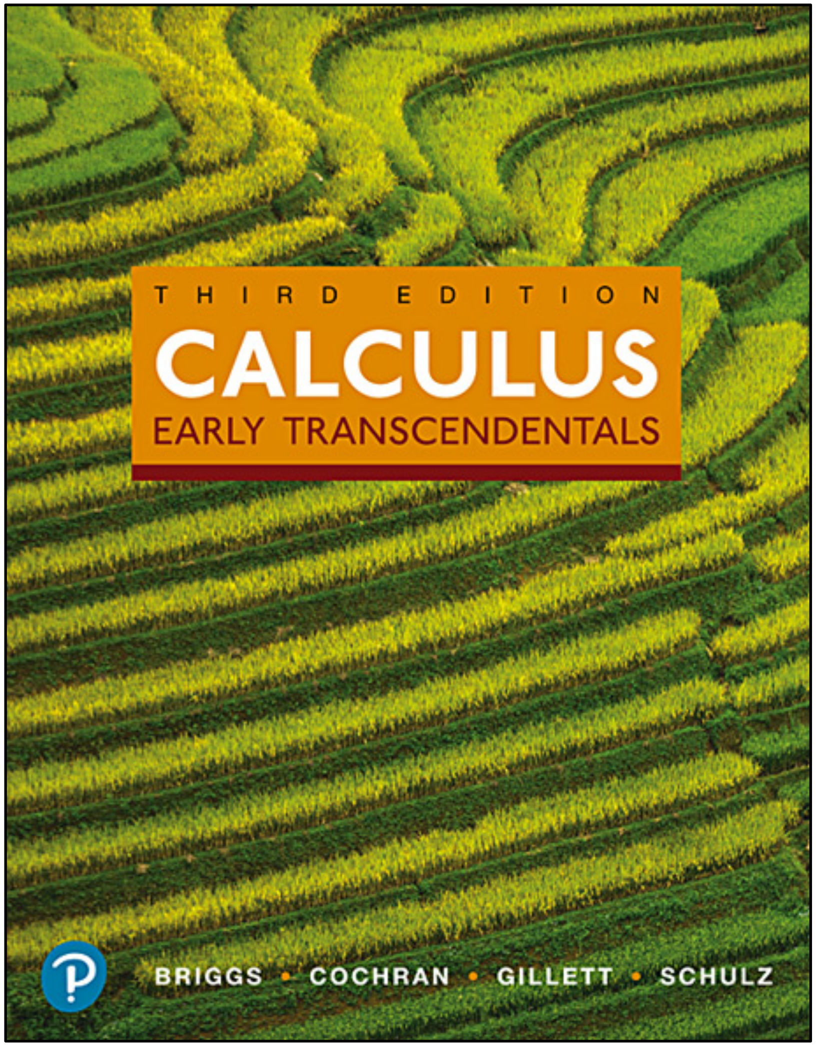
Calculus Early Transcendentals

Third Edition



Chapter 17

Vector Calculus

Lesson	Section	
25	§17.1	Vector Field
26, 27	§17.2	Line Integral < scalar-valued func / vector field
28	§17.3	Conservative VF
29	§17.4	Green's Theorem
30	§17.5	Divergence and Curl
31, 32, 33	§17.6	Surface Integrals
34, 35	§17.7	Stokes' Theorem
36, 37	§17.8	Divergence Theorem

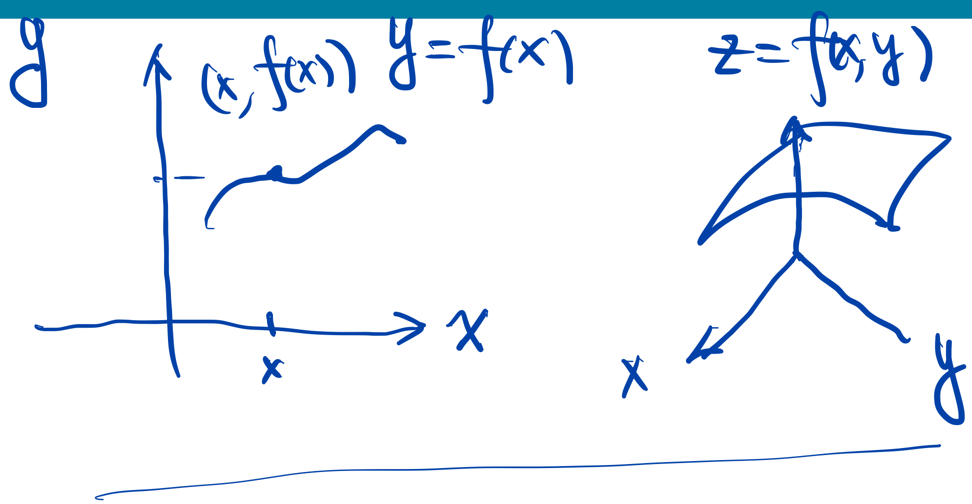
$y = f(x)$ scalar-valued function

$z = f(x, y)$ " "

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ vector-valued function

$\vec{F}(x, y) = \langle f(x,y), g(x,y) \rangle$ vector-valued function

Section 17.1 Vector Fields



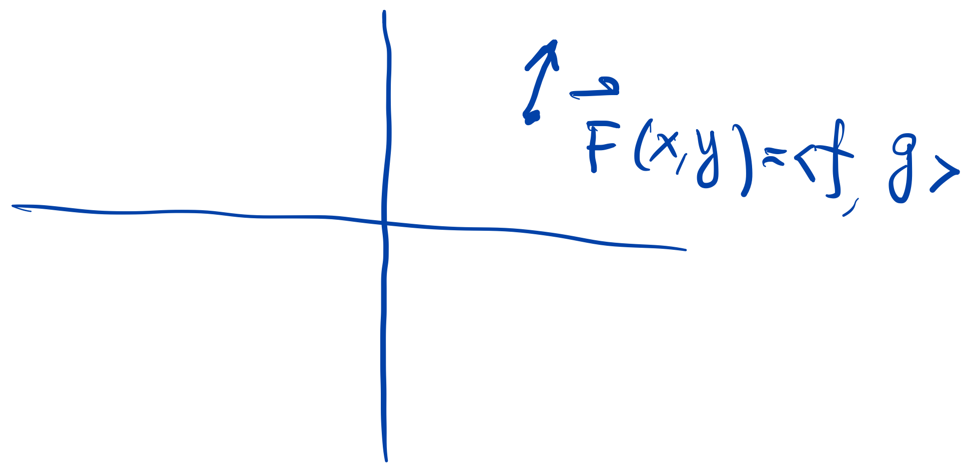


Figure 17.1

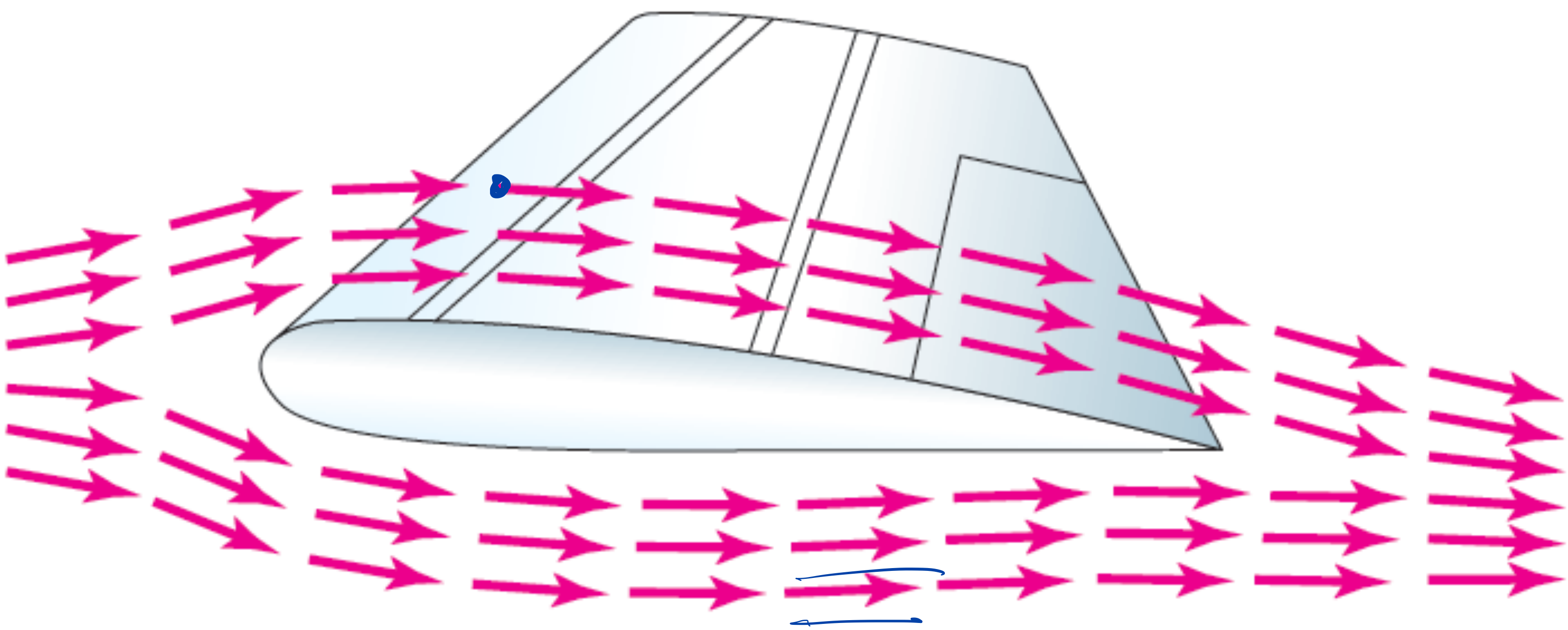


Figure 17.2 (a & b)

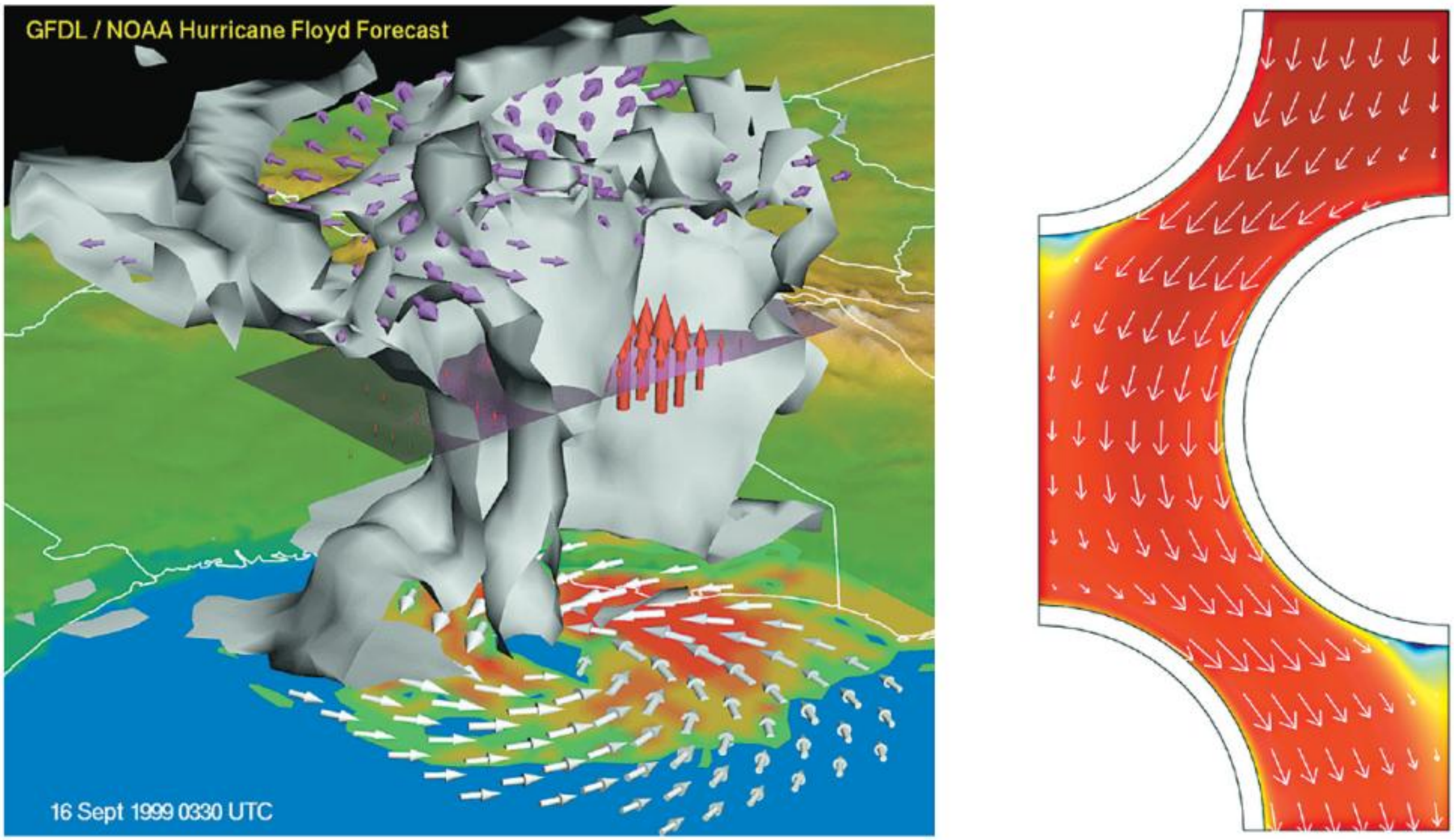
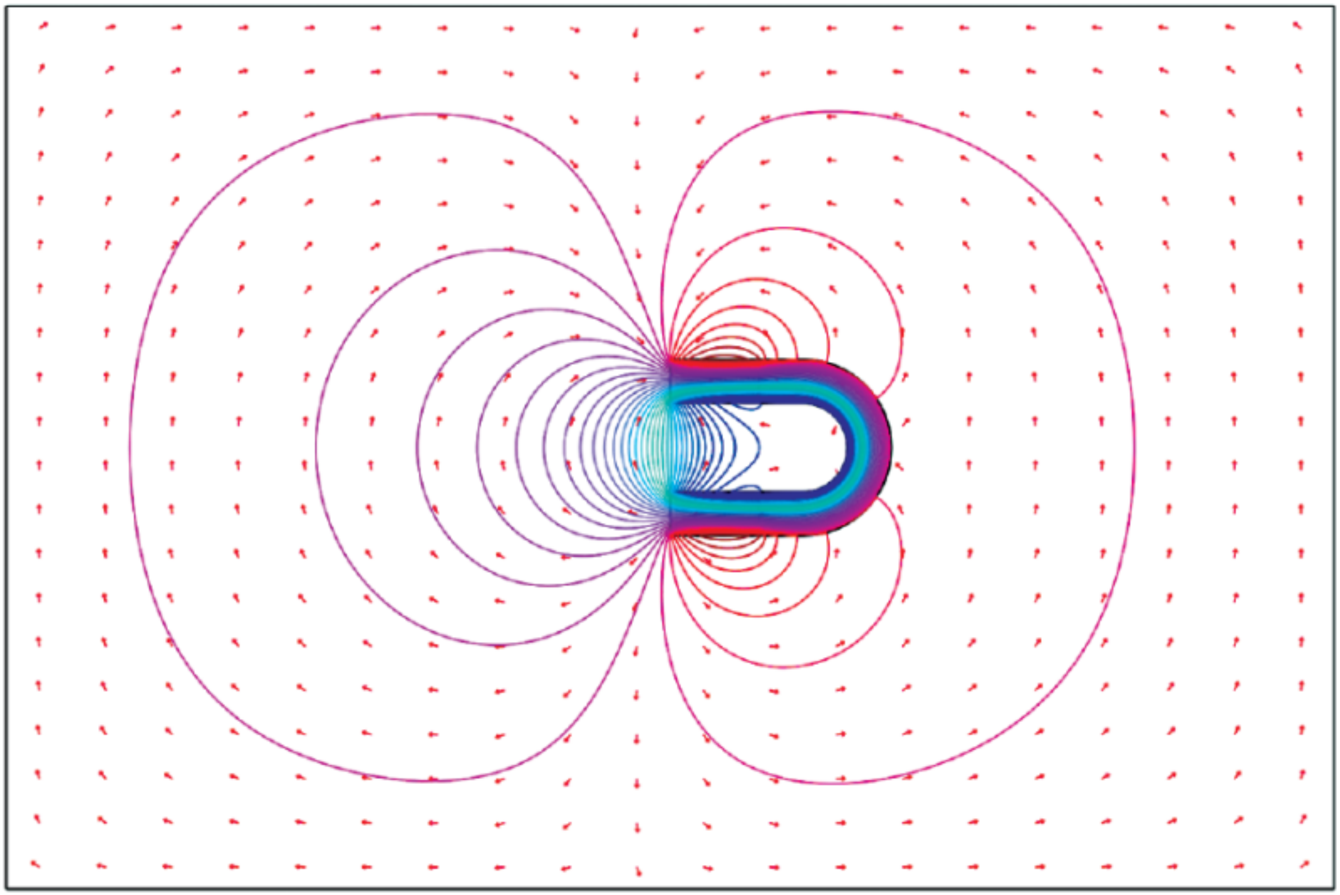


Figure 17.2 (c)



Definition Vector Fields in Two Dimensions

$\vec{F}(x_1, \ldots, x_d) = \langle f_1, \ldots, f_c \rangle \quad d = c$

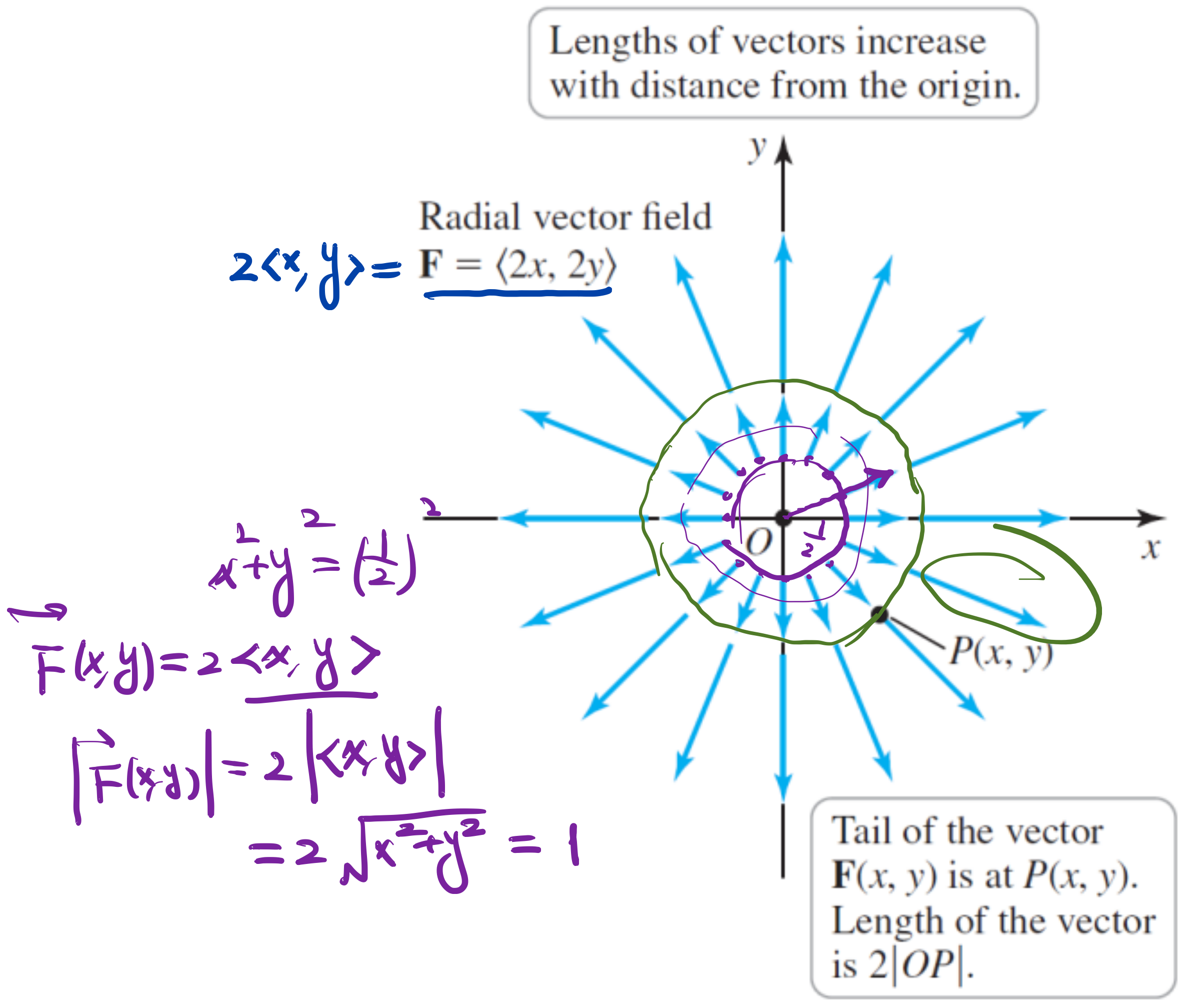
Let *f* and *g* be defined on a region *R* of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function **F** that assigns to each point in *R* a vector $\langle f(x,y), g(x,y) \rangle$. The vector field is written as

$$\mathbf{F}(x,y) = \langle f(x,y), g(x,y) \rangle \text{ or } : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\mathbf{F}(x,y) = f(x,y)\,\mathbf{i} + g(x,y)\,\mathbf{j}.$$

A vector field $\mathbf{F} = \langle f, g \rangle$ is continuous or differentiable on a region *R* of $\mathbb{R}^2$ if *f* and *g* are continuous or differentiable on *R*, respectively.

