Chapter 17 Vector Calculus (8 sections, 13 lectures)

§17.1 Vector Fields

- vector fields in two dimensions

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle = f(x,y)\,\vec{i} + g(x,y)\,\vec{j}$$

Ex. 1

- radial vector field in R^2

$$\vec{F}(x,y) = f(x,y)\,\vec{r}, \text{ where } \vec{r} = \langle x, y \rangle, \text{ e.g., } \vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$$

Ex. 2 Let $C: x^2+y^2=a^2$, where $a>0$.

(a) Show that at each pt of C, $\vec{F} = \frac{\vec{r}}{|\vec{r}|}$ is orthogonal to the line tangent to C.

(b) Show that rotation field $\vec{G}(x,y) = \frac{\langle -y, x \rangle}{\sqrt{x^2+y^2}}$ is parallel to the line tangent to C.

- vector fields in three dimensions

$$\vec{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle = f\,\vec{i} + g\,\vec{j} + h\,\vec{k}$$

$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$ — radial vector fields.

$\vec{F} = \nabla \varphi$ — gradient field, φ – potential function of $\vec{F}$

$\varphi = k$–const. — equi-potential curves

flow curves (streamlines) $\perp$ equi-potential curves

§17.2 Line Integrals ⟨ scalar function / vector field

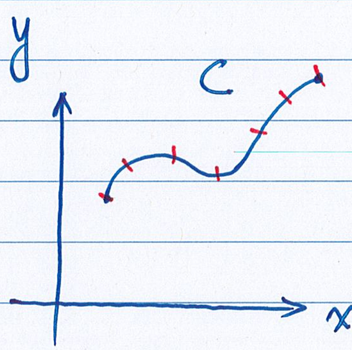
- $$\int_C f\,ds \quad \text{and} \quad \int_C \vec{F}\cdot d\vec{s} = \int_C \left(\vec{F}\cdot\vec{T}\right) ds$$

C: a smooth curve parametrized by $\vec{r}(t) = \langle x(t), y(t)\rangle$, $\forall t \in [a, b]$

f: a function defined on C

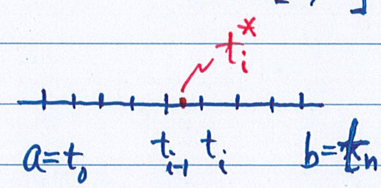
$\vec{F}$: a vector field defined on C, $\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ — unit vector tangent to C

- <u>curve</u>



<u>parametrization</u> $\vec{r}(t) = \langle x(t), y(t)\rangle$

$t \in [a, b]$



uniform partition: $a = t_0 < t_1 < \cdots < t_n = b$

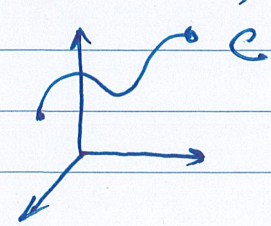
$t_i = a + i\Delta t$, $\Delta t = \dfrac{b-a}{n}$.

- <u>line integrals</u>

$$\int_C f\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\vec{r}(t_i^*)\right)\Delta s_i$$

$$= \int_a^b f(\vec{r}(t))\,\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

- <u>line integrals in R^3</u> $\quad C: \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $t \in [a, b]$



$$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))\,|\vec{r}\,'(t)|\,dt$$

Ex. 1, 2, 3

- <u>line integrals of a vector field $\vec{F} = \langle f, g, h \rangle$</u>

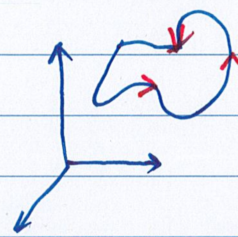
$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C \left(\vec{F} \cdot \vec{T}\right) ds} = \int_a^b \vec{F}\left(\vec{r}(t)\right) \cdot \underbrace{\frac{\vec{r}'(t)}{|\vec{r}'(t)|}}_{\vec{T}(t)} \underbrace{|\vec{r}'(t)|\, dt}_{ds}$$

$$\boxed{= \int_a^b \vec{F}\left(\vec{r}(t)\right) \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}}$$

$$\boxed{= \int_a^b f\, dx + g\, dy + h\, dz}$$

Ex. 4, 5.

