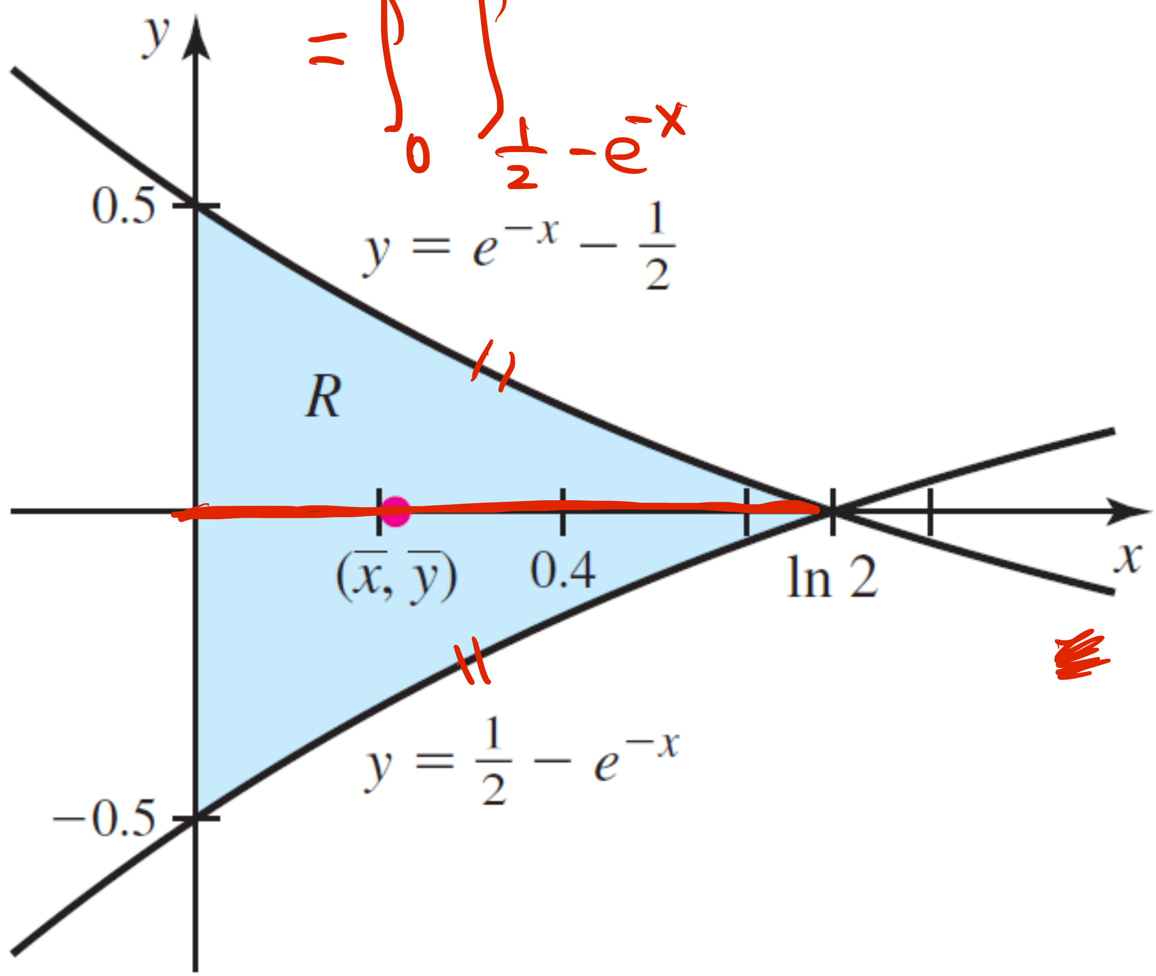
where $m = \iint_R \rho(x,y)dA$ is the mass, and M_y and M_x are the moments with respect to the *y*-axis and *x*-axis, respectively. If ρ is constant, the center of mass is called the **centroid** and is independent of the density.

Example 3 Find the centroid of the unit-density, dart-shaped region bounded by the y-axis and the curves $y = e^{-x} - \frac{1}{2}$ and $y = \frac{1}{2} - e^{-x}$.