Figure 17.3



Lengths of vectors increase with distance from the origin.
y
Radial vector field
$\mathbf{F} = \langle 2x, 2y \rangle$
O
x
P(x, y)
Tail of the vector $\mathbf{F}(x, y)$ is at $P(x, y)$. Length of the vector is $2|OP|$.

Example 1 Sketch representative vectors of the following VFs:

(a) $\vec{F}(x,y) = \langle 0, x\rangle$ (a shear field); (b) $\vec{F}(x,y) = \langle 1-y^2, 0\rangle$ for $|y| \le 1$ (channel flow)

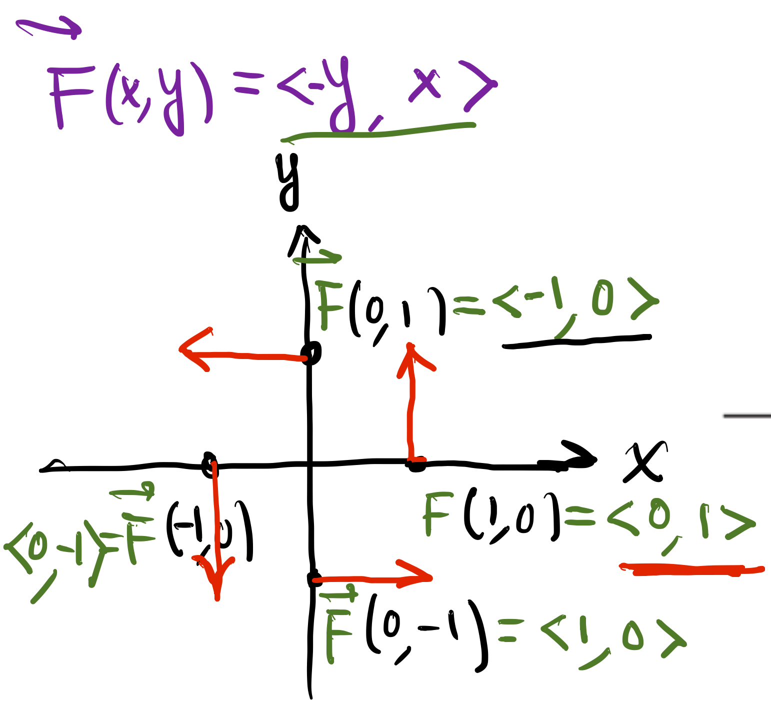
(c) $\vec{F}(x,y) = \langle -y, x\rangle$ (a rotation field)

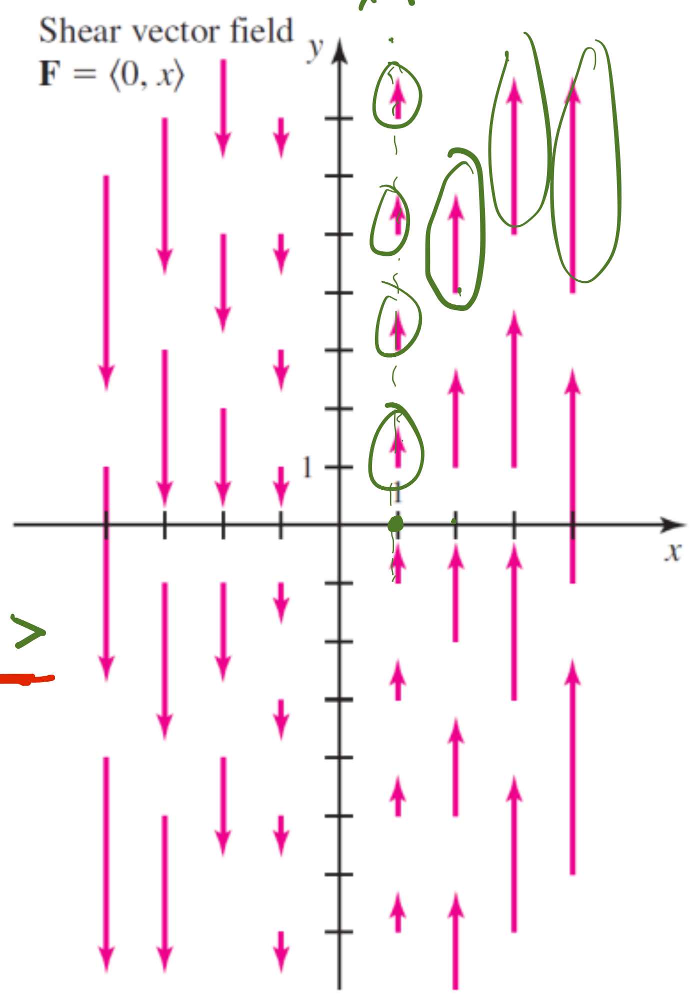
Figure 17.4

$\vec{F} = \langle 0, x\rangle$ $\begin{cases} \uparrow & x>0 \\ \downarrow & x<0 \end{cases}$

$\vec{F}(1,y) = \langle 0,1\rangle$, $x=1$

$\vec{F}(x,y) = \langle 1-y^2, 0\rangle$, $1-y^2 \ge 0$





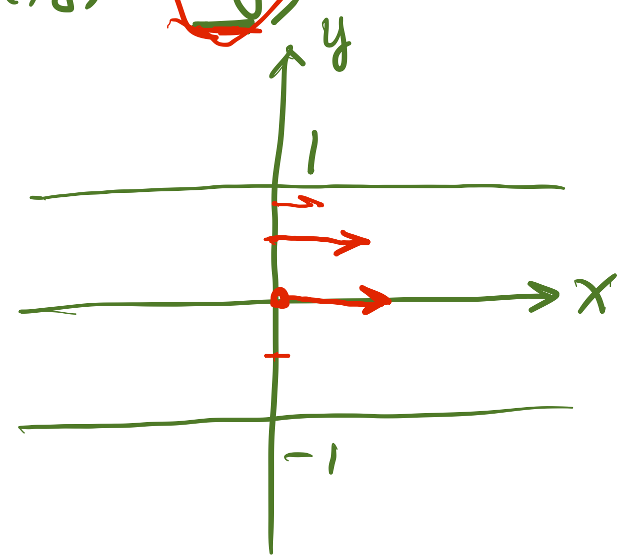


Figure 17.5

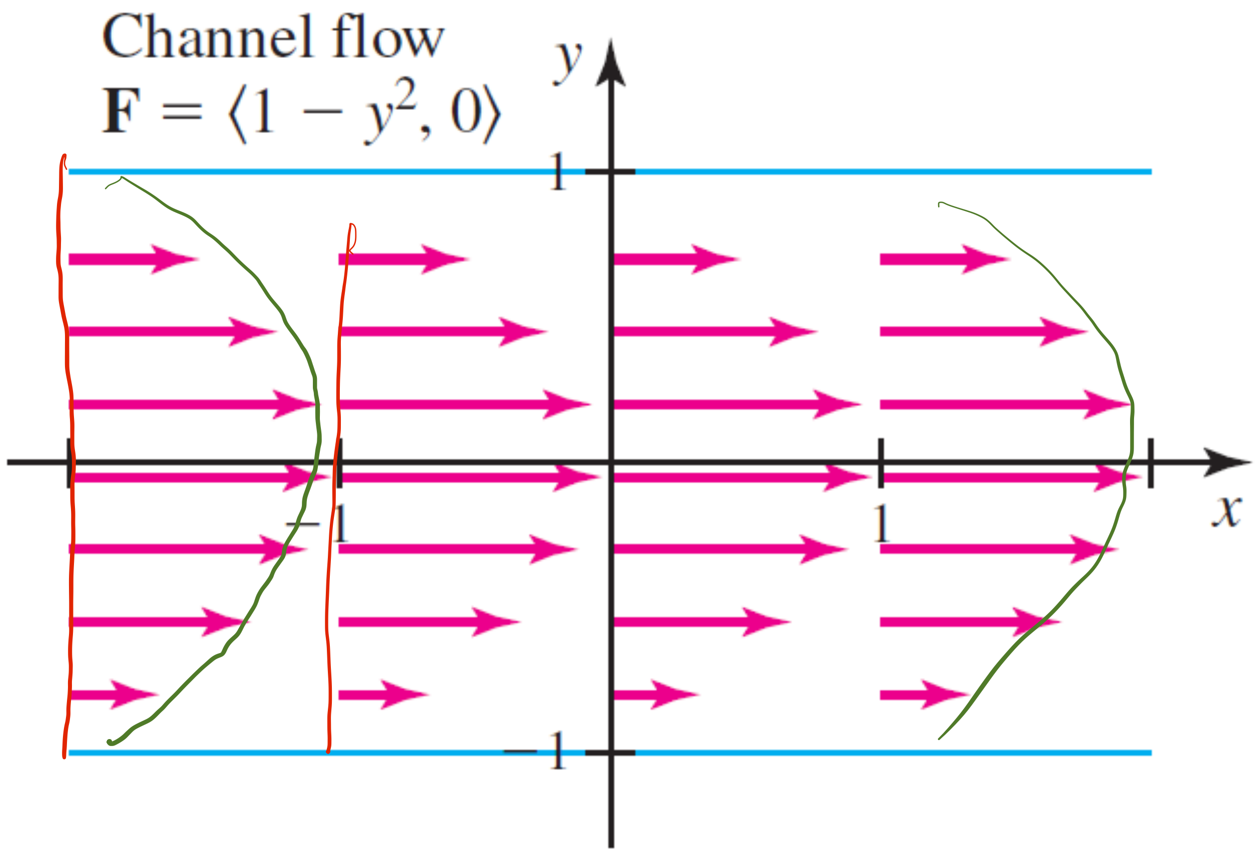


Figure 17.6

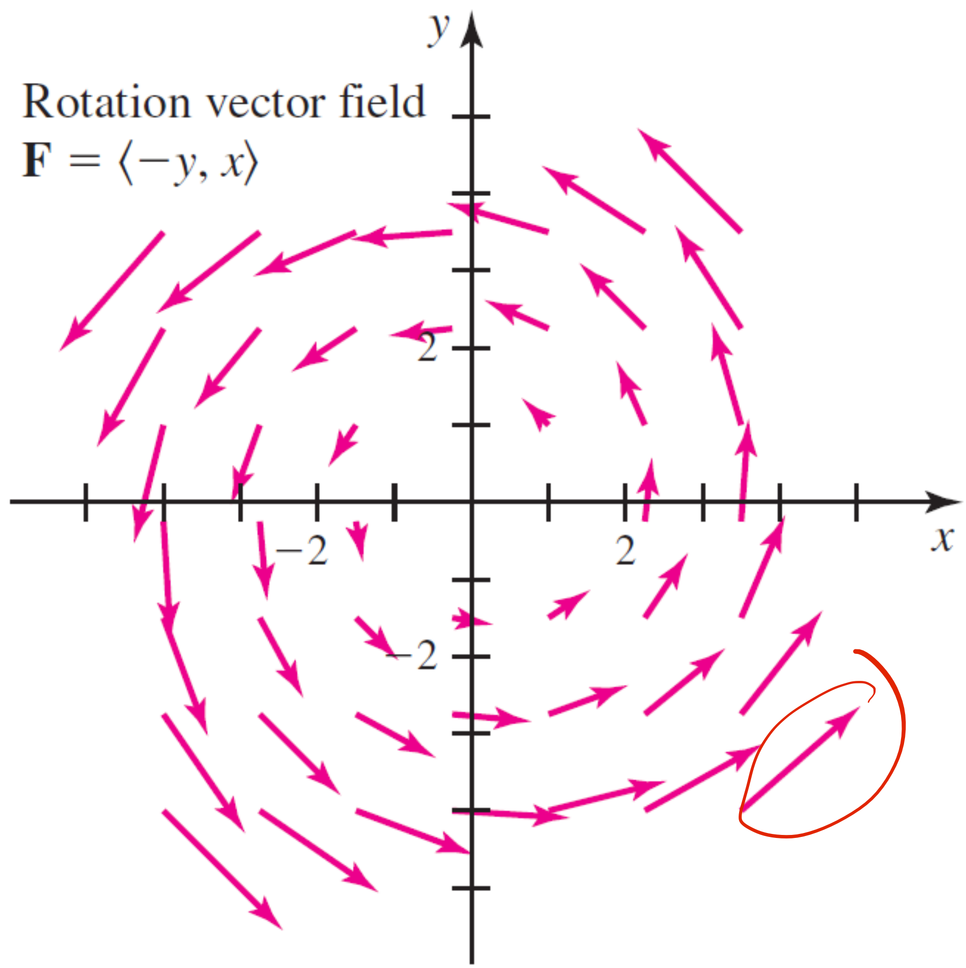
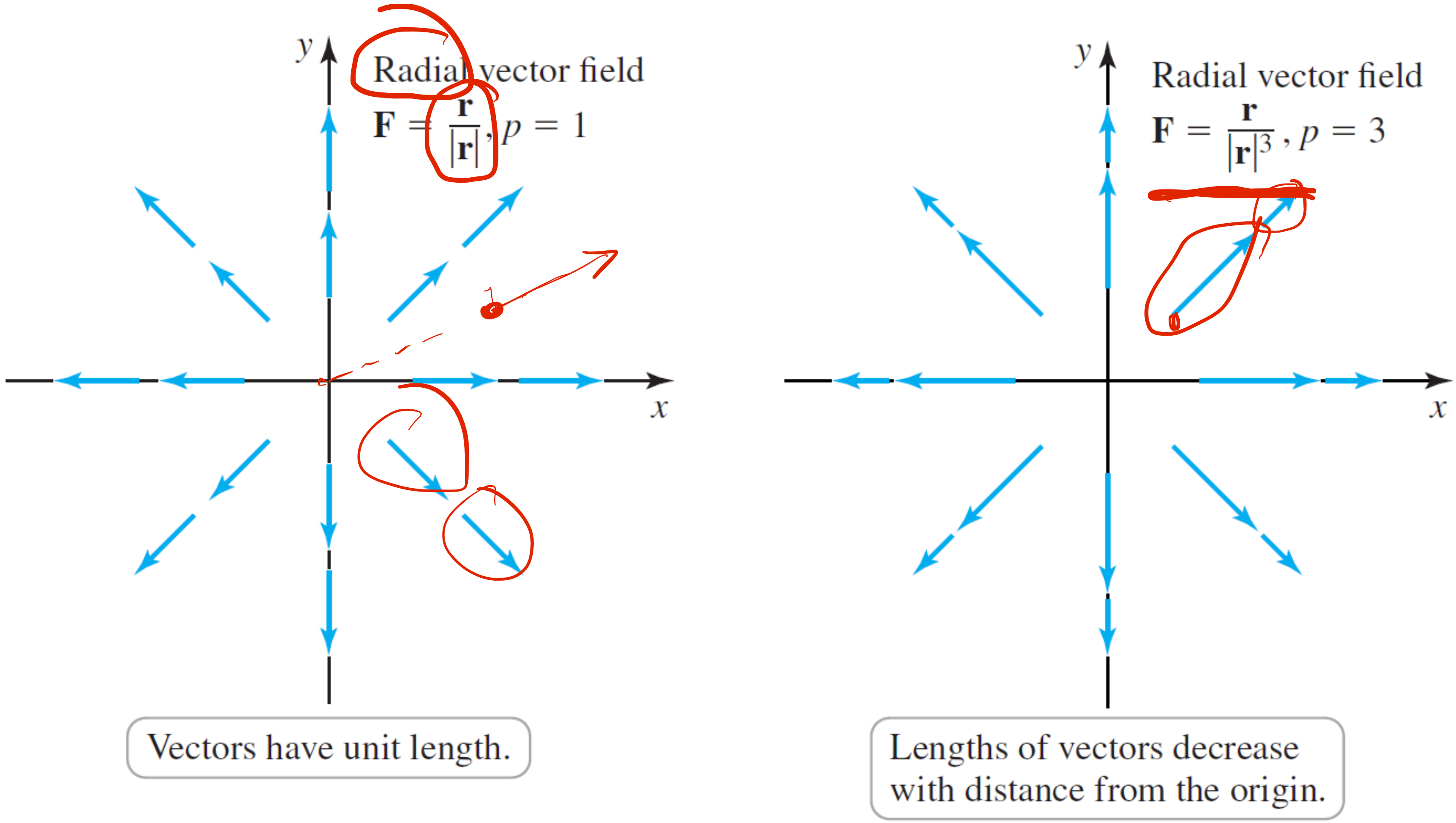


Figure 17.7


Radial vector field
$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|}, p = 1$
y
x
Vectors have unit length.
Radial vector field
$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3}, p = 3$
y
x
Lengths of vectors decrease with distance from the origin.

Definition Radical Vectors Fields in *R* Squared

Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x, y)\,\mathbf{r}$,

where *f* is a scalar-valued function, is a **radial vector field**. Of specific interest are the radial vector fields

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^{p}} = \frac{\langle x, y \rangle}{|\mathbf{r}|^{p}},$$

where *p* is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with a

Magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$.

Example 2 Let C be the circle $x^2+y^2=a^2$, where $a>0$.

(a) Show that at each pt of C, $\vec{F}=\vec{r}/|\vec{r}|$ is orthogonal to the line tangent to C at that pt.

(b) " " " " " " " $\vec{G}=\langle -y, x\rangle/\sqrt{x^2+y^2}$ is parallel " " " " "

Figure 17.8

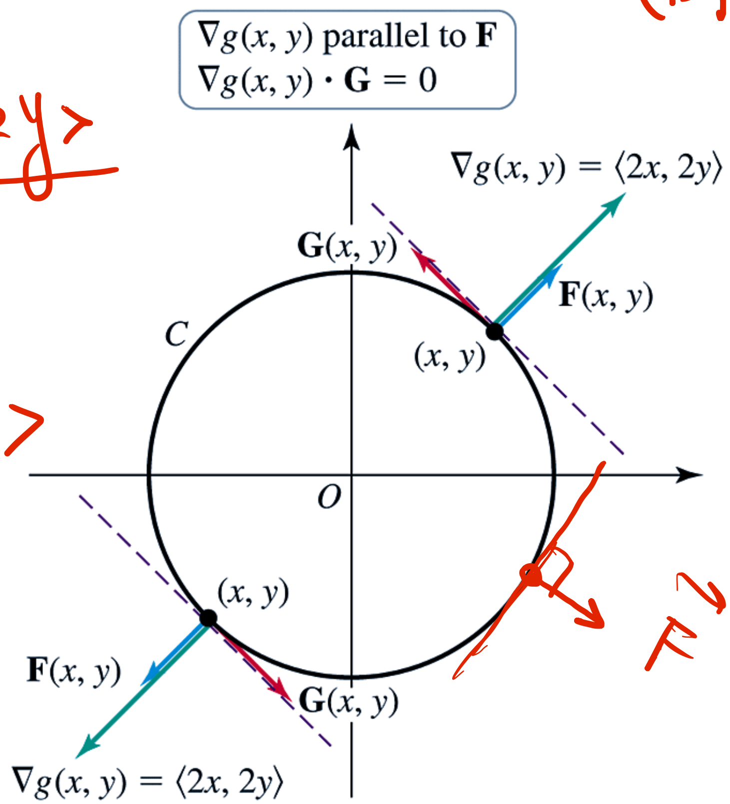
$C: g = x^2+y^2 = a^2$

$\nabla g = \langle g_x, g_y\rangle = \langle 2x, 2y\rangle$

$= 2\langle x, y\rangle$

$\vec{F} = \dfrac{\langle x, y\rangle}{\sqrt{x^2+y^2}} = \dfrac{1}{a}\langle x, y\rangle$

$= \dfrac{1}{2a}\nabla g$



Radial field $\mathbf{F} = \dfrac{\langle x, y\rangle}{\sqrt{x^2+y^2}}$

Rotation field $\mathbf{G} = \dfrac{\langle -y, x\rangle}{\sqrt{x^2+y^2}}$

(b) $\nabla g \perp \vec{G}$

$2\langle x, y\rangle \cdot \dfrac{\langle -y, x\rangle}{\sqrt{x^2+y^2}}$

$= \dfrac{2}{a}(-xy+yx) = 0$

$\Rightarrow \nabla g \perp \vec{G}$

Definition Vector Fields and Radical Vector Fields in *R* Cubed

Let *f*, *g*, and *h* be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function **F** that assigns to each point in D a vector $\langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$. The vector field is written as

$$\mathbf{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle \text{ or}$$

$$\mathbf{F}(x,y,z) = f(x,y,z)\,\mathbf{i} + g(x,y,z)\,\mathbf{j} + h(x,y,z)\,\mathbf{k}.$$

A vector field $\mathbf{F} = \langle f, g, h\rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if *f*, *g* and *h* are continuous or differentiable on *D*, respectively. Of particularly importance are the **radical vector fields**

$$\mathbf{F}(x,y,z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,y,z\rangle}{|\mathbf{r}|^p},$$

where *p* is a real number.