- <u>circulation of $\vec{F}$</u> $\qquad \int_C \left(\vec{F} \cdot \vec{T}\right) ds$, $\quad C$: a closed smooth oriented curve



Ex. 6, 7

- <u>flux of $\vec{F} = \langle f, g \rangle$ across C</u>: $\vec{r}(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$

$$\int_C \left(\vec{F} \cdot \vec{n}\right) ds = \int_a^b \left(f\left(\vec{r}(t)\right) y'(t) - g\left(\vec{r}(t)\right) x'(t) \right) dt$$

<u>Ex. 8</u>

§17.3 Conservative Vector Fields

- <u>Types of Curves and Regions</u>

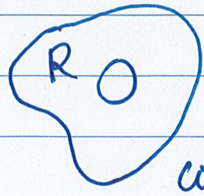
<u>simple and closed curve</u>

C: $\vec{r}(t)$, $t \in [a,b]$

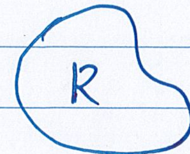
simple: $\vec{r}(t_1) \neq \vec{r}(t_2)$ if $t_1 \neq t_2$

closed: $\vec{r}(a) = \vec{r}(b)$

connected and simply connected regions



connected



simply connected

- **Conservative Vector Field**

$\vec{F}$ is conservative on a region $\iff \exists\, \varphi$ such that $\vec{F} = \nabla\varphi$.

$$\vec{F} = \langle f, g, h \rangle = \nabla\varphi = \langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \rangle .$$

- **the curl**

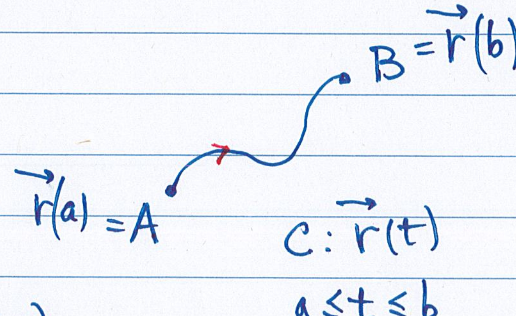
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} = \langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \rangle$$

- **test for conservative vector field** (Thrm 17.3)

$\vec{F}$ is conservative $\iff$ $\nabla \times \vec{F} = \vec{0}$

Ex. 1, 2

- **fundamental theorem for line integrals**



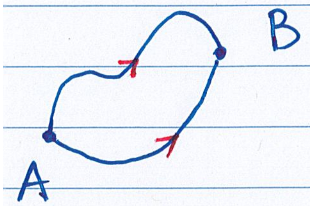
$$\int_C \nabla\varphi \cdot d\vec{r} = \varphi(B) - \varphi(A)$$

Proof $\displaystyle\int_C \nabla\varphi \cdot d\vec{r} = \int_a^b \nabla\varphi \cdot \vec{r}\,'(t)\, dt = \int_a^b \frac{d\varphi(\vec{r}(t))}{dt}\, dt$

$$= \varphi(\vec{r}(t)) \Big|_{t=a}^{t=b} .$$

- <u>independence of path</u>: $\int_C \vec{F}\cdot d\vec{r}$ is independent of path

$$\iff \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} \quad \forall \text{ curves } C_1 \,\&\, C_2 \text{ having the same initial and terminal points}$$



<u>Thrm 17.5</u> $\int_C \vec{F}\cdot d\vec{r}$ is indep of path $\Longrightarrow \vec{F} = \nabla\varphi$.

Ex. 3, 4

- <u>line integrals on closed curves</u>

<u>Thrm 17.6</u> $\vec{F}$ is conservative $\iff \oint_C \vec{F}\cdot d\vec{r} = 0$ $\forall$ simply close oriented piecewise-smooth curve.

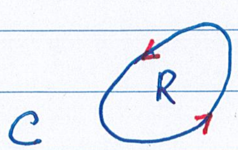
Ex. 5

<u>§17.4 Green's Theorem</u>

- <u>fundamental thrms</u>

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a) \quad \text{and} \quad \int_C \nabla\varphi\cdot d\vec{r} = \varphi(B) - \varphi(A)$$

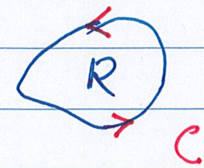
- <u>Gree's Thrm</u> $\vec{F} = \langle f, g\rangle$, $\nabla\times\vec{F} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$ ~ two-dim curl.



$$\iint_R \nabla\times\vec{F}\, dA = \oint_C \vec{F}\cdot d\vec{r}$$

- $\vec{F} = \langle f, g \rangle$ is irrotational $\Longleftrightarrow$ $\nabla \times \vec{F} = 0$.