$m = \iint_R dA$, $M_y = \iint_R x\,dA$, $M_x = \iint_R y\,dA$

$= \int_0^2 \int_{\frac{1}{2} - e^{-x}}^{e^{-x} - \frac{1}{2}}$

Figure 16.71



Example 4 Find the center of mass of the rectangular plate $R = \{(x,y) : -1 \le x \le 1,\ 0 \le y \le 1\}$ with a density of $\rho(x,y) = 2 - y$.

$$m = \int_{-1}^{1} dx \int_0^1 (2-y)\,dy = \left(\int_{-1}^{1} dx\right)\left(\int_0^1 (2-y)\,dy\right)$$

Figure 16.72

$$M_y = \left(\int_{-1}^{1} x\,dx\right)\left(\int_0^1 (2-y)\,dy\right)$$

$$M_x = \int_{-1}^{1} dx \int_0^1 y(2-y)\,dy$$

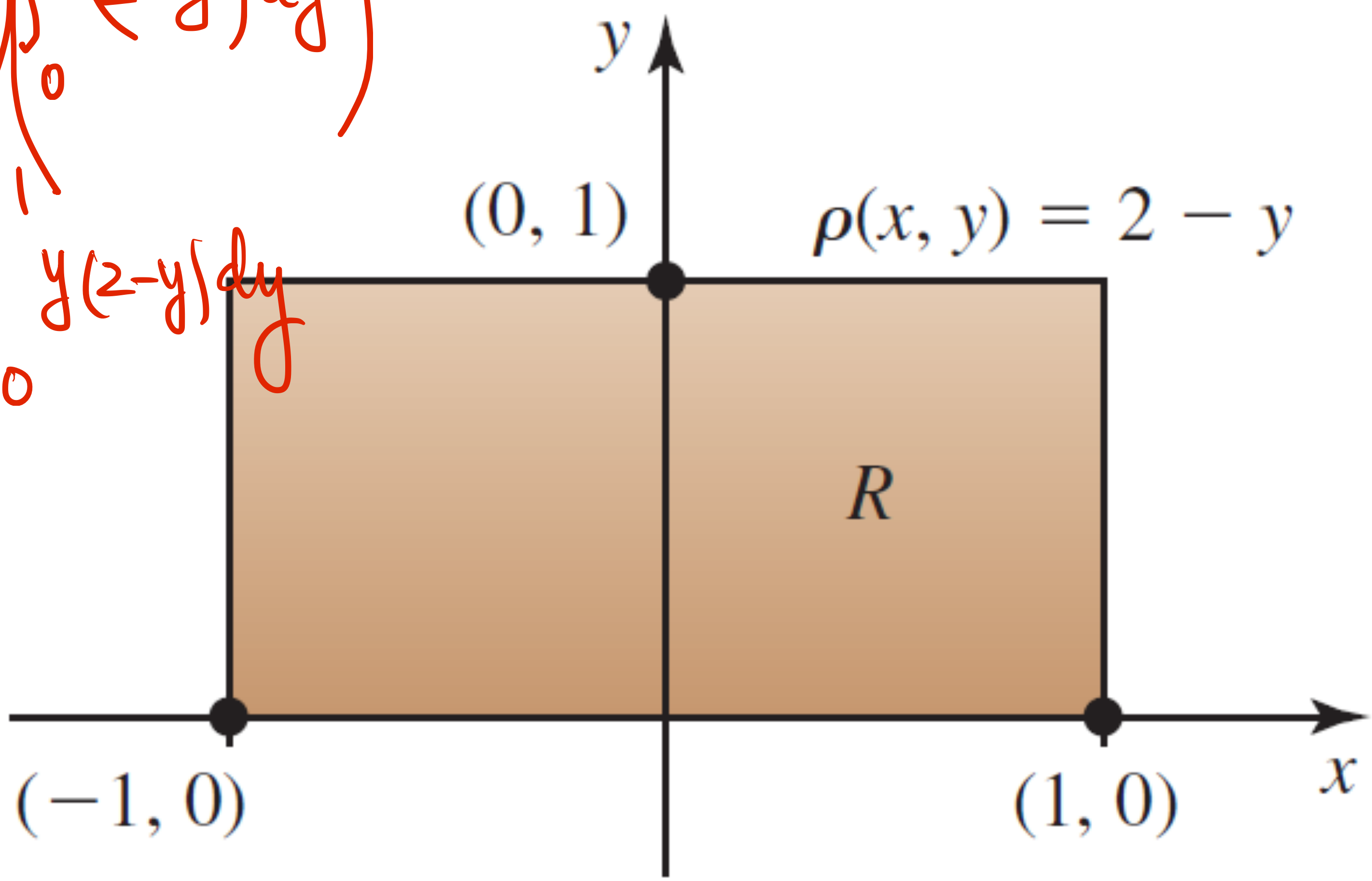
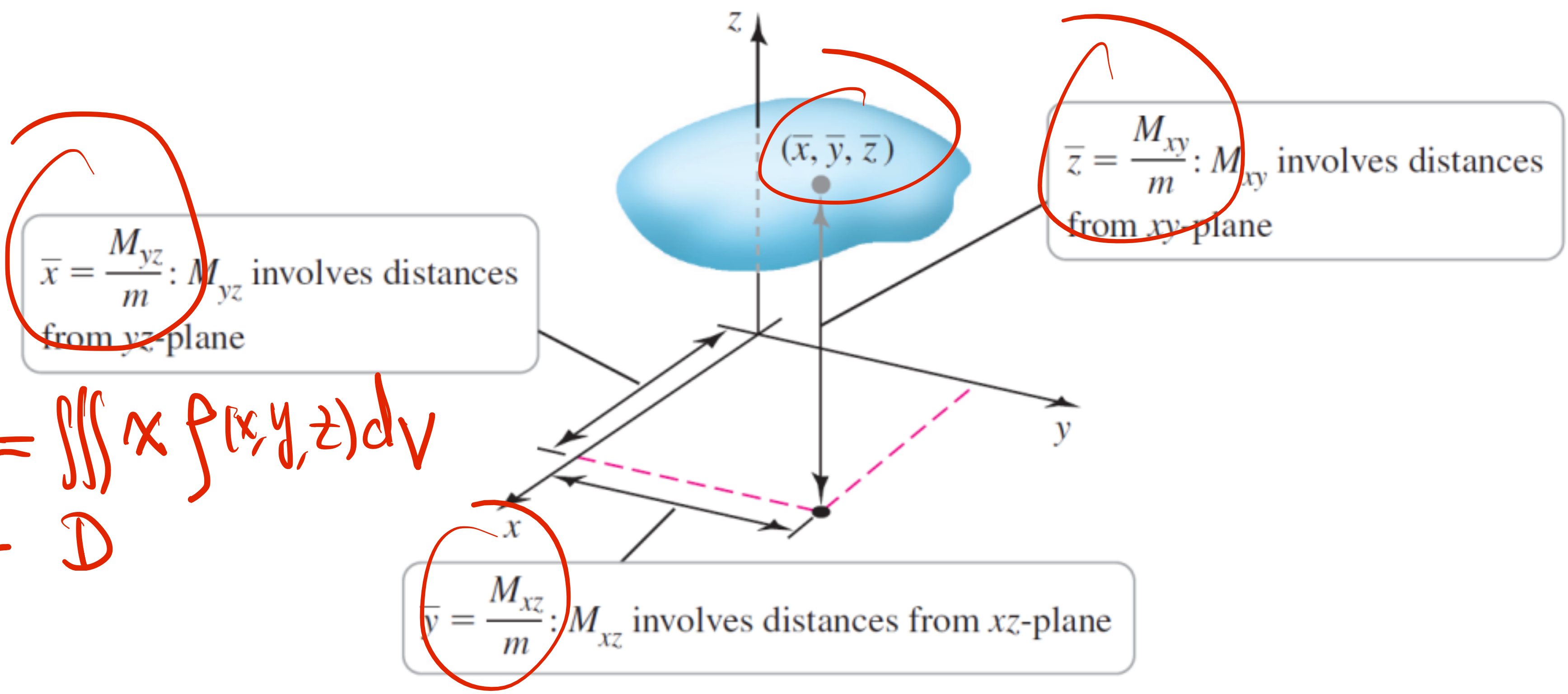