Example 3 Sketch and discuss the following VFs: ≥ 0

(a) $\vec{F} = \langle x, y, e^{-z} \rangle$, for $z \geq 0$;

(b) $\vec{F} = \langle 0, 0, 1 - x^2 - y^2 \rangle$, for $x^2 + y^2 \leq 1$

$= \langle 0, 0, 1 - r^2 \rangle$

$r \leq 1$

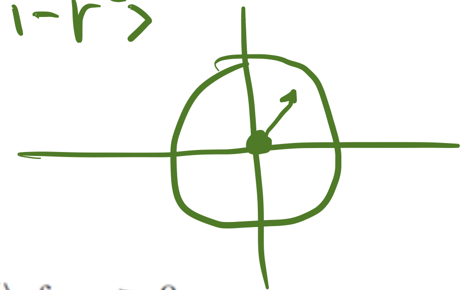
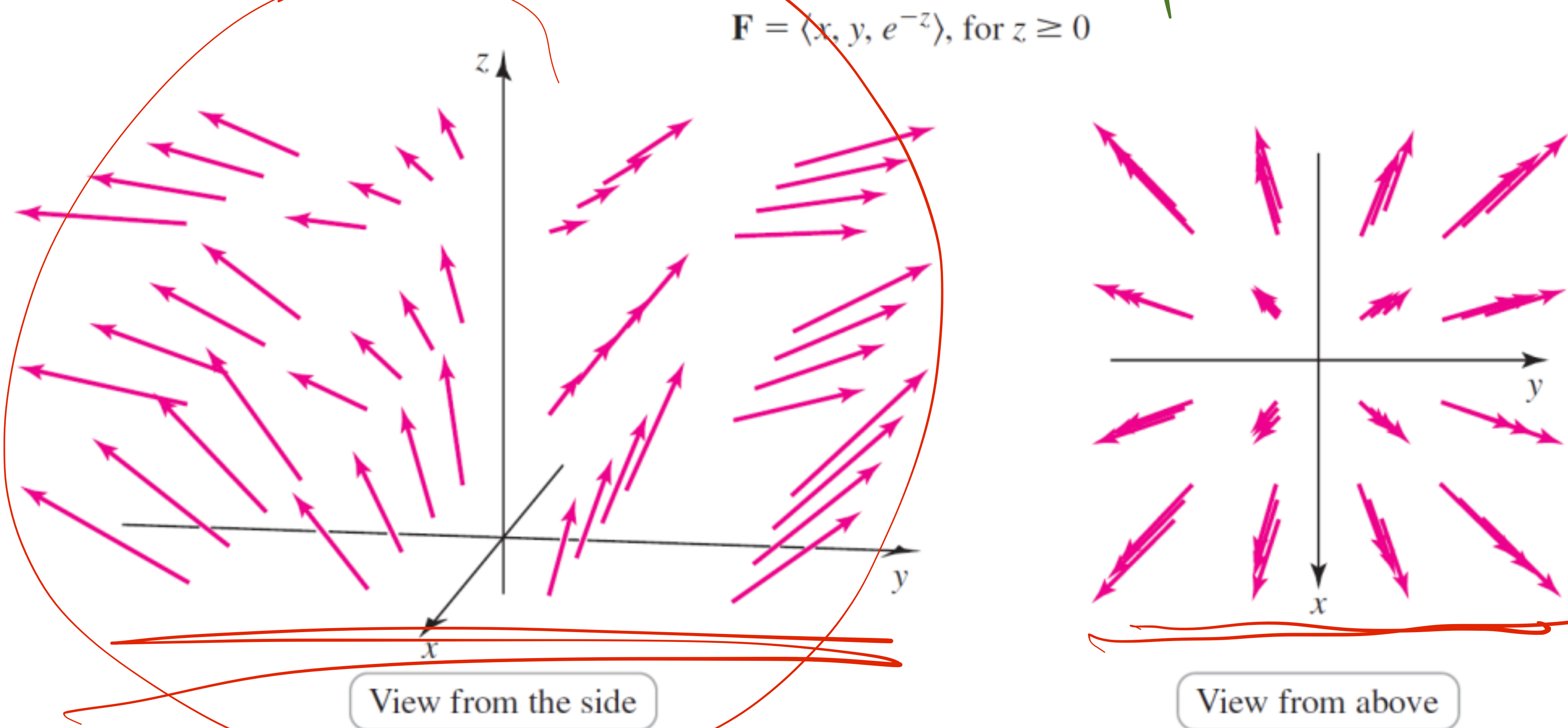
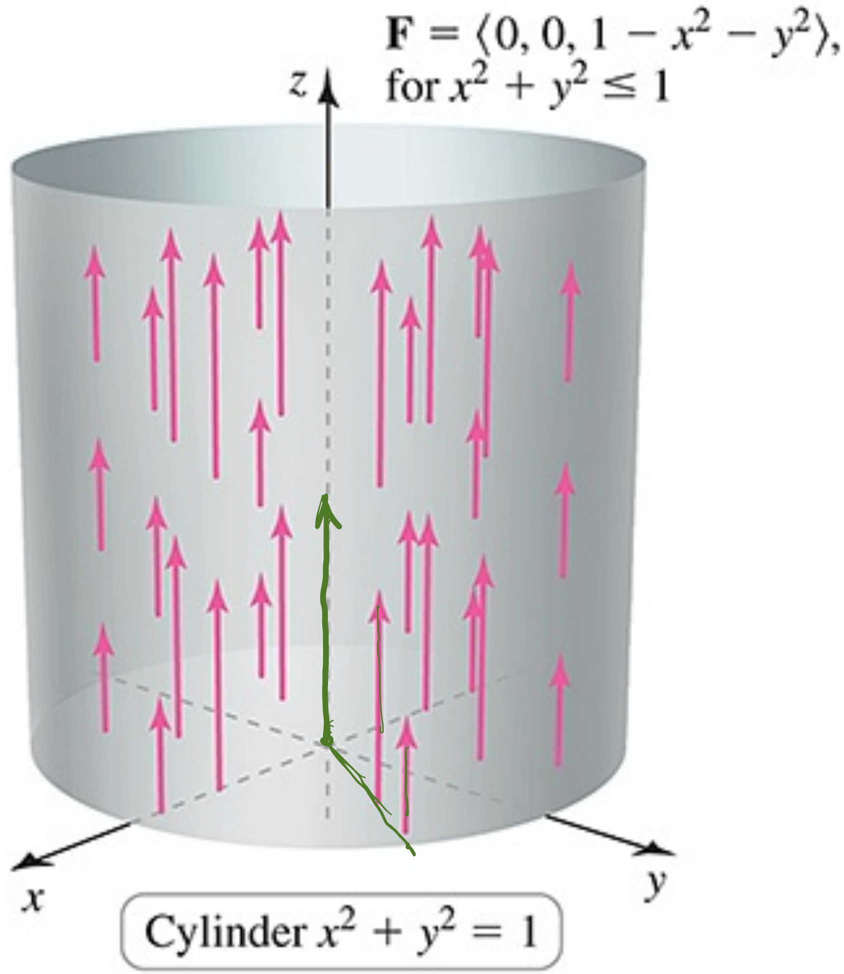


Figure 17.9



Pearson

Figure 17.10

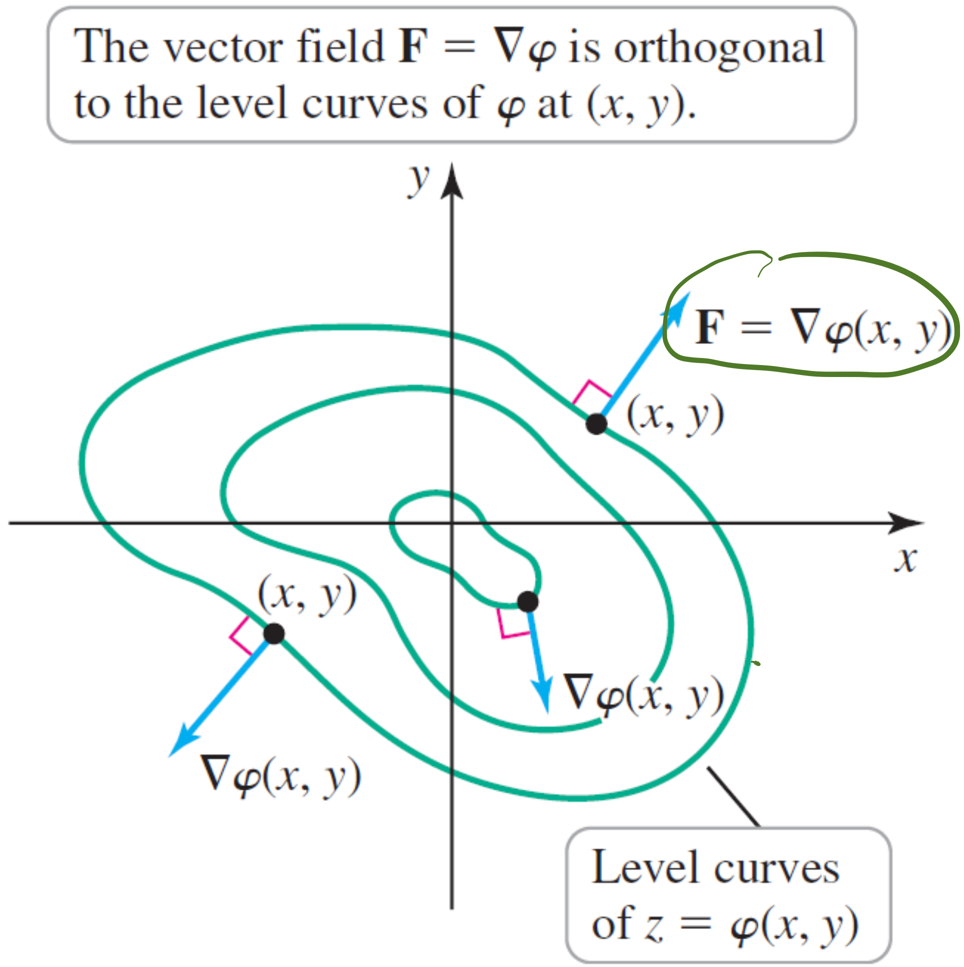

$\mathbf{F} = \langle 0, 0, 1 - x^2 - y^2 \rangle$,
for $x^2 + y^2 \leq 1$
z
x
y
Cylinder $x^2 + y^2 = 1$

Gradient Fields and Potential Functions

$\vec{F} = \nabla \varphi(x,y,z) = \langle \varphi_x, \varphi_y, \varphi_z \rangle$

Figure 17.11

$\vec{F} = \nabla \varphi$



Definition Gradient Fields and Potential Functions

Let φ be differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$.

The vector field $\mathbf{F} = \nabla\varphi$ is a **gradient field** and the function φ is a **potential function** for **F**.

Example 4 (a) Sketch and interprete the gradient field associated with the Temperature function $T = 200 - x^2 - y^2$ on $R = \{(x,y) : x^2 + y^2 \le 25\}$.

(b) $\varphi = \tan^{-1}(xy)$

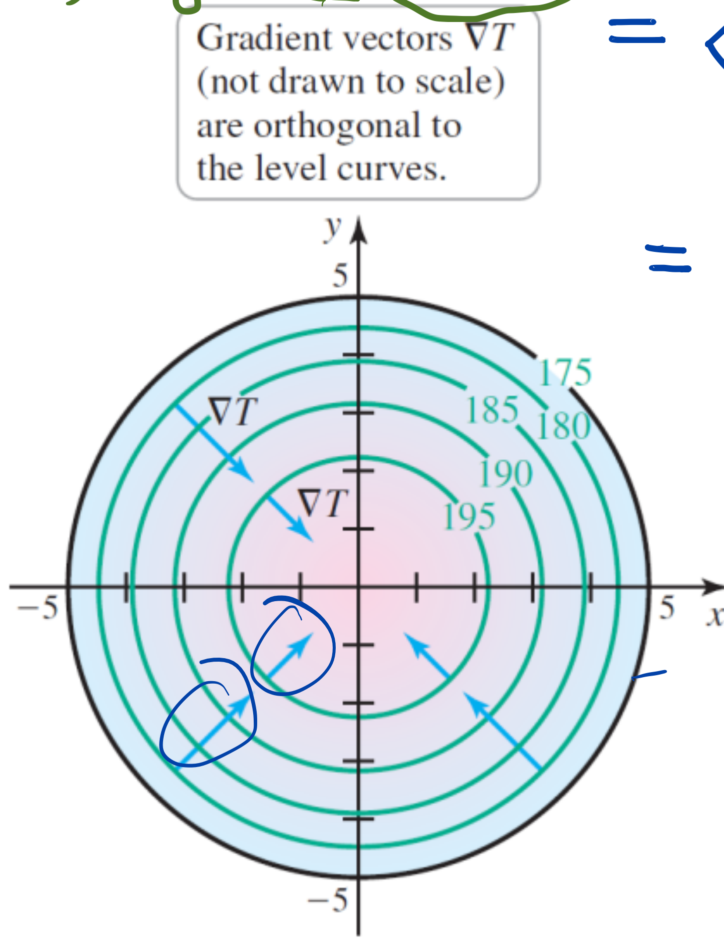
$(\tan^{-1} t)' = \frac{1}{1+t^2}$

Figure 17.12

(a) $\nabla T = \langle T_x, T_y \rangle = \langle -2x, -2y \rangle = -2\langle x, y \rangle$

$T(x,y) = 200 - x^2 - y^2 = C$

$\Rightarrow x^2 + y^2 = 200 - C$



Level curves of $T(x, y) = 200 - x^2 - y^2$

(b) $\nabla \varphi = \langle \varphi_x, \varphi_y \rangle$

$= \left\langle \frac{1}{1+(xy)^2} \cdot y, \frac{x}{1+(xy)^2} \right\rangle$

$= \frac{1}{1+(xy)^2} \langle y, x \rangle$

$\varphi = \tan^{-1}(xy) = C$

$xy = \tan C$

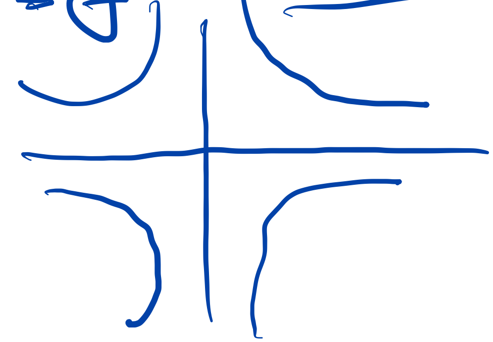
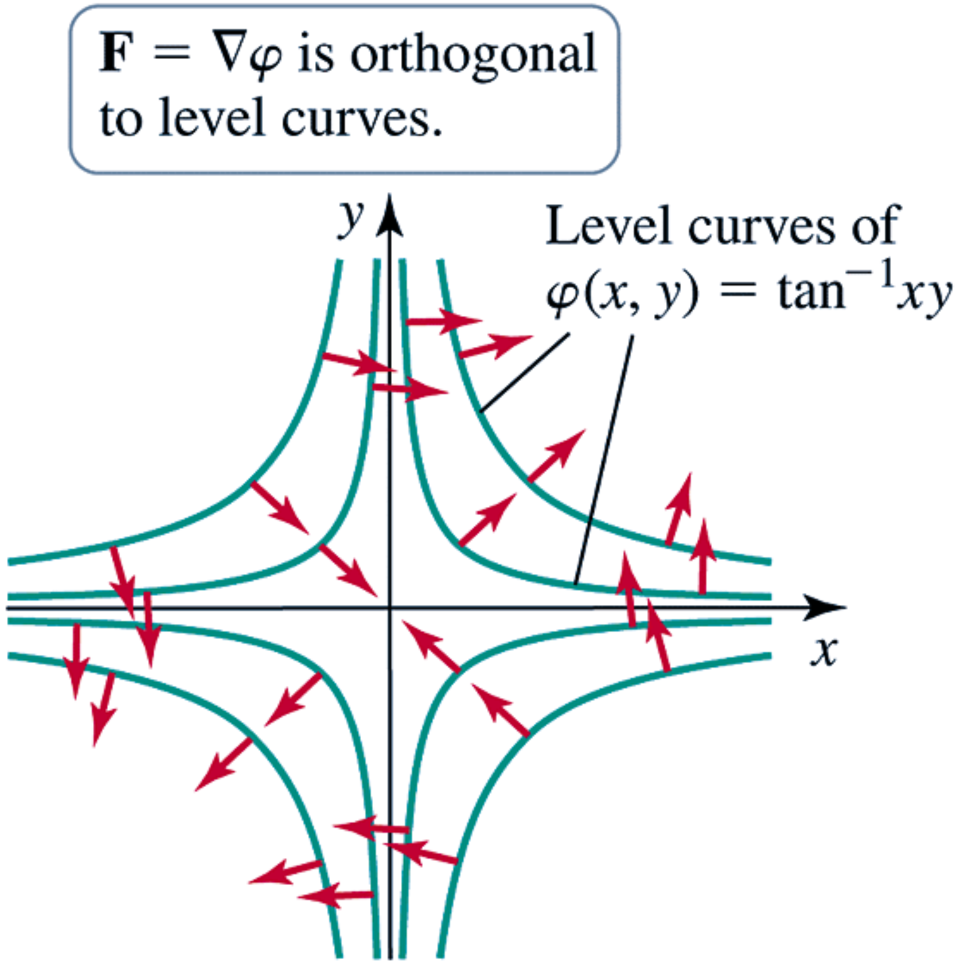


Figure 17.13


$\mathbf{F} = \nabla\varphi$ is orthogonal to level curves.
Level curves of $\varphi(x, y) = \tan^{-1}xy$
y
x

Equipotential Curves and Flow Curves

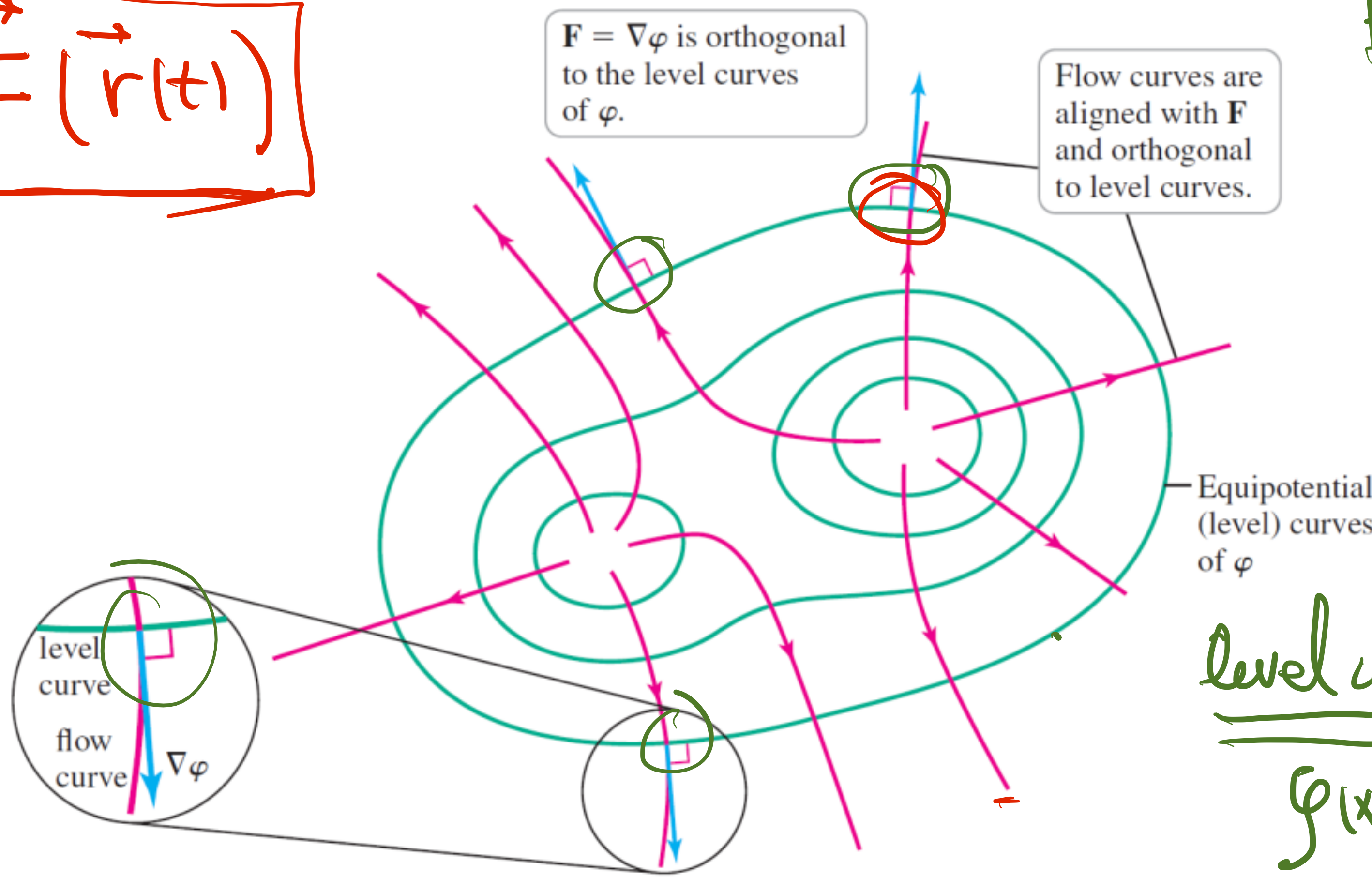
$\vec F(x,y) = \langle f(x,y), g(x,y)\rangle$

gradient VF: $\vec F = \nabla \varphi$ — φ scalar-valued potential

Figure 17.14

$\vec r(t) = \langle x(t), y(t)\rangle$

$$\vec r\,'(t) = \vec F(\vec r(t))$$



level curve of φ

$\varphi(x,y) = c$

Example 5 The equipotential curves for the potential function $\varphi = \frac{1}{2}(x^2 - y^2)$ are shown in green.
(a) Find the gradient field associated with φ and verify that $\vec{F} = \nabla\varphi$ is orthogonal to the equipotential curve at $(2,1)$; and (b) at all pts.