Ex. 1



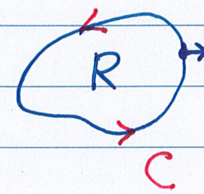
- area of a region $R \subset \mathbb{R}^2$

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (-y\,dx + x\,dy)$$

Ex. 2

- <u>Green's Thm</u> $\vec{F} = \langle f, g \rangle$, $\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ ~ divergence

$$\iint_R \nabla \cdot \vec{F}\, dA = \oint_C (\vec{F} \cdot \vec{n})\, ds$$



$\vec{n}$ unit normal

Ex. 3, 4.

- <u>general regions</u>

Ex. 5, 6.

<u>§17.5 Divergence and Curl</u>

- <u>the divergence</u> $\vec{F} = \langle f, g, h \rangle$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Ex. 1, 2, 3

$$\nabla \cdot \left(\frac{\vec{r}}{|\vec{r}|^p} \right) = \frac{3-p}{|\vec{r}|^p} \quad \text{where } \vec{r} = \langle x, y, z \rangle$$

- <u>the curl</u> $\vec{F} = \langle f, g, h \rangle$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

- <u>rotation vector field</u> $\vec{F} = \vec{a} \times \vec{r}$, $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\neq \vec{0}$ const

$\vec{r} = \langle x, y, z \rangle$

$\nabla \cdot \vec{F} = 0$, $\nabla \times \vec{F} = 2\vec{a}$, angular speed $\omega = |\vec{a}| = \frac{1}{2} |\nabla \times \vec{F}|$

Ex. 4

- <u>working with divergence and curl</u>

$$\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$$

$$\nabla \cdot (c\vec{F}) = c\,(\nabla \cdot \vec{F})$$

$$\nabla \cdot (u\vec{F}) = \nabla u \cdot \vec{F} + u\, \nabla \cdot \vec{F}$$

<u>identities</u>

$$\begin{cases} \nabla \times \nabla \varphi = 0 \\ \nabla \cdot (\nabla \times \vec{F}) = 0 \end{cases}$$

<u>Ex. 5</u>

<u>§17.6 Surface Integrals</u>

parametrization of surface
scalar function
vector field

(3 lectures)

- <u>representations of surfaces</u>

graph: $z = f(x, y)$, level surface: $f(x, y, z) = 0$

parametrization: $\vec{r}(u, v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, $(u, v) \in R$.

Examples

cylinders $\{(x,y,z):\ x = a\cos\theta,\ y = a\sin\theta,\ 0 \le \theta \le 2\pi,\ 0 \le z \le h\}$

parametric rep. $\vec{r}(u,v) = \langle a\cos u, a\sin u, v\rangle$

$0 \le u \le 2\pi,\ 0 \le v \le h.$

cone $\{(r,\theta,z):\ 0 \le r \le a,\ 0 \le \theta \le 2\pi,\ z = \frac{h}{a} r\}$

sphere $\{(\rho,\varphi,\theta):\ \rho = a,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi\}$

Ex. 1

- surface integrals of scalar-valued functions $f(x,y,z)$

$$\iint_S f(x,y,z)\,dS = \iint_R f(\vec{r}(u,v))\,|\vec{t}_u \times \vec{t}_v|\,du\,dv$$

$S:\ \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle,\ (u,v) \in R$

$\vec{t}_u = \frac{\partial \vec{r}}{\partial u},\ \vec{t}_v = \frac{\partial \vec{r}}{\partial v}$

Ex. 2, 3, 4,

$S:\ z = g(x,y),\ (x,y) \in R$

$$\iint_S f\,dS = \iint_R f(x,y,g(x,y))\sqrt{z_x^2 + z_y^2 + 1}\,dx\,dy$$

Ex. 5, 6, Table 17.3

- surface integrals of vector fields $\vec{F} = \langle f, g, h \rangle$

(a) surface $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \; (u,v) \in R$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \left(\vec{F} \cdot \vec{n}\right) dS = \iint_R \vec{F} \cdot \left(\vec{t}_u \times \vec{t}_v\right) dA$$

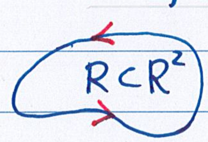
(b) graph $S: z = \overset{s}{\cancel{f}}(x,y), \; (x,y) \in R$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_R \left(-f \frac{\partial z}{\partial x} - g \frac{\partial z}{\partial y} + h\right) dA$$