Figure 16.73



$$M_{yz} = \iiint_D x\,\rho(x,y,z)\,dV$$

Definition Center of Mass in Three Dimensions

Let ρ be an integrable density function on a closed bounded region *D* in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\rho(x,y,z)\,dV, \quad \bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\rho(x,y,z)\,dV, \text{ and}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\rho(x,y,z)\,dV,$$

where $m = \iiint_D \rho(x,y,z)\,dV$ is the mass, and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.

Example 5 Find the center of mass of the constant-density solid cone D bounded by the surface $z = 4 - \sqrt{x^2+y^2}$ and $z = 0$.

Figure 16.74

$$m = \iiint_D dV = \iint_R \left(\int_0^{4-\sqrt{x^2+y^2}} dz \right) dA = \iint_R \left(4 - \sqrt{x^2+y^2}\right) dA$$

$$= \int_0^4 dr \int_0^{2\pi} (4-r)\, r\, d\theta = 2\pi \int_0^4 r(4-r)\, dr$$

R: $\begin{cases} z = 4 - \sqrt{x^2+y^2} \\ z = 0 \end{cases} \Rightarrow \sqrt{x^2+y^2} = 4 \Rightarrow x^2+y^2 = 4^2 \Rightarrow r = 4$

$$M_{yz} = \iiint_D x\, dV = \iint_R \left(\int_0^{4-\sqrt{x^2+y^2}} x\, dz \right) dA$$

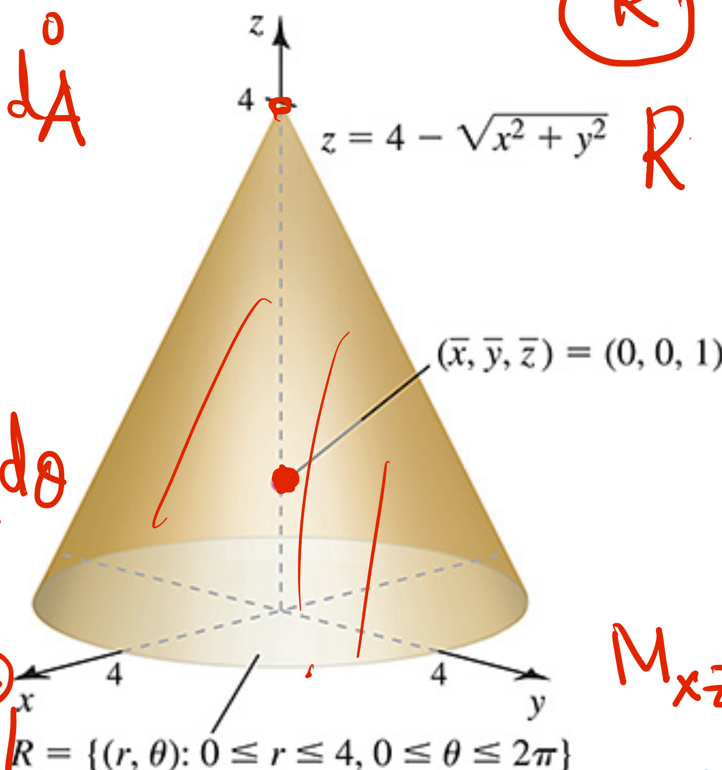
$$= \iint_R x\left(4 - \sqrt{x^2+y^2}\right) dA$$

$$= \int_0^4 dr \int_0^{2\pi} r\cos\theta\,(4-r)\, r\, d\theta$$

$$= \int_0^4 (4r^2 - r^3)\, dr \int_0^{2\pi} \cos\theta\, d\theta$$

$$\sin\theta \Big|_0^{2\pi} = 0$$

$$= 0$$



$$M_{xz} = \iiint_D y\, dV = 0$$

$$M_{xy} = \iiint_D z\, dz = \iint_R \frac{1}{2}(4-r)^2\, dA$$

$$= \int_0^4 \frac{1}{2} r(4-r)^2\, dr \int_0^{2\pi} d\theta$$

Example 6 Find the center of mass of the interior of the hemisphere D of radius a with its base on the xy-plane. The density is $f(\rho, \varphi, \theta) = 2 - \rho/a$.

Figure 16.75

$$m = \iiint_D \left(2 - \frac{\rho}{a}\right) dV = \int_0^a \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left(2 - \frac{\rho}{a}\right) \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$\bar{x}$$

$$M_{yz} = \int_0^a \rho\left(2 - \frac{\rho}{a}\right)\rho^2 d\rho \int_0^{\frac{\pi}{2}} \sin^2\varphi \, d\varphi \int_0^{2\pi} \cos\theta \, d\theta = 0$$

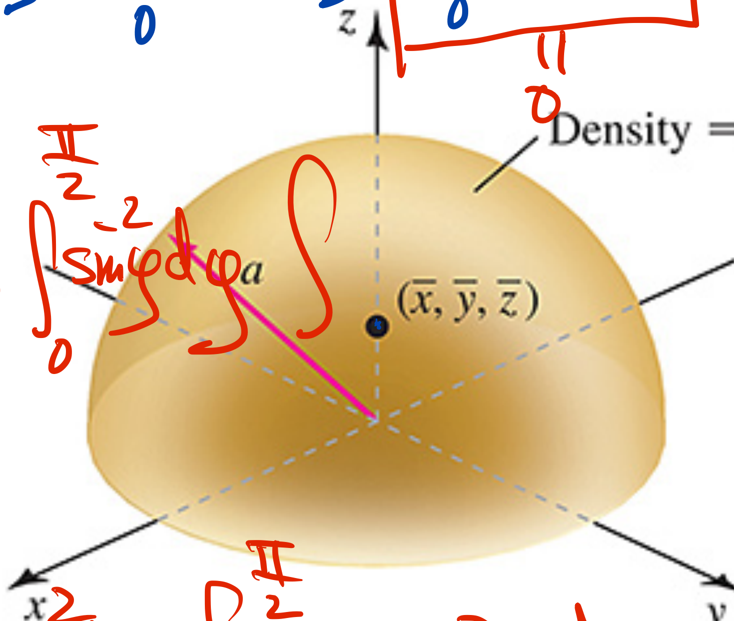
$$\int_0^{2\pi} \cos\theta \, d\theta = 0$$