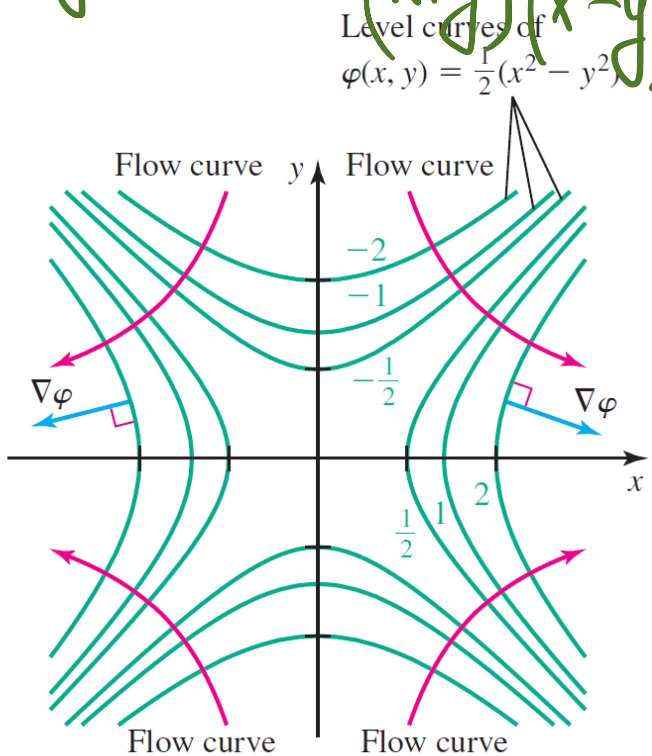
Figure 17.15

(a) $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle x, -y \rangle$

level curves of φ

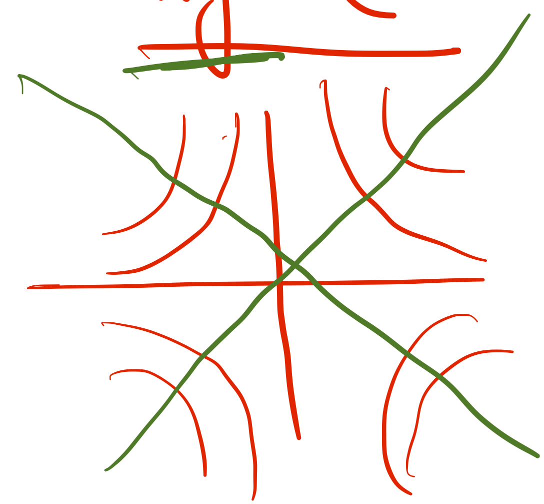
$\varphi = c \quad \frac{1}{2}(x^2 - y^2) = c$

$(x+y)(x-y) = x^2 - y^2 = c$



Flow curves of $\mathbf{F} = \nabla\varphi$ are orthogonal to level curves of φ everywhere.

$xy = c$



$\int_C f\,ds$

$f(x, y)$

$\int_C \vec{F}\cdot d\vec{r} = \int_C (\vec{F}\cdot\vec{T})\,ds$

$\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle$

Section 17.2 Line Integrals

Parametrization of a smooth curve C

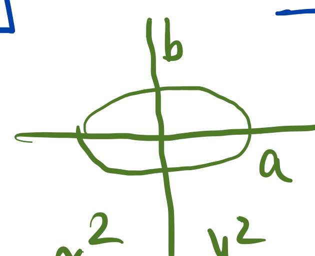
- graph in R^2 $y = f(x)$, $x\in[a,b]$

$\vec{r}(t) = \langle x(t), y(t)\rangle$

$= \langle t, f(t)\rangle$

$t\in[a,b]$

- level curve in R^2



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\vec{r}(t) = \langle a\cos t, b\sin t\rangle$

$0 \le t \le 2\pi$

$$\int_C f\,ds = \lim_{\Delta s_k \to 0} \sum_k f(x(t_k^*), y(t_k^*))\,\Delta s_k$$

Figure 17.16

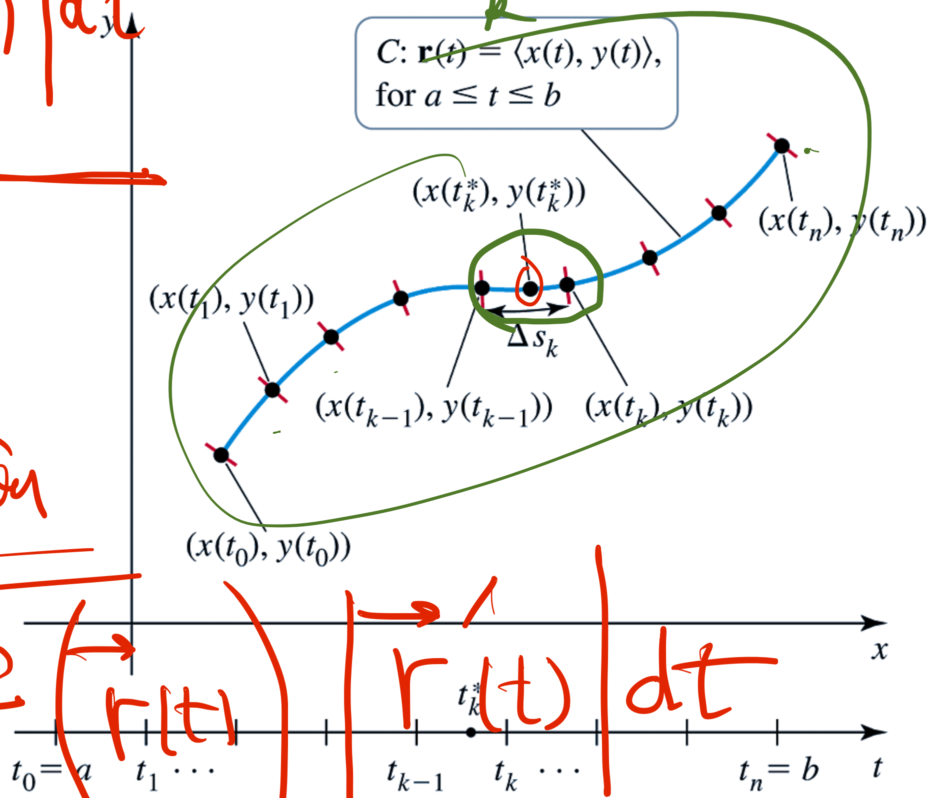
$$\int_C 1\,ds = \int_a^b |\vec{r}\,'(t)|\,dt$$

$\vec{r}(t) = ?$

parametrization

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$\int_a^b f(x)\,dx$$



The parameter t resides on the t-axis. As t varies from a to b, the curve C in the xy-plane is generated from $(x(a), y(a))$ to $(x(b), y(b))$.

Definition Scalar Line Integral in the Plane

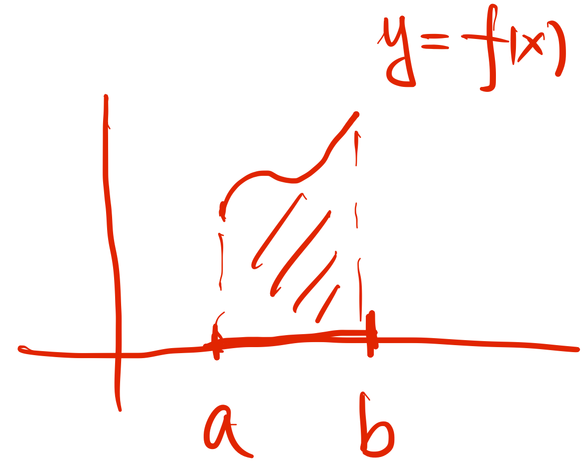
Suppose the scalar-valued function *f* is defined on a region containing the smooth curve *C* given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. The **line integral of *f* over *C*** is

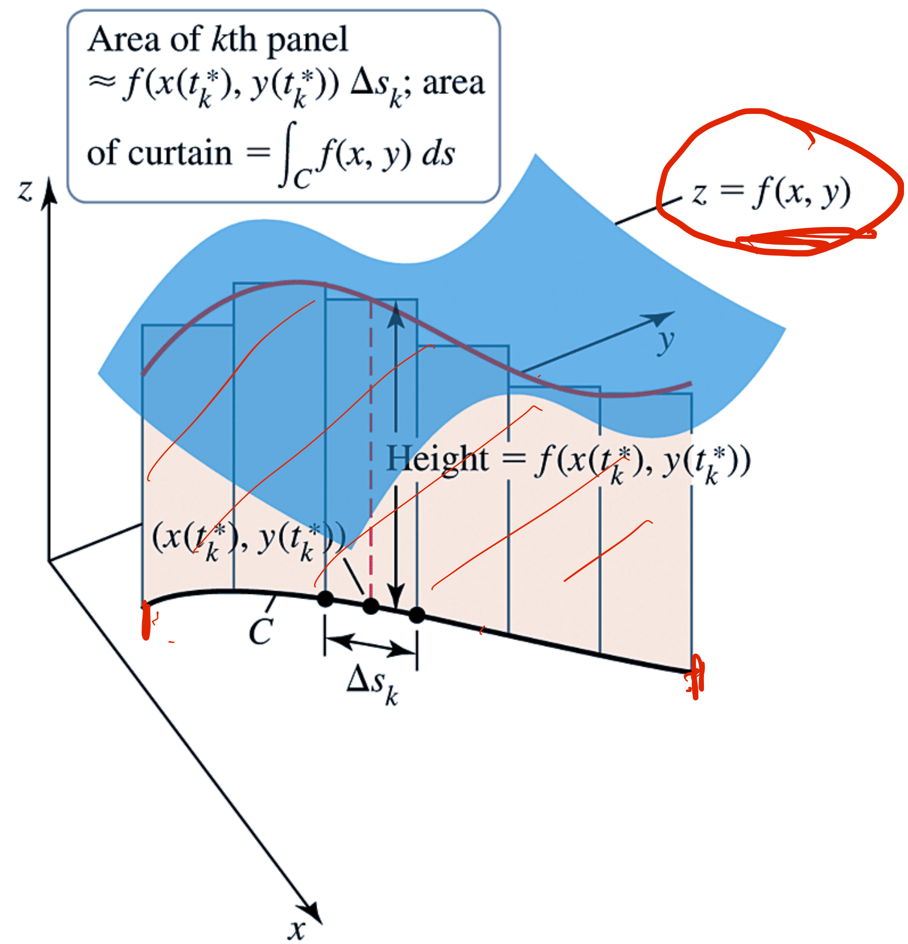
$$\int_C f(x(t), y(t))\,ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\Delta s_k,$$

provided this limit exists over all partitions of [*a*, *b*]. When the limit exists, *f* is said to be **integrable** on *C*.

Figure 17.17

$\int_a^b f(x)\,dx$





$\iint_R f\,dA = V$

Theorem 17.1 Evaluating Scalar Line Integrals in *R* Squared

Let *f* be continuous on a region containing a smooth curve $C: \mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. Then

$$\int_C f\, ds = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt. = \int_a^b f(\vec{r}(t)) |\vec{r}\,'(t)|\, dt$$

$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, a \le t \le b$

$$\int_C f\, ds = \int_a^b f(\vec{r}(t)) |\vec{r}\,'(t)|\, dt$$

Procedure Evaluating the Line Integral Integral *c* of *f* d s

1. Find a parametric description of *C* in the form

$\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$.

2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.

3. Make substitution for *x* and *y* in the integrand and evaluate an ordinary integral:

$$\int_C f \, ds = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)| \, dt.$$

Example 1 The temperature of the circular plate $R = \{(x,y): x^2+y^2 \le 1\}$ is $T(x,y) = 100(x^2+2y^2)$. Find the average temperature along the edge of the plate.

Figure 17.18

$\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

$C: x^2+y^2=1$

$|\vec{r}\,'(t)| = |\langle -\sin t, \cos t \rangle| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$$\bar{T} = \int_C T\,ds \Big/ \int_C 1\,ds$$

$$= \int_0^{2\pi} 100(x^2+2y^2)\cdot 1\,dt \Big/ \int_0^{2\pi} 1\cdot 1\cdot dt$$

$$= \frac{100}{2\pi}\int_0^{2\pi} (x^2+2y^2)\,dt$$

$x^2+y^2+y^2 = 1+y^2$

$$= \frac{50}{\pi}\int_0^{2\pi} (1+\sin^2 t)\,dt$$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$\cos 2t = \cos^2 t - \sin^2 t$

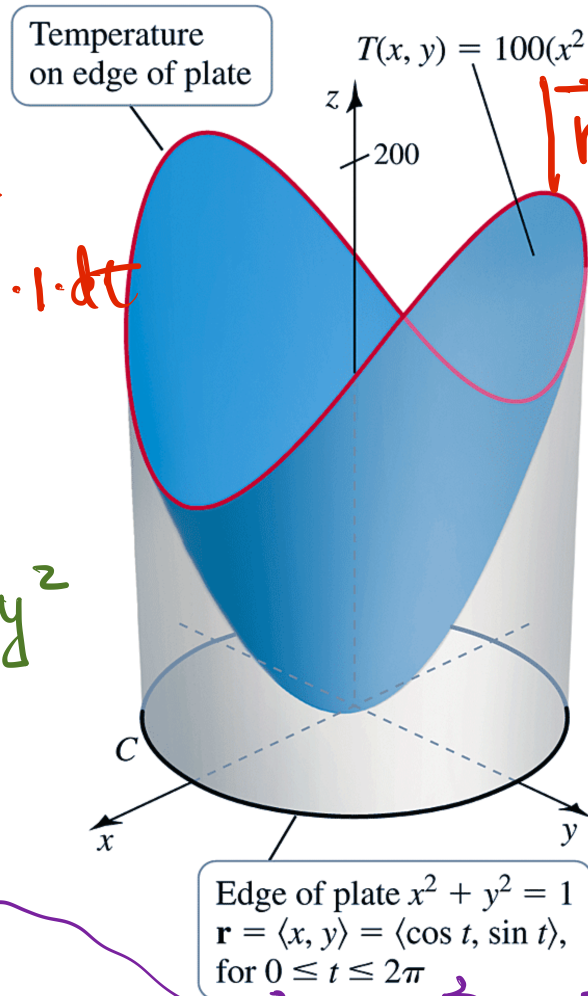
$1+\cos^2 t - \sin^2 t = 2\cos^2 t$

$f(x,y)$ on R

$$\bar{f} = \frac{\iint_R f(x,y)\,dA}{|R|}$$

$f(x)$ on $[a,b]$

$$\bar{f} = \int_a^b f\,dx \Big/ (b-a)$$

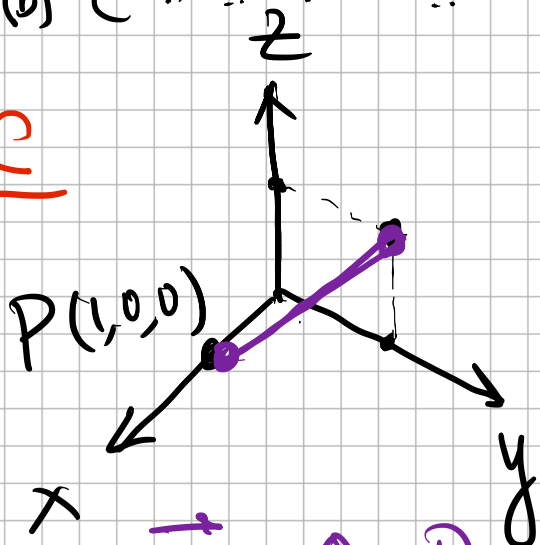


Example 2 Evaluate $\int_C (xy+2z)\,ds = \int_0^1 \left[(1-t)t+2t\right]\sqrt{3}\,dt$

(a) C is the line segment from P(1,0,0) to Q(0,1,1)

(b) C ·· ·· ·· ·· ·· Q(0,1,1) to P(1,0,0)

C



$\vec{v} = Q - P$

$= \langle 0,1,1\rangle - \langle 1,0,0\rangle$

$= \langle -1,1,1\rangle$

$\vec{r}(t) = \langle 1,0,0\rangle + t\vec{v}$

$= \langle 1,0,0\rangle + t\langle -1,1,1\rangle$

$= \langle 1-t, t, t\rangle$

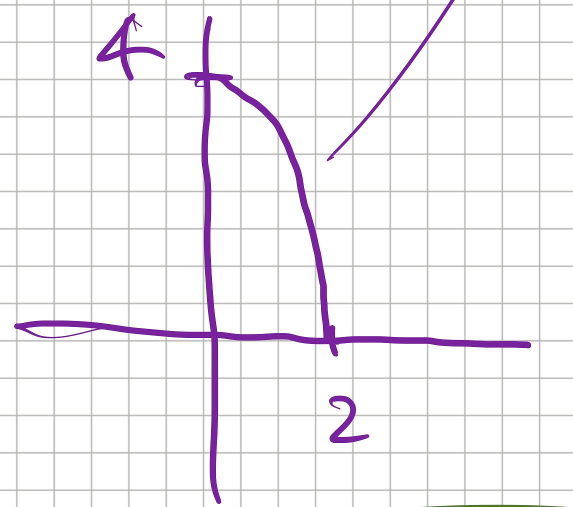
$0 \le t \le 1$

$|\vec{r}\,'(t)| = |\langle -1,1,1\rangle|$

$= \sqrt{3}$

Example 3 (#29) C is a portion of the ellipse $\frac{x^2}{4}+\frac{y^2}{16}=1$ in the first quadrant, oriented counterclockwise.

$$\int_C xy\,ds$$



① $\vec{r}(t)=\langle 2\cos t,\ 4\sin t\rangle$

$0\le t\le \frac{\pi}{2}$

② $|\vec{r}\,'(t)|=|\langle -2\sin t,\ 4\cos t\rangle|=\sqrt{4\sin^2 t+16\cos^2 t}$