Ex. 7, 8

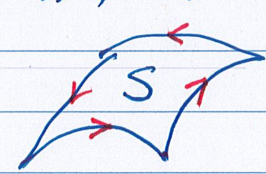
§17.7 Stoke's Theorem

$\vec{F} = \langle f(x,y), g(x,y) \rangle$



$$\iint_R \left(\nabla \times \vec{F}\right) \cdot \vec{k} \, dA = \oint_C \vec{F} \cdot d\vec{r}$$

$\vec{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$



$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} \, dS = \oint_C \vec{F} \cdot d\vec{r}$$

Ex. 1, 2, 3,

- interpreting the curl (Ex. 4)

proof of Stoke's Thrm in a special case $S: \ z = s(x,y)$.

Remarks

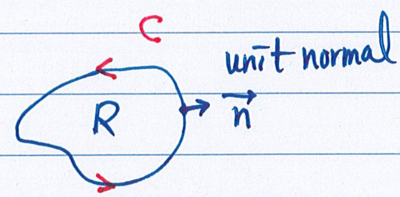
(1) $C = \partial S_1 = \partial S_2$ with the same orientation

$$\iint_{S_1} \nabla \times \vec{F} \cdot \vec{n}_1 \, dS = \iint_{S_2} \nabla \times \vec{F} \cdot \vec{n}_2 \, dS.$$

(2) $\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla \varphi$.

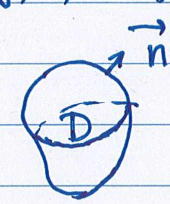
§17.8 Divergence Theorem

$\vec{F}(x,y) = \langle f, g \rangle$



$$\iint_R \nabla \cdot \vec{F} \, dA = \oint_C \vec{F} \cdot \vec{n} \, ds$$

$\vec{F}(x,y,z) = \langle f, g, h \rangle$



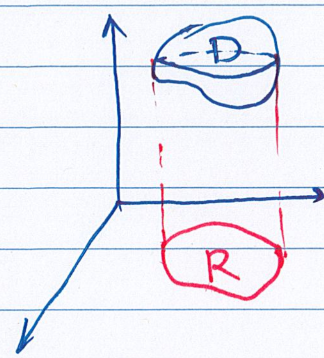
$S = \partial D$

$$\iiint_D \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot \vec{n} \, dS$$

Ex. 1, 2, 3

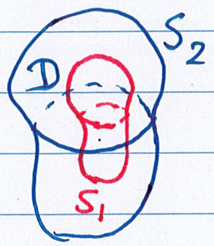
- interpretation of the divergence theorem using mass transport

- <u>proof of the Divergence Thrm</u>



$$\iiint_D \frac{\partial h}{\partial z}\, dV \overset{?}{=} \iint_S h\left(\vec{k}\cdot\vec{n}\right) dS$$

- hollow regions   D — bounded by surfaces S_1 and S_2

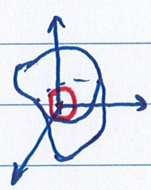


$$\iiint_D \vec{\nabla}\cdot\vec{F}\, dV = \iint_{S_2} \vec{F}\cdot\vec{n}_2\, dS - \iint_{S_1} \vec{F}\cdot\vec{n}_1\, dS$$

Ex. 4

- <u>Gauss' Law</u>   electric field $\vec{E}(x,y,z) = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\vec{r}}{|\vec{r}|^3}$

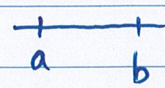
S — surface (closed) enclosed the origin



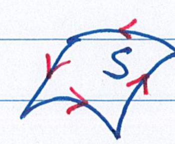
$$\iint_S \vec{E}\cdot\vec{n}\, dS = \frac{Q}{\varepsilon_0}$$

- <u>Final Perspective</u>

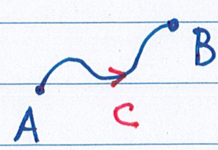
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$



$$\iint_S \left(\vec{\nabla}\times\vec{F}\right)\cdot\vec{n}\, dS = \oint_C \vec{F}\cdot d\vec{r}$$



$$\int_C \nabla f\cdot d\vec{r} = f(B) - f(A)$$



$$\iiint_D \vec{\nabla}\cdot\vec{F}\, dV = \iint_S \vec{F}\cdot\vec{n}\, dS$$