$$x = \rho \sin\varphi \cos\theta$$

$$M_{xz} = \int_0^a \rho\left(2 - \frac{\rho}{a}\right)\rho^2 d\rho \int_0^{\frac{\pi}{2}} \sin^2\varphi \, d\varphi \int_0^{2\pi} \sin\theta \, d\theta = 0$$

$$\int_0^{2\pi} \sin\theta \, d\theta = 0$$

$$y = \rho \sin\varphi \sin\theta$$

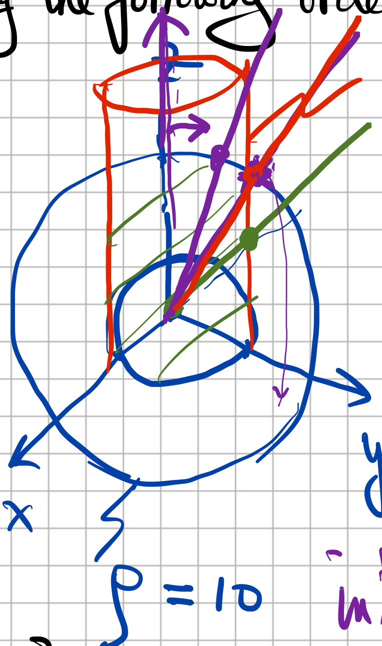


$$M_{xy} = \int_0^a \rho\left(2 - \frac{\rho}{a}\right)\rho^2 d\rho \int_0^{\frac{\pi}{2}} \cos\varphi \sin\varphi \, d\varphi \int_0^{2\pi} d\theta$$

$$\cos\varphi \sin\varphi = \frac{1}{2}\sin 2\varphi$$

$$z = \rho\cos\varphi$$

#9 Let D be the region bounded below by the plane $z=0$, above by the sphere $x^2+y^2+z^2=100$, and on the sides by the cylinder $x^2+y^2=25$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration: (a) $d\rho\, d\varphi\, d\theta$, (b) $d\varphi\, d\rho\, d\theta$.



$r=5 \Leftrightarrow \rho \sin\varphi = 5$

$0 \le \rho \le \dfrac{5}{\sin\varphi}$ $\qquad$ $0 \le \rho \le 10$

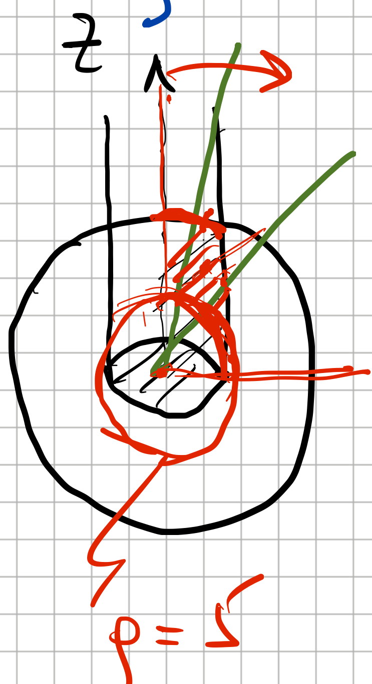
$\dfrac{\pi}{6} \le \varphi \le \dfrac{\pi}{2}$ $\qquad$ $0 \le \varphi \le \dfrac{\pi}{6}$

$0 \le \theta \le 2\pi$

intersection

$\rho = 10$, $\rho\sin\varphi = 5 \Rightarrow \sin\varphi = \dfrac{1}{2}$

$\varphi = \dfrac{\pi}{6}$



$5 \le \rho \le 10$ $\qquad$ $0 \le \rho \le 5$

$0 \le \varphi \le \sin^{-1}\left(\dfrac{5}{\rho}\right)$ $\qquad$ $0 \le \varphi \le \dfrac{\pi}{2}$

$\rho\sin\varphi = 5$