$=2\sqrt{\sin^2 t+4\cos^2 t}=2\sqrt{1+3\cos^2 t}$

③ $\int_C xy\,ds=\int_0^{\pi/2} 2\cos t\,4\sin t\cdot 2\sqrt{1+3\cos^2 t}\,dt$

$=16\int_0^{\frac{\pi}{2}} \cos t\sin t\sqrt{1+3\cos^2 t}\,dt$

$u=1+3\cos^2 t$ $\quad 16\int_4^1 -\frac{1}{6}\sqrt{u}\,du$

$du=6\cos t\cdot(-\sin t)$

Theorem 17.2 Evaluating Scalar Line Integrals in *R* Cubed

Let *f* be continuous on a region containing a smooth curve $C: \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \le t \le b$. Then

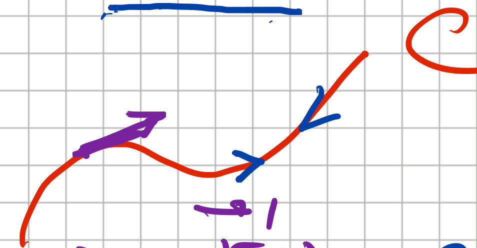
$$\int_C f \, ds = \int_a^b f(x(t), y(t), z(t)) \left|\mathbf{r}'(t)\right| dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.$$

Line Integral of Vector-valued Function

- $\vec{F}(x,y,z) = \langle f(x,y,z),\ g(x,y,z),\ h(x,y,z) \rangle$ vector-valued funct
- C is a smooth curve parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$a \le t \le b$



$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left(\vec{F} \cdot \vec{T} \right) ds$$

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} - C$$

unit tangent vector

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \underbrace{|\vec{r}'(t)|\, dt}_{ds}$$

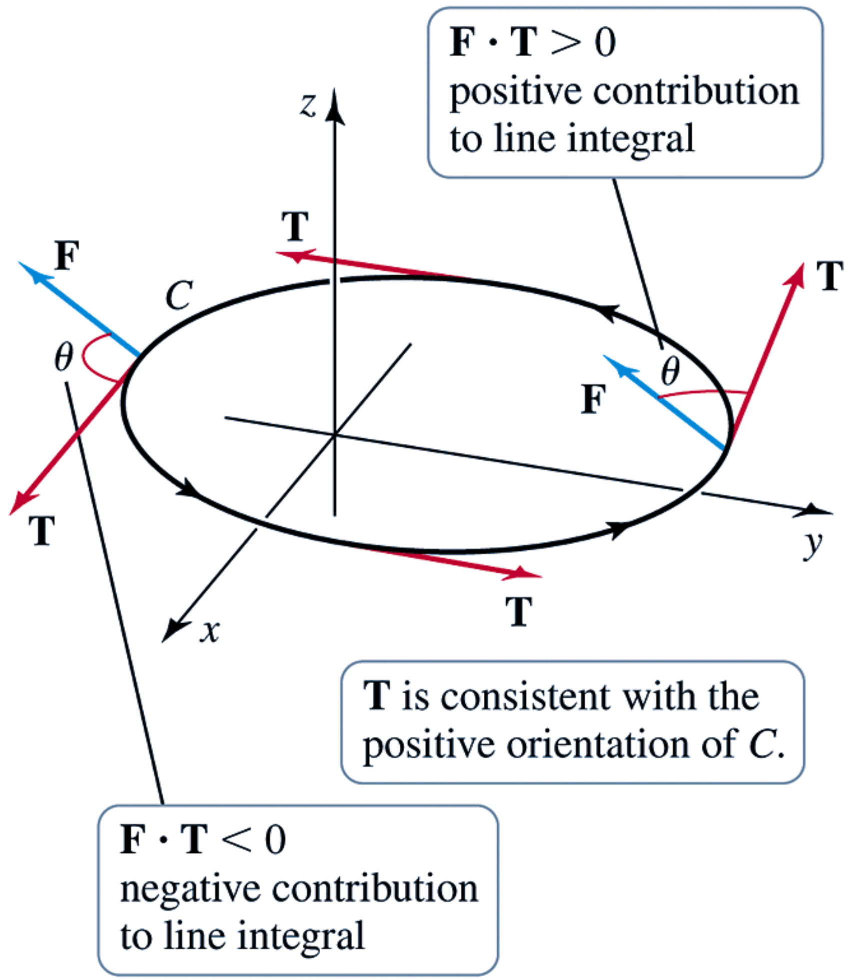
$$= \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_C \vec{F} \cdot d\vec{r}}$$

$$\int_a^b \langle f, g, h \rangle \cdot \langle x', y', z' \rangle\, dt$$

$$= f x' dt + g y' dt + h z' dt$$

$$\boxed{\int_C f\,dx + g\,dy + h\,dz}$$

Figure 17.19



$\mathbf{F} \cdot \mathbf{T} > 0$
positive contribution
to line integral
z
T
F
C
θ
T
θ
F
y
T
x
T
T is consistent with the
positive orientation of C.
$\mathbf{F} \cdot \mathbf{T} < 0$
negative contribution
to line integral

Definition Line Integral of a Vector Field

Let **F** be a vector field that is continuous on a region containing a smooth oriented curve *C* parameterized by arc length. Let **T** be the unit tangent vector at each point of *C* consistent with the orientation. The line integral of **F** over *C* is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\, ds$ may be expressed in the following forms, where $\mathbf{F}=\langle f,g,h\rangle$ and *C* has a parameterization $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle,$ for $a\le t\le b$:

$$\int_b^a \mathbf{F}\cdot\mathbf{r}'(t)dt=\int_b^a (f(t)x'(t)+g(t)y'(t)+h(t)z'(t)dt)$$

$$=\int_C f\,dx+g\,dy+h\,dz=\int_C \mathbf{F}\cdot d\mathbf{r}.$$

For line integrals in the plane, we let $\mathbf{F}=\langle f,g\rangle$ and assume *C* is parameterized in the form $\mathbf{r}(t)=\langle x(t),y(t)\rangle,$ for $a\le t\le b.$

Then $\int_a^b \mathbf{F}\cdot\mathbf{r}'\; dt=\int_a^b (f(t)x'(t)+g(t)y'(t))dt=\int_C f\,dx+g\,dy=\int_C \mathbf{F}\cdot d\mathbf{r}.$

Example 4 $I=\int_C \vec{F}\cdot d\vec{r}$, $\vec{F}=\langle y-x,\ x\rangle$.

(a) The quarter-circle C_1 from $P(0,1)$ to $Q(1,0)$

(b) " " " $-C_1$ from $Q(1,0)$ to $P(0,1)$

(c) The path C_2 from $P(0,1)$ to $Q(1,0)$ via two line segments through $O(0,0)$.

Figure 17.20

(a) $I=\int_{\frac{\pi}{2}}^{0}\langle \sin t-\cos t,\ \cos t\rangle\cdot\langle -\sin t,\ \cos t\rangle dt$ $\quad \vec{r}_1{}'=\langle -\sin t,\ \cos t\rangle$

$C_1:\ \vec{r}_1(t)=\langle \cos t,\ \sin t\rangle$, $t:\frac{\pi}{2}\to 0$

$=-\int_0^{\frac{\pi}{2}}\left[-\sin^2 t+\cos t\sin t+\cos^2 t\right]dt$

$=-\int_0^{\frac{\pi}{2}}\left[\cos 2t+\frac{1}{2}\sin 2t\right]dt$

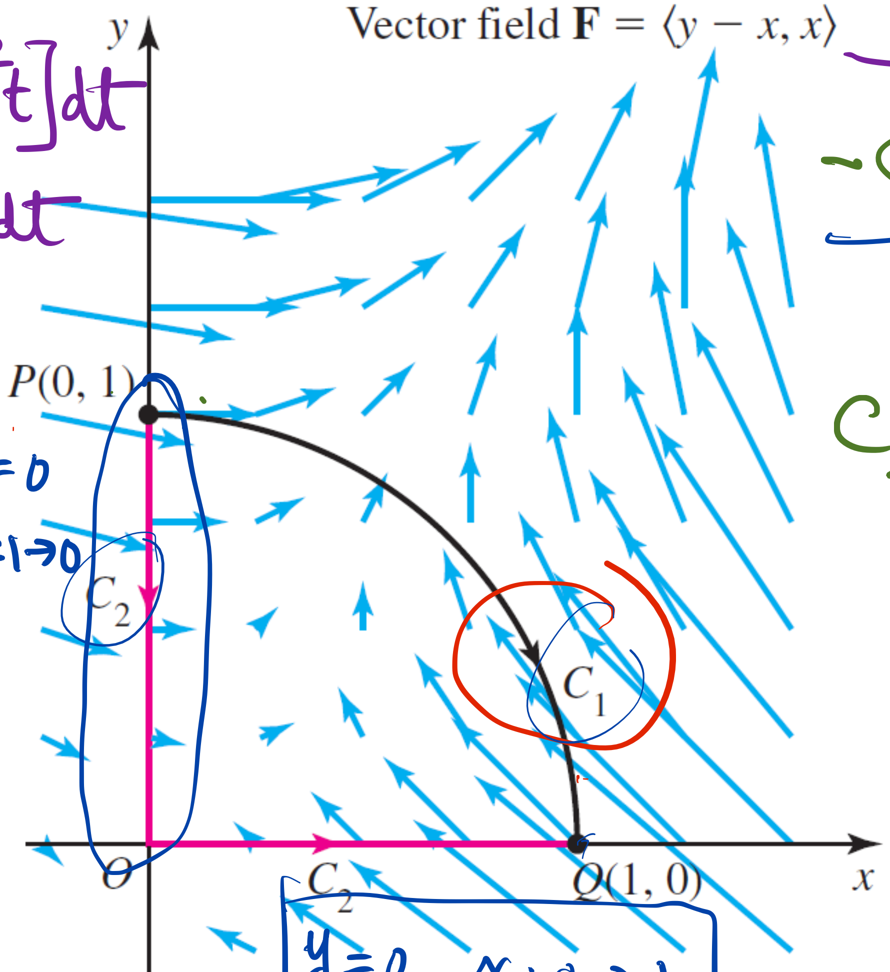
$=-\left[\frac{1}{2}\sin 2t-\frac{1}{4}\cos 2t\right]_0^{\frac{\pi}{2}}$

$=\frac{1}{4}\cdot(-2)=-\frac{1}{2}$

$-C_1:\ \vec{r}_1(t)=\langle \cos t,\ \sin t\rangle$, $t:0\to\frac{\pi}{2}$

$C_2:\ \vec{r}_2(t)=\langle 0,\ t\rangle$, $t:1\to 0$

$\vec{r}_3(t)=\langle t,\ 0\rangle$, $t:0\to 1$



$\begin{cases} x=0 \\ y:1\to 0\end{cases}$

$y=0,\quad x:0\to 1$

$\int_{C_2}\langle y-x,\ x\rangle\cdot d\vec{r}$

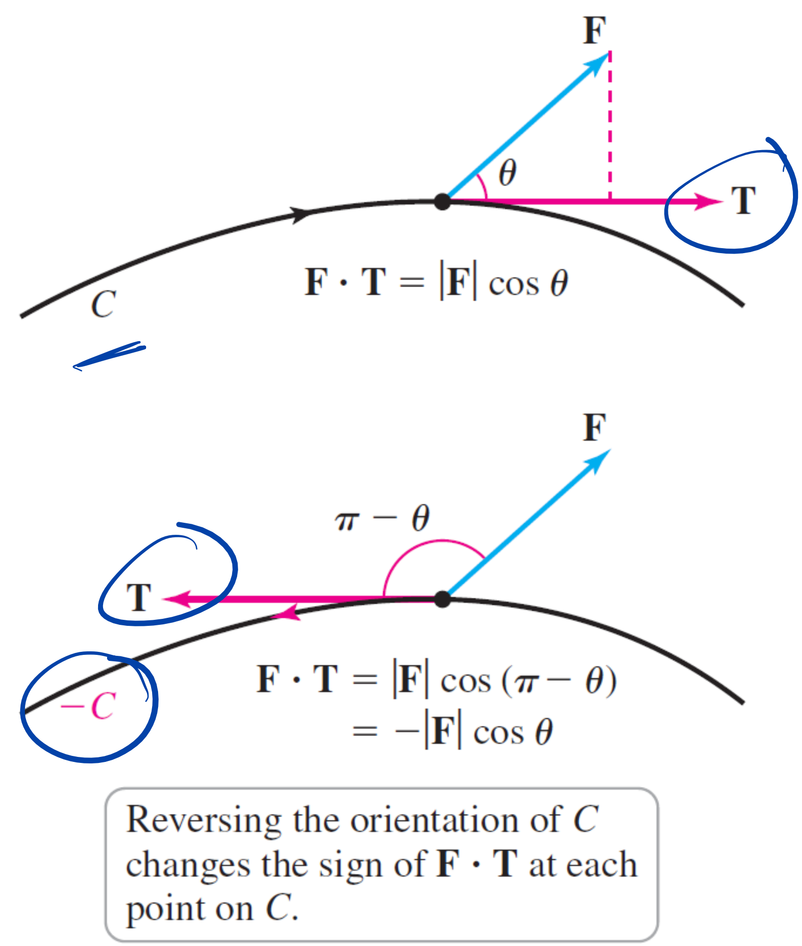
$=\int_{C_2}(y-x)dx+x\,dy$

$=0+\int_0^1 -x\,dx=-\frac{1}{2}x^2\Big|_0^1$

$=-\frac{1}{2}$

Figure 17.21

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$$



Work Done in a Force Field

Figure 17.22 (a & b)

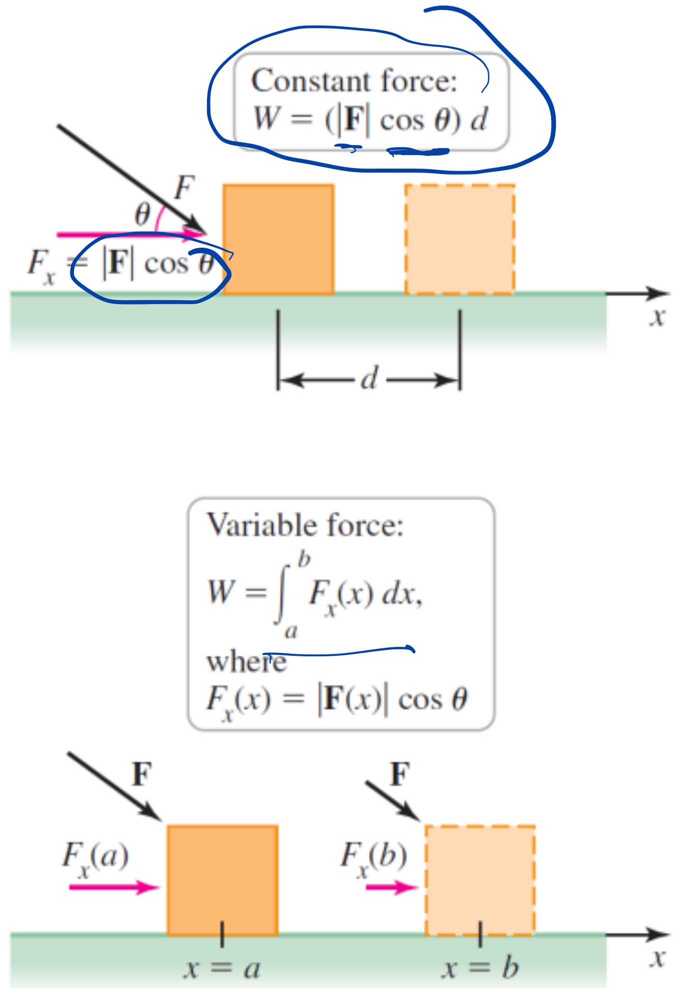
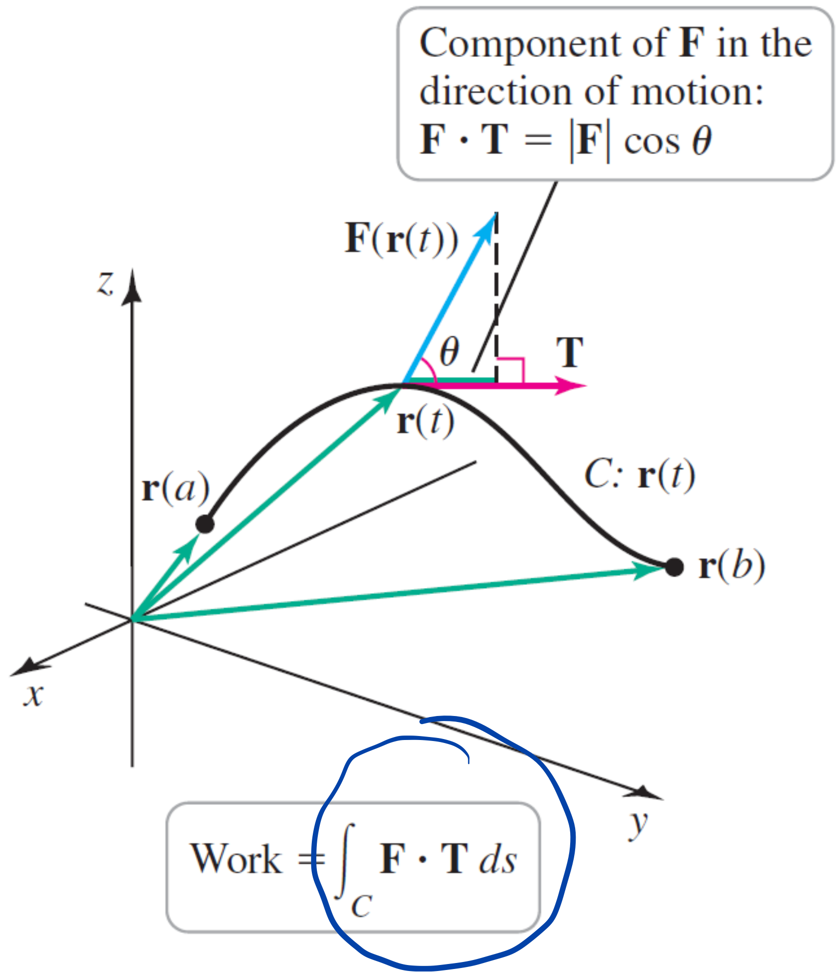


Figure 17.23



Component of **F** in the direction of motion:
$\mathbf{F} \cdot \mathbf{T} = |\mathbf{F}| \cos \theta$
$\mathbf{F}(\mathbf{r}(t))$
z
θ
$\mathbf{T}$
$\mathbf{r}(t)$
$\mathbf{r}(a)$
C: $\mathbf{r}(t)$
$\mathbf{r}(b)$
x
y
Work $= \int_C \mathbf{F} \cdot \mathbf{T}\, ds$

Definition Work Done in a Force Field

Let **F** be a continuous force field in a region *D* of $\circ^{3}$.

Let $C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$ for $a \le t \le b,$ be a smooth curve in *D* with a unit tangent vector **T** consistent with the orientation. The work done in moving an object along *C* in the positive direction is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

Define Circulation

Let **F** be a continuous vector field on a region *D* of $\mathbb{R}^3$ and let *C* be a closed smooth oriented curve in *D*. The **circulation** of ***F*** on *C* is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, $= \int_C \vec{F} \cdot d\vec{r}$ where **T** is the unit vector tangent to *C* consistent with the orientation

Example 6 Let C be the unit circle with counterclockwise orientation. Find the circulation on C of the following vector fields. (a) the radial field $\vec{F} = \langle x, y \rangle$
(b) the rotation field $\vec{F} = \langle -y, x \rangle$

Figure 17.24 (a & b)

(a) $\int_C \langle x, y \rangle \cdot d\vec{r} = 0$

$= \int_C \langle x, y \rangle \cdot \vec{T}\, ds$

$= \int_C 0\, ds$

$= 0$

(b) $\int_C \langle -y, x \rangle \cdot d\vec{r} = ?$

$= \int_C \langle -y, x \rangle \cdot \langle x', y' \rangle\, ds$

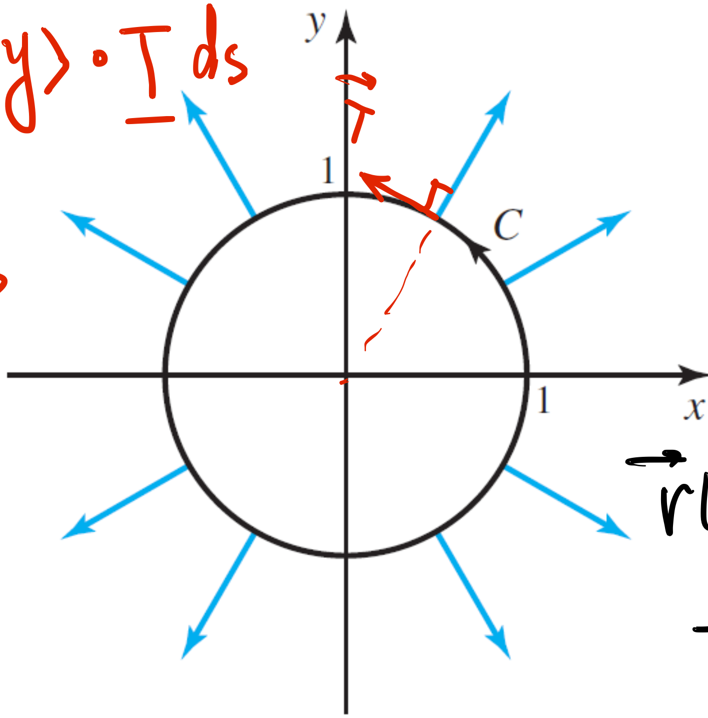
$= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$

$= \int_0^{2\pi} 1\, dt$

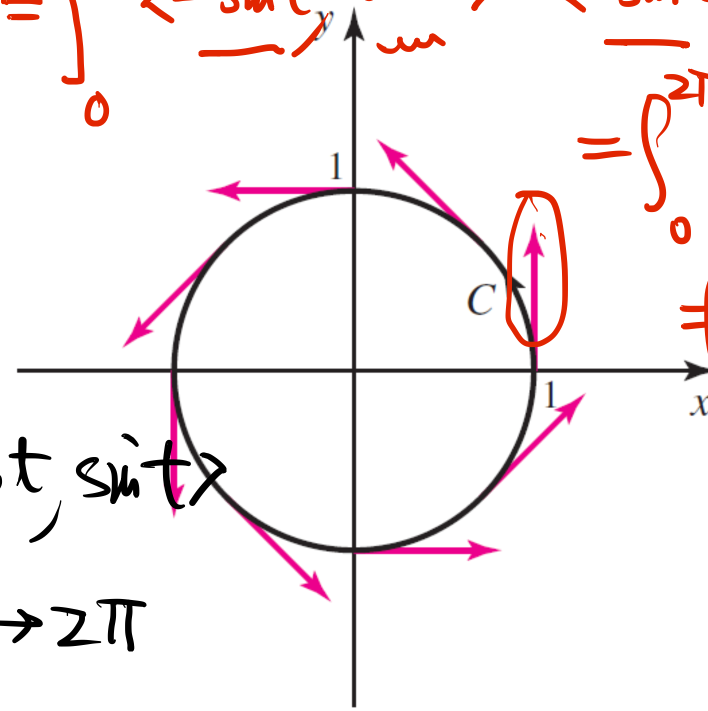
$= 2\pi$

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$t: 0 \to 2\pi$



On the unit circle, $\mathbf{F} = \langle x, y \rangle$ is orthogonal to C and has zero circulation on C.



On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has positive circulation on C.

Example 7 $I = \int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle z, x, -y \rangle$ and $C: \vec{r}(t) = \langle \cos t, \sin t, \cos t \rangle$, $0 \le t \le 2\pi$.

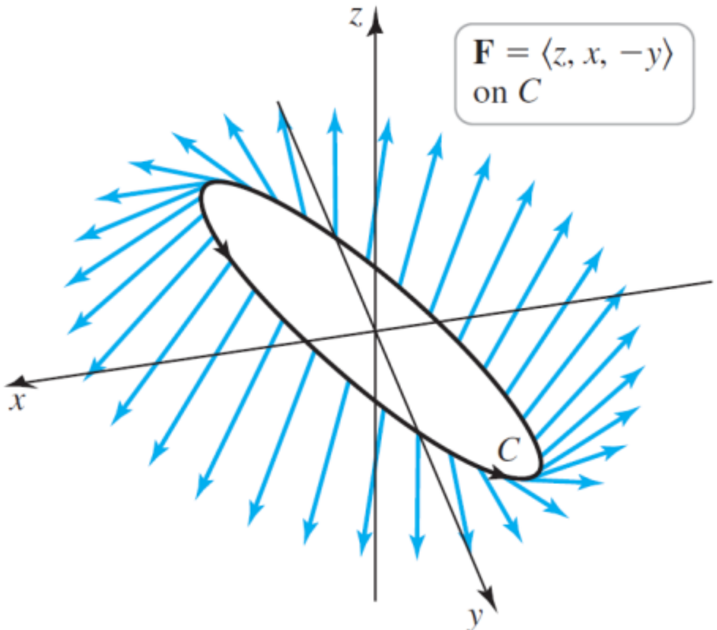
$$= \int_0^{2\pi} \langle \cos t, \cos t - \sin t \rangle \cdot \langle -\sin t, \cos t, -\sin t \rangle \, dt$$

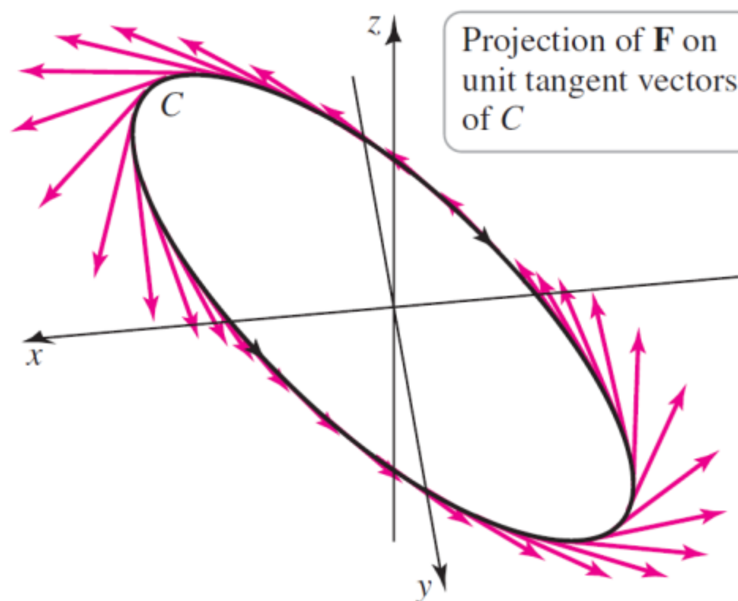
Figure 17.25 (a & b)

$$= \int_0^{2\pi} \left[-\cos t \sin t + \cos^2 t + \sin^2 t \right] dt = \int_0^{2\pi} \left[1 - \frac{1}{2} \sin 2t \right] dt$$

$$= 2\pi + \frac{1}{4} \cos 2t \Big|_0^{2\pi}$$

$$= 2\pi$$



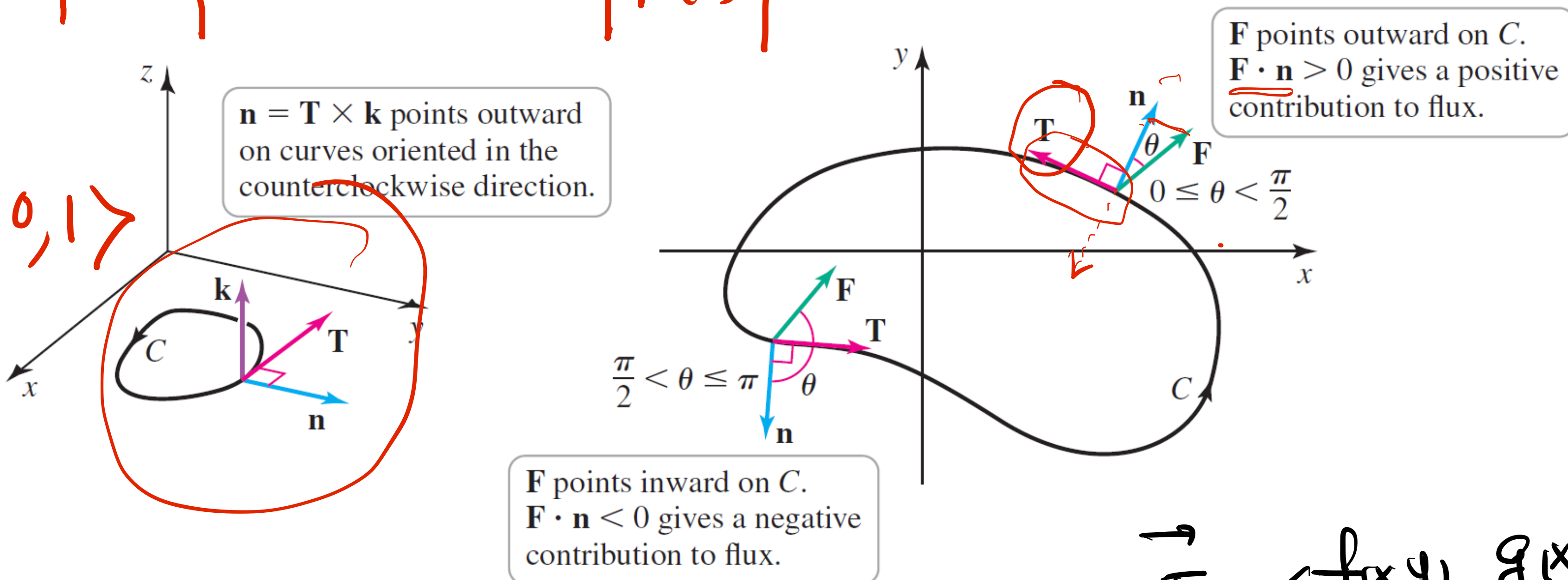


$$\int_C (\vec{F} \cdot \vec{n})\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\langle y', x' \rangle}{|\vec{r}\,'(t)|} |\vec{r}\,'(t)|\, dt$$

Figure 17.26 (a & b)

$$\vec{T} = \frac{\langle x', y' \rangle}{|\vec{r}\,'(t)|} \qquad \vec{n} = \frac{\langle y'(t), -x'(t) \rangle}{|\vec{r}\,'(t)|}$$

$$k = \langle 0, 0, 1 \rangle$$



$$\vec{F} = \langle f(x,y), g(x,y) \rangle$$

Definition Flux

$\vec{n} = \langle y', -x' \rangle$

Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field on a region *R* of $\mathbb{R}^2$.

Let $C : \mathbf{r}(t) = \langle x(t),\ y(t) \rangle$, for $a \le t \le b$, be a smooth oriented curve in *R* that does not intersect itself. The **flux** of the vector field **F** across *C* is

$$\int_C \mathbf{F} \cdot \mathbf{n}\ dS = \int_a^b \left(f(t)y'(t) - g(t)x'(t) \right) dt,$$

$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \langle y', x' \rangle dt$

where $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the unit normal vector and **T** is the unit tangent vector consistent with the orientation. If *C* is a closed curve with counterclockwise orientation, **n** is the outward normal vector and the flux integral gives the **outward flux** across *C*.

Example 8 Find the outward flux across the unit circle with counterclockwise orientation for

(a) $\vec{F} = \langle x, y \rangle$

(b) $\vec{F} = \langle -y, x \rangle$

Figure 17.27 (a & b)

(a) $\int_C \langle x, y\rangle \cdot \vec{n}\, ds$

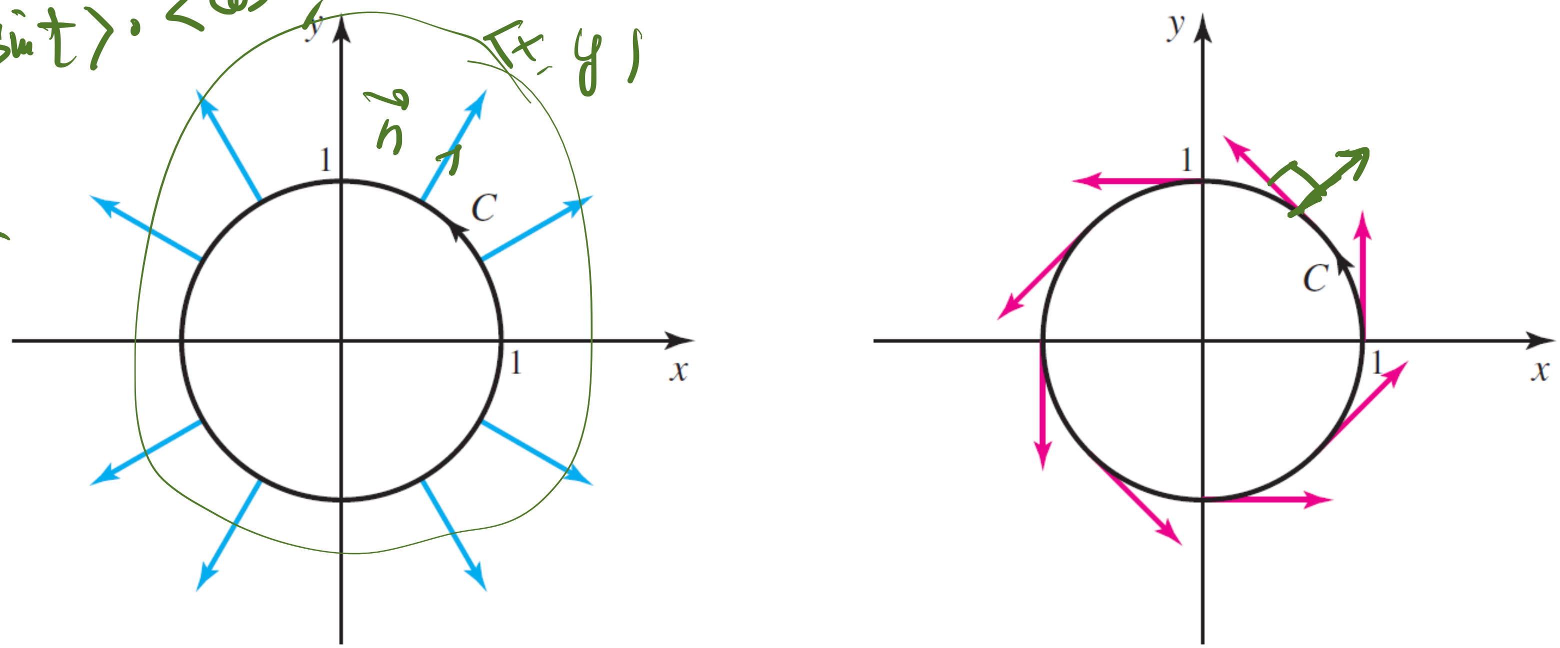
$= \int_0^{2\pi} \langle \cos t, \sin t\rangle \cdot \langle \cos t, \sin t \rangle\, dt$

$= \int_0^{2\pi} 1\, dt$

$= 2\pi$

(b) $\int_C \langle -y, x\rangle \cdot \vec{n}\, ds$

$= \int_C 0\, ds = 0$



On the unit circle, $\mathbf{F} = \langle x, y\rangle$ is orthogonal to C and has positive outward flux on C.

On the unit circle, $\mathbf{F} = \langle -y, x\rangle$ is tangent to C and has zero outward flux on C.

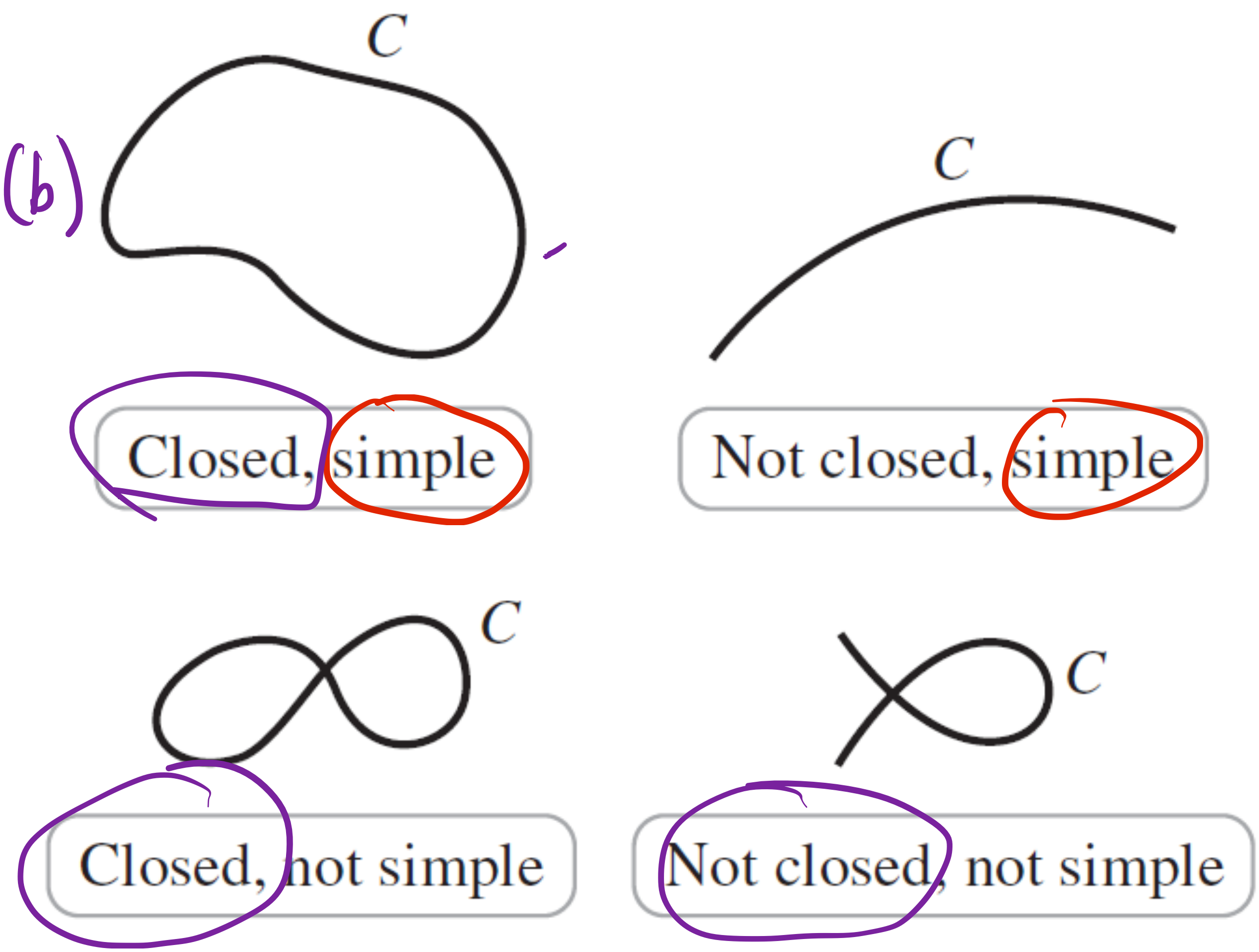
Section 17.3 Conservative Vector Fields

Figure 17.28

$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle,\ a \le t \le b$

$\forall\, t_1, t_2 \in (a, b)$ and $t_1 \ne t_2$ s.t. $\vec{r}(t_1) \ne \vec{r}(t_2)$

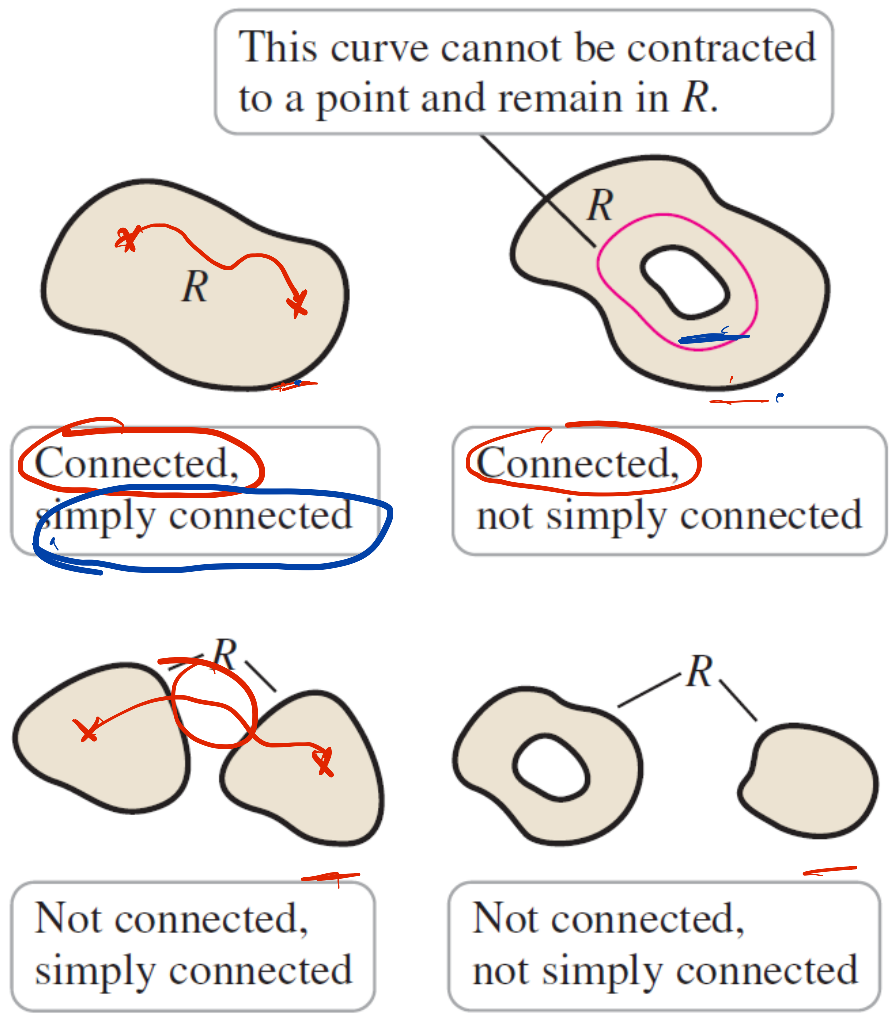
$\vec{r}(a) = \vec{r}(b)$ closed



Definition Simple and Closed Curves

Suppose a curve *C* (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$. Then *C* is a **simple curve** if $\mathbf{r}(t_1) \ne \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, *C* never intersects itself between its endpoints. The curve *C* is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of *C* are the same (Figure 17.28).

Figure 17.29


This curve cannot be contracted to a point and remain in R.
R
R
Connected, simply connected
Connected, not simply connected
R
R
Not connected, simply connected
Not connected, not simply connected

Definition Connected and Simply Connected Regions

An open region *R* in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$)

is **connected** if it is possible to connect any two points of *R* by a continuous curve lying in *R*. An open region *R* is **simply connected** if every closed simple curve in *R* can be deformed and contracted to a point in *R* (Figure 17.29).

Definition Conservative Vector Field

A vector field **F** is said to be **conservative** on a region

(in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that

$\mathbf{F} = \nabla \varphi$ on that region.

Curl of $\vec{F} = \langle f, g, h \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

$\vec{F}(x,y) = \langle f(x,y), g(x,y), 0 \rangle$

$\vec{F} = \nabla \varphi \iff \nabla \times \vec{F} = \langle 0, 0, 0 \rangle$

$$\nabla \times \vec{F}(x,y) = \left\langle 0, 0, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

Theorem 17.3 Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region *D* of $\mathbb{R}^3$, where *f*, *g*, and *h* have continuous first partial derivatives on *D*. Then **F** is a conservative vector field *D* (there is a potential function φ such that $\mathbf{F} = \nabla \varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \text{ and } \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Procedure Finding Potential Functions in *R* Cubed

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla \varphi$, use the following steps:

1. Integrate $\varphi_x = f$ with respect to *x* to obtain φ, which includes an arbitrary function $c(y, z)$.

2. Compute φ_y and equate it to *g* to obtain an expression for $c_y(y, z)$.

3. Integrate $c_y(y, z)$. with respect to *y* to obtain $c(y, z)$. including an arbitrary function $d(z)$.

4. Compute φ_z and equate it to *h* to get $d(z)$.

A simple procedure beginning with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Example 1

(a) $\vec{F} = \langle e^x \cos y, -e^x \sin y \rangle = \langle f, g \rangle$

(b) $\vec{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

(i) Is $\vec{F}$ a conservative field?

(ii) If so, find a potential function.

(a) (i) $\frac{\partial f}{\partial y} \overset{?}{=} \frac{\partial g}{\partial x}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y$

$\parallel$

$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(-e^x \sin y) = -e^x \sin y$

$\vec{F} = \nabla \varphi$

(ii) $\varphi = ?$

$\nabla \varphi = \langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \rangle = \langle f, g \rangle = \langle e^x \cos y, -e^x \sin y \rangle$

$\varphi(x, y) = ?$ p. differential eq.

$$\begin{cases} \frac{\partial \varphi}{\partial x} = e^x \cos y \\ \frac{\partial \varphi}{\partial y} = -e^x \sin y \end{cases}$$

$\rightarrow \varphi = \int \frac{\partial \varphi}{\partial x} dx = \int e^x \cos y \, dx = e^x \cos y + C(y)$

$\Rightarrow \varphi(x, y) = e^x \cos y + C(y)$

$C(y) = ?$

$\parallel$

$\frac{\partial}{\partial y}[e^x \cos y + C(y)]$

$= -e^x \sin y + C'(y) = -e^x \sin y$

$\Rightarrow C'(y) = 0$

$\Rightarrow C(y) = \text{const.}$

$\varphi(x, y) = e^x \cos y + \text{const.}$

Theorem 17.4 Fundamentals Theorem for Line Integrals

Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\mathbf{F} = \nabla \varphi$ (which means that **F** is conservative), then

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

$\vec{F}(x,y) = \langle f(x,y), g(x,y), 0 \rangle$; $\nabla \times \vec{F} = \langle 0, 0, g_x - f_y \rangle$

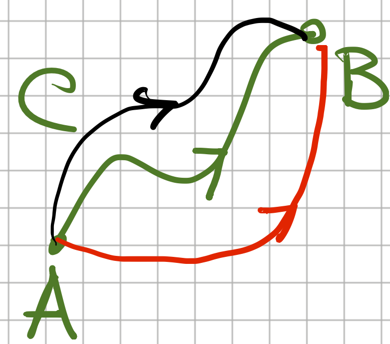
$\vec{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$

$\vec{F}$ is a conservative field $\Longleftrightarrow$ $\vec{F} = \nabla \varphi$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A)$$

$\nabla \varphi = \vec{F}$



$$\int_a^b F\,dx = f(b) - f(a)$$

$f' = F$

$$\vec{F}(x,y,z) = \langle f(x,y,z),\ g(x,y,z),\ h(x,y,z) \rangle$$

The following statements are equivalent

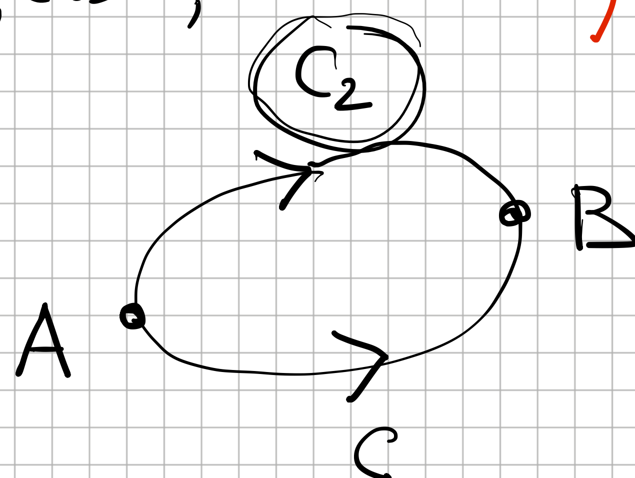
(1) $\vec{F}$ is a conservative field $\Leftrightarrow \vec{F} = \nabla \varphi$

(2) $\nabla \times \vec{F} = \vec{0}$

(3) $\int_C \vec{F} \cdot d\vec{r}$ is independent of path

(4) Let C be a simple, closed, oriented curve.

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$



$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r}$$

$$\|$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$$

Definition Independence of Path

Let **F** be a continuous vector field with domain *R*. If

$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for all piecewise-smooth curves

C_1 and C_2 in *R* with the same initial and terminal points,

then the line integral is **independent of path**.

Theorem 17.5

Let **F** be a continuous vector field on an open connected region *R* in $\mathbb{R}^2$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then **F** is is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla \varphi$ on *R*.

Example 3 Consider the potential function $\varphi(x,y)=\frac{1}{2}(x^2-y^2)$ and its gradient field $\vec{F}=\langle x,-y\rangle$.

- Let C_1 be $\vec{r}(t)=\langle \cos t, \sin t\rangle$, $0\le t\le \frac{\pi}{2}$ from $A(1,0)$ to $B(0,1)$
- Let C_2 be $\vec{r}(t)=\langle 1-t, t\rangle$, $0\le t\le 1$, also from A to B.

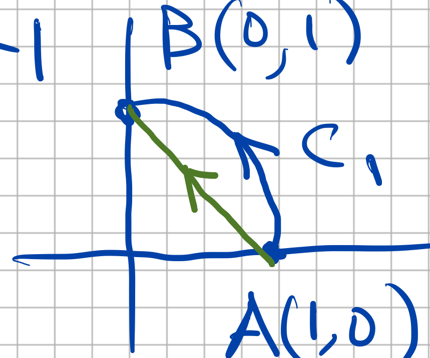
Evaluate $\int_{C_1}\vec{F}\cdot d\vec{r}$, $\int_{C_2}\vec{F}\cdot d\vec{r}$, $\varphi(B)-\varphi(A)$.

$$\int_{C_1}\vec{F}\cdot d\vec{r}=\int_0^{\frac{\pi}{2}}\langle x,-y\rangle\cdot\vec{r}\,'(t)\,dt=\int_0^{\frac{\pi}{2}}\langle \cos t,-\sin t\rangle\cdot\langle -\sin t,\cos t\rangle\,dt$$

$$=-\int_0^{\frac{\pi}{2}}2\cos t\sin t\,dt=-\int_0^{\frac{\pi}{2}}\sin 2t\,dt=\frac{\cos 2t}{2}\Big|_0^{\frac{\pi}{2}}=-1$$

$$\int_{C_2}\vec{F}\cdot d\vec{r}=\int_0^{1}\langle x,-y\rangle\cdot\vec{r}\,'(t)\,dt=\int_0^{1}\langle 1-t,-t\rangle\cdot\langle -1,1\rangle\,dt$$

$$=\int_0^1\left[-(1-t)-t\right]dt=-\int_0^1 dt=-1$$



$\varphi=?$

$$\begin{cases}\dfrac{\partial\varphi}{\partial x}=x \Rightarrow \varphi(x,y)=\frac{1}{2}x^2+C(y)\\[2mm] \dfrac{\partial\varphi}{\partial y}=-y=\dfrac{\partial}{\partial y}\left(\frac{1}{2}x^2+C(y)\right)=C'(y)\end{cases}$$

$$\Rightarrow C(y)=-\frac{1}{2}y^2+D \Rightarrow \varphi=\frac{1}{2}x^2-\frac{1}{2}y^2+D$$

$$\varphi(B)-\varphi(A)=\left[-\frac{1}{2}+D\right]-\left[\frac{1}{2}+D\right]=-1$$

Example 4 Let C be a simple curve from $A(-3,-2,-1)$ to $B(1,2,3)$.

Evaluate $\int_C \langle 2xy - z^2,\ x^2 + 2z,\ 2y - 2xz \rangle \cdot d\vec{r}$

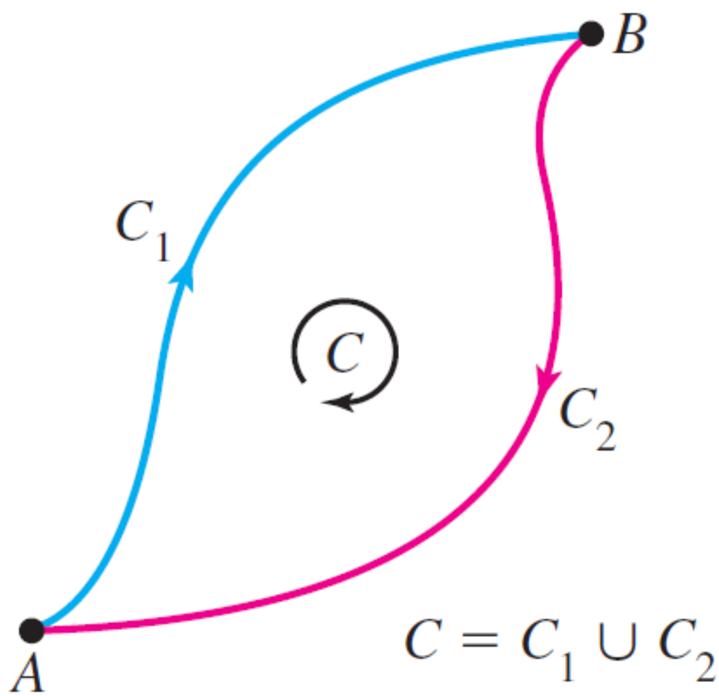
$\underbrace{\langle 2xy - z^2,\ x^2 + 2z,\ 2y - 2xz \rangle}_{\nabla \varphi}$

$\boxed{\varphi = ?}$

$= \varphi(B) - \varphi(A)$

$= \varphi(1,2,3) - \varphi(-3,-2,-1)$

Figure 17.30



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Theorem 17.6 Line Integrals on Closed Curves

Let R be an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Then **F** is a conservative vector field on R if and only if

$\int_c \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

Example 5 Without using Theorems 17.4 & 17.6, evaluate $\int_C \nabla(-xy + xz + yz)\cdot d\vec{r} = 0$

where $C: \vec{r}(t) = \langle \sin t, \cos t, \sin t\rangle$, for $0 \le t \le 2\pi$.

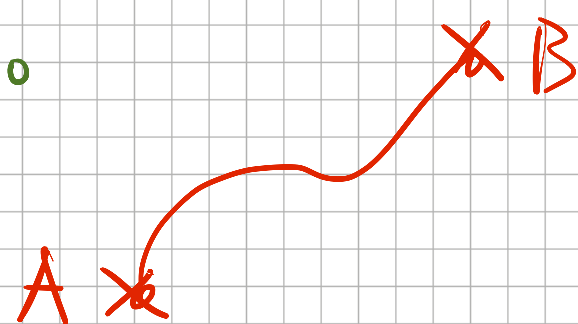
$$= \int_0^{2\pi} \langle -y+z, -x+z, x+y\rangle \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \langle -\cos t + \sin t, 0, \sin t + \cos t\rangle \cdot \langle \cos t, -\sin t, \cos t\rangle\,dt$$

$$= \int_0^{2\pi} \left[-\cos^2 t + \sin t\cos t\right] + 0 + \left[\sin t\cos t + \cos^2 t\right] dt$$

$$= \int_0^{2\pi} \sin 2t\,dt = -\frac{\cos 2t}{2}\Big|_0^{2\pi}$$

$$= -\frac{1}{2}\left[\cos 4\pi - \cos 0\right] = 0$$



$$\int_C \nabla\varphi \cdot d\vec{r} = \varphi(B) - \varphi(A)$$

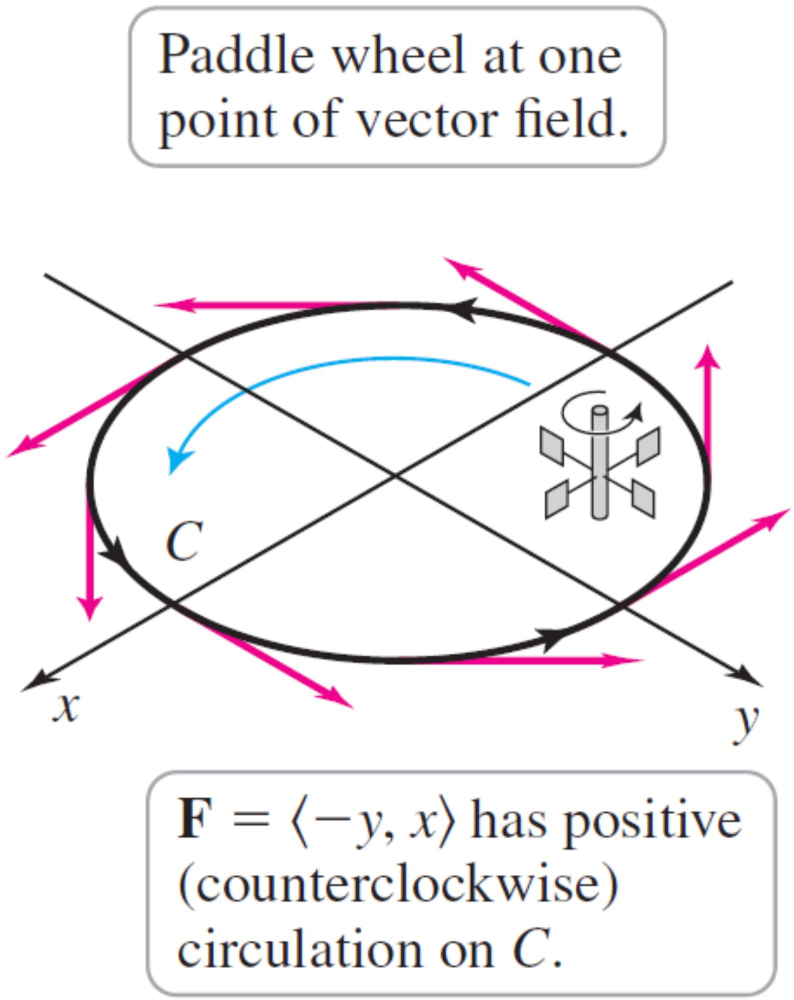
Section 17.4 Green's Theorem

Stokes Theorem

Gauss Divergence Theorem.

Circulation Form of Green's Theorem

Figure 17.31



Theorem 17.7 Green's Theorem-Circulation Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

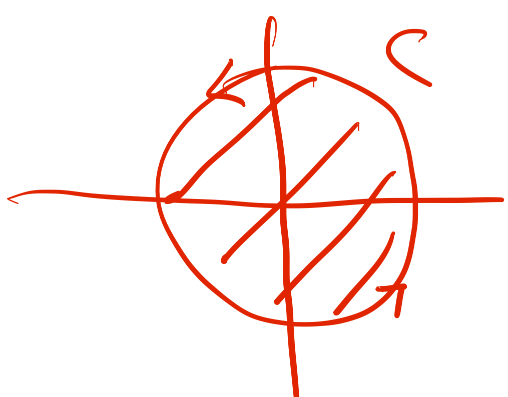
$\vec{F}$ ⟨ conservative, gradient, irrotational

$\vec{F} = \nabla \varphi$

Definition Two-Dimensional Curl

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\nabla \times \vec{F} = \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is **irrotational** on that region.

Example: Using Green's Thm to compute $\oint_C \langle -y, x \rangle \cdot d\vec{r}$, where C is the unit circle oriented counterclockwisely.



$$= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_R [1 - (-1)]\, dA = 2 \iint_R dA = 2\pi$$

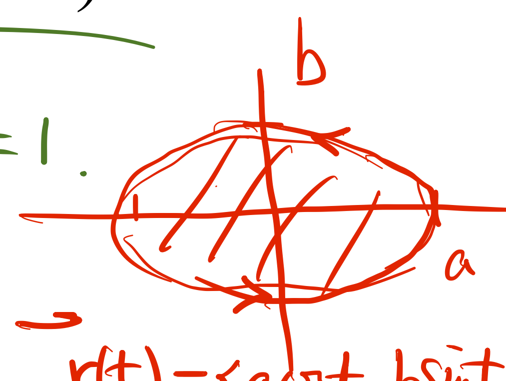
Theorem 17.8 Area of a Plane Region by Line Integrals

$A(R) = \iint_R 1\, dA = \oint_C \langle -y, 0\rangle \cdot d\vec{r} = \oint_C \langle 0, x\rangle \cdot d\vec{r}$

Under the conditions of Green's Theorem, the area of a region *R* enclosed by a curve *C* is

$$\int_C x\, dy = -\int_C y\, dx = \frac{1}{2}\int_C (x\, dy - y\, dx).$$

Example 2 Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



$\vec{r}(t) = \langle a\cos t, b\sin t\rangle$

$t: 0 \to 2\pi$

$A(R) = \int_C x\, dy = \int_0^{2\pi} a\cos t\, d(b\sin t) = ab\int_0^{2\pi} \cos^2 t\, dt$

$= \int_C -y\, dx = -\int_0^{2\pi} b\sin t\, d(a\cos t) = +ab\int_0^{2\pi} \sin^2 t\, dt$

$= \frac{1}{2}\left[ab\int_0^{2\pi} \cos^2 t\, dt + ab\int_0^{2\pi} \sin^2 t\, dt\right] = \frac{ab}{2}\int_0^{2\pi} 1\, dt = ab\pi$

$$\iint_R \nabla \times \vec{F}\, dA = \oint_{\partial R} \vec{F}\cdot d\vec{r} = \oint_{\partial R} f\,dx + g\,dy$$

Stoke's Thrm $\vec{F} = \langle f, g, h \rangle$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{r}$$

∂R

Definition Two-Dimensional Divergence

$$\nabla \times \vec{F} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$$

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is **source free** on that region.

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle f, g \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

scalar-valued function

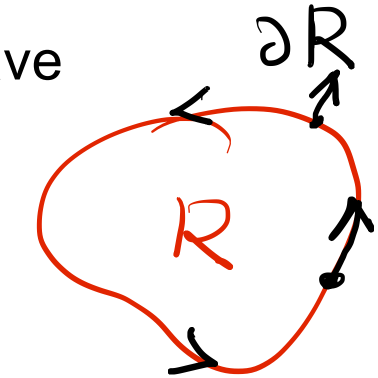
$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$$

vector-valued

Theorem 17.9 Green's Theorem—Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then



$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{Outward flux}} = \underbrace{\oint_C f\,dy - g\,dx}_{\text{Outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA,$$

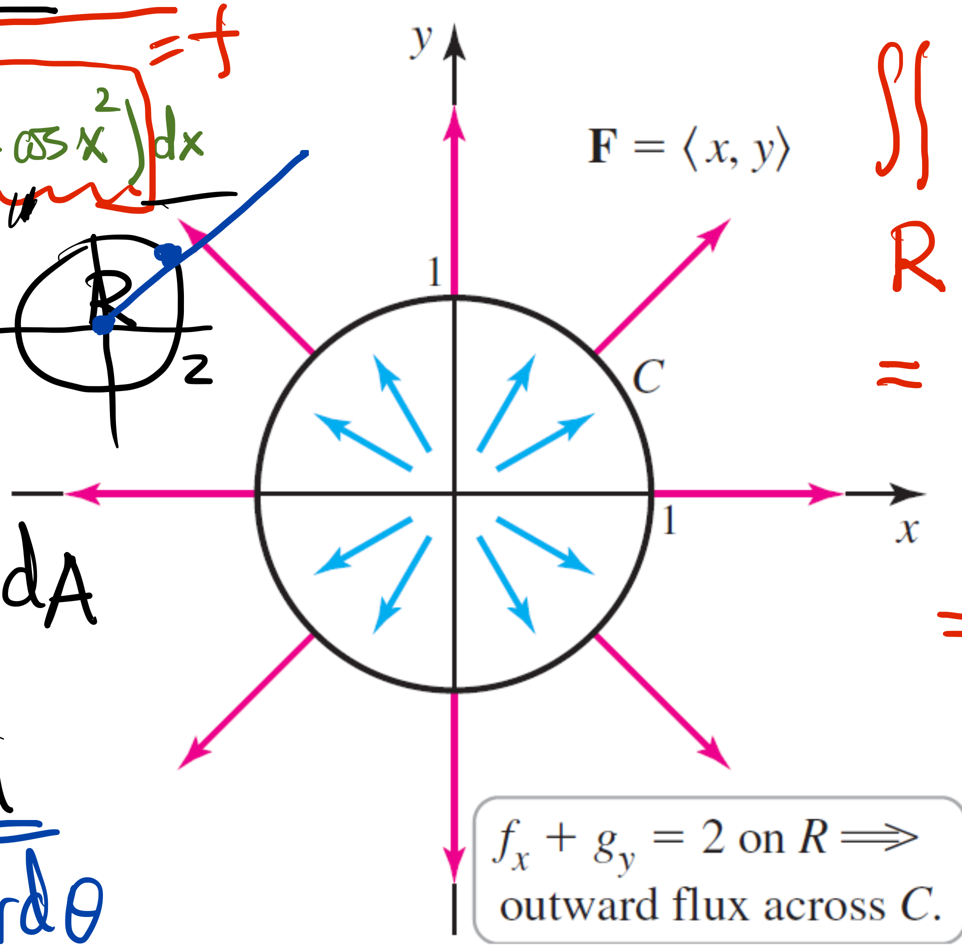
where $\mathbf{n}$ is the outward unit normal vector on the curve.

Example 3 Use Green's Thm to compute the outward flux of $\vec F = \langle x, y\rangle$ across the unit circle.

$$\oint_C (\vec F\cdot\vec n)\,ds = \oint_C x\,dy - y\,dx$$

$$= \iint_R \nabla\cdot\vec F\,dA = \iint_R \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right) dA = 2\iint_R dA = 2\,\text{area}(R) = 2\pi$$

Figure 17.34



Example 4 $C: x^2 + y^2 = 4$ oriented counterclockwise.

$$\oint_C (4x^3 + \sin y^2)\,dy - (4y^3 + \cos x^2)\,dx$$

$g = 4x^3 + \sin y^2$, $f = -(4y^3 + \cos x^2)$

$$= \oint_C f\,dx + g\,dy$$

$$= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= \iint_R (12x^2 + 12y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^2 12r^2 \cdot r\,dr\,d\theta$$

Example 5 C is the boundary of $R = \{(x,y) : 1 \le x^2+y^2 \le 9,\ y \ge 0\}$

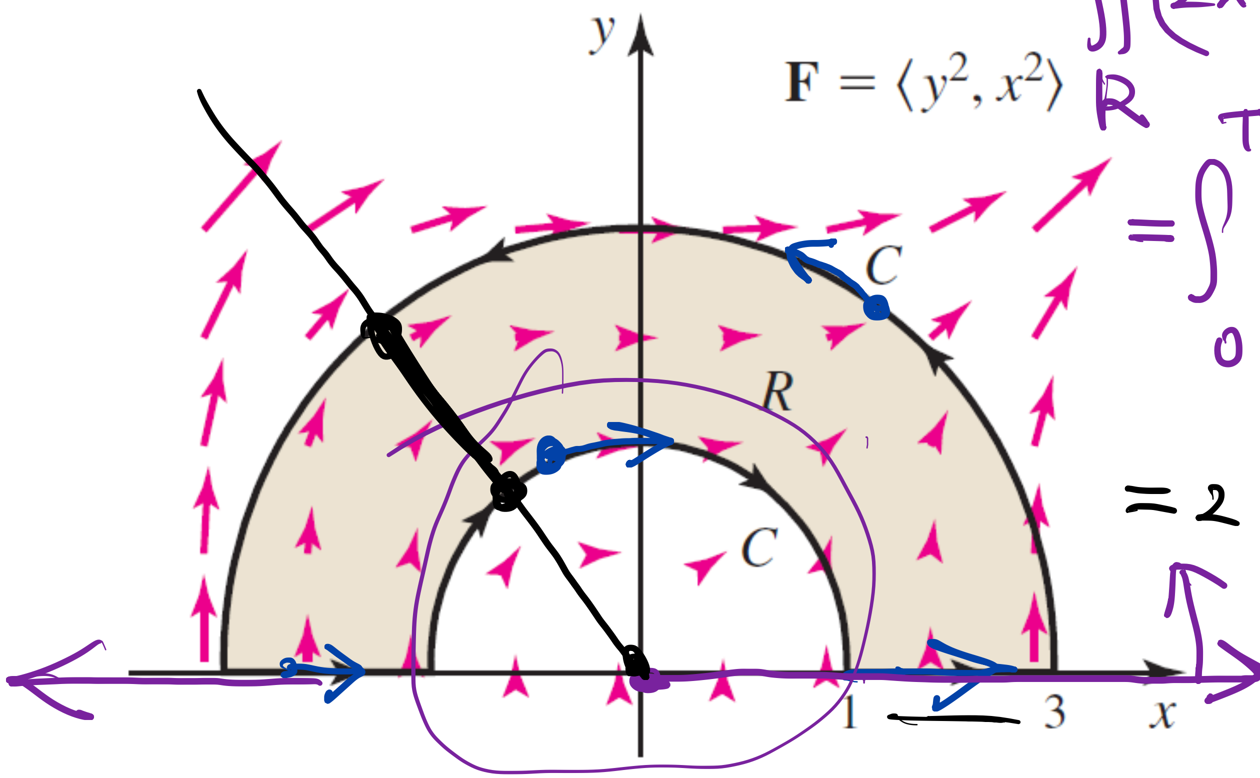
Figure 17.35

$$\oint_C y^2\,dx + x^2\,dy = \iint_R \left(\frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial y}\right) dA$$

$$= \iint_R (2x - 2y)\,dA$$

$$= \int_0^{\pi} \int_1^3 2r(\cos\theta - \sin\theta) \cdot r\,dr\,d\theta$$

$$= 2\int_0^{\pi} (\cos\theta - \sin\theta)\,d\theta \cdot \int_1^3 r^2\,dr$$



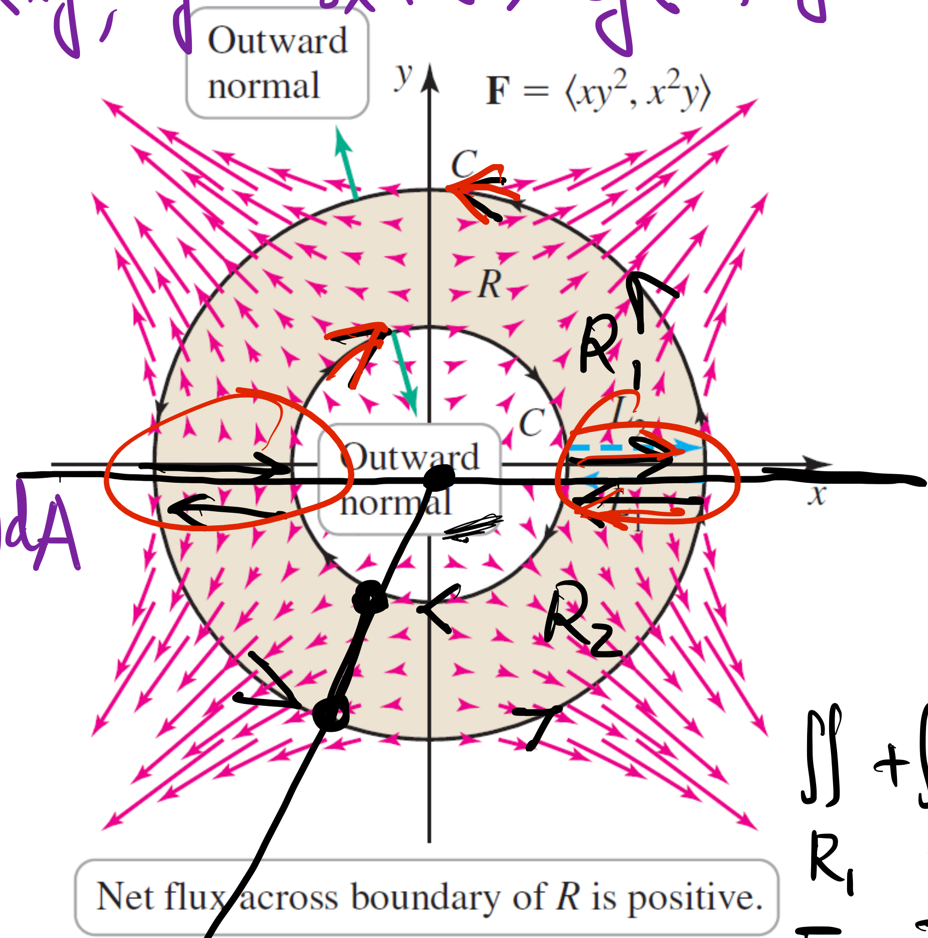
Circulation on boundary of R is negative.

Example 6 Find the outward flux of $\vec{F} = \langle xy^2, x^2y \rangle$ across the boundary of $R = \{(x,y) : 1 \le x^2+y^2 \le 4\}$.

$$\iint_R \nabla\times\vec{F}\,dA = \iint_{R_1} + \iint_{R_2} \nabla\times\vec{F}\,dA = \oint_{\partial R_1} \vec{F}\cdot d\vec{r} + \int_{\partial R_2} \vec{F}\cdot d\vec{r}$$

Figure 17.36

$$\nabla\cdot\langle xy^2, x^2y\rangle = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(x^2y) = y^2 + x^2$$



$$\oint_{\partial R} (\vec{F}\cdot\vec{n})\,ds = \iint_R \nabla\cdot\vec{F}\,dA = \iint_R (x^2+y^2)\,dA$$

$$= \int_0^{2\pi}\int_1^2 r^2\cdot r\,dr\,d\theta$$

$$\iint_{R_1} + \iint_{R_2} = \iint_R$$

Table 17.1

Conservative Fields $\mathbf{F} = \langle f, g \rangle$	Source-Free Fields $\mathbf{F} = \langle f, g \rangle$
• curl $= \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$	• divergence $= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$
• Potential function φ with $\mathbf{F} = \nabla \varphi$ or $f = \frac{\partial \varphi}{\partial x}, \quad g = \frac{\partial \varphi}{\partial y}$	• Stream function ψ with $f = \frac{\partial \psi}{\partial y}, \quad g = -\frac{\partial \psi}{\partial x}$
• Circulation $= \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed curves C.	• Flux $= \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$ on all closed curves C.
• Path independence $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$	• Path independence $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$

Table 17.2

Circulation/work integrals: $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f\,dx + g\,dy$		
	C closed	**C not closed**
F conservative ($\mathbf{F} = \nabla\varphi$)	$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$	$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$
F not conservative	Green's Theorem $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)\,dA$	Direct evaluation $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b (fx' + gy')\,dt$
Flux integrals: $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C f\,dy - g\,dx$		
	C closed	**C not closed**
F source free ($f = \psi_y, g = -\psi_x$)	$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$	$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \psi(B) - \psi(A)$
F not source free	Green's Theorem $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R (f_x + g_y)\,dA$	Direct evaluation $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (fy' - gx')\,dt$

Section 17.5 Divergence and Curl

$\vec{F} = \langle f, g, h \rangle$ $\quad \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle f, g, h \rangle$

$= f_x + g_y + h_z$ — scalar

$$\nabla \times \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$$ vector

Definition Divergence of a Vector Field

The **divergence** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.$$

If $\nabla \cdot \mathbf{F} = 0$, the vector field is **source free.**

rotational VF

Example 1 (a) $\vec{F} = \langle x, y, z \rangle$; (b) $\vec{F} = \langle -y, x - z, y \rangle$; (c) $\vec{F} = \langle -y, x, z \rangle$

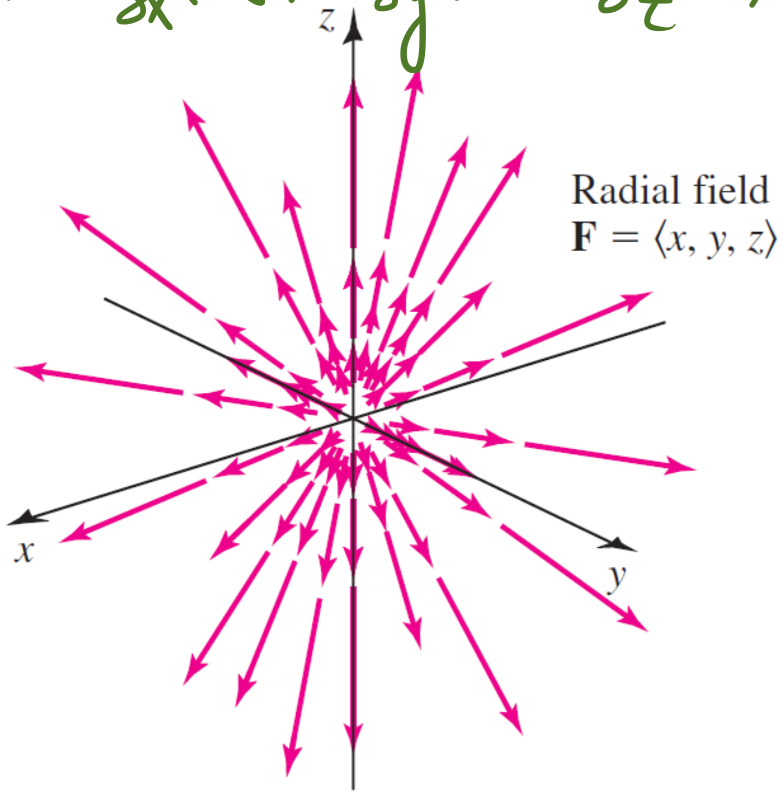
$? = \nabla \cdot \vec{F}$

(a) $\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$, (b) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x - z) + \frac{\partial}{\partial z}(y)$

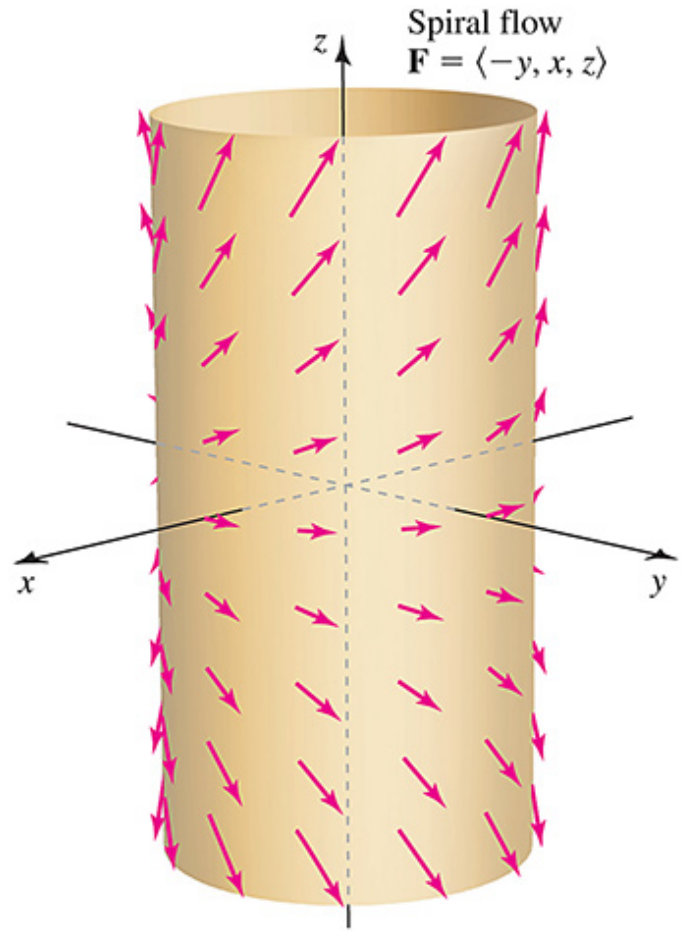
$= 0 + 0 + 0 = 0$

Figure 17.38 (a & b)

(c) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z) = 1$



$\nabla \cdot \mathbf{F} = 3$ at all points $\Longrightarrow$ vector field expands outward at all points.



Theorem 17.10 Divergence of Radial Vector Fields

For a real number *p*, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} \quad \text{is} \quad \nabla \cdot \mathbf{F} = \frac{3 - p}{|\mathbf{r}|^p}.$$

$(p=1)$

Example 2 $(p=1)$ $\nabla \cdot \vec{F} = ?$, $\vec{r} = \langle x, y, z \rangle$, $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\vec{F} = \left\langle \frac{x}{|\vec{r}|}, \frac{y}{|\vec{r}|}, \frac{z}{|\vec{r}|} \right\rangle$

$\frac{\partial}{\partial x}|\vec{r}| = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} \cdot 2x$

$= x|\vec{r}|^{-1}$

$\frac{\partial}{\partial x}\left(\frac{x}{|\vec{r}|}\right) = \frac{1 \cdot |\vec{r}| - x \frac{\partial}{\partial x}|\vec{r}|}{|\vec{r}|^2}$

$= \frac{|\vec{r}| - x^2|\vec{r}|^{-1}}{|\vec{r}|^2} = \frac{|\vec{r}|^2 - x^2}{|\vec{r}|^3} = \frac{y^2+z^2}{|\vec{r}|^3}$

$\nabla \cdot \vec{F} = \frac{y^2+z^2}{|\vec{r}|^3} + \frac{x^2+z^2}{|\vec{r}|^3} + \frac{x^2+y^2}{|\vec{r}|^3}$

$= \frac{2|\vec{r}|^2}{|\vec{r}|^3} = \frac{2}{|\vec{r}|}$

Example 3 $\vec{F} = \langle x^2, y \rangle$ and C is a circle of radius 2 centered at the origion.

(a) At Q(1,1), Is $\nabla\cdot\vec{F}$ positive or negative? Why? without computing.

(b) Compute $\nabla\cdot\vec{F}(1,1)$.

(c) Inside C, determine regions that $\nabla\cdot\vec{F}$ is pos. or neg.

Figure 17.39

$$\nabla\cdot\vec{F}(1,1) = \frac{\partial x^2}{\partial x} + \frac{\partial y}{\partial y}\Big|_{(1,1)} = (2x+1)_{(1,1)} = 3 > 0$$

$$\vec{F} = \langle f, g, h \rangle$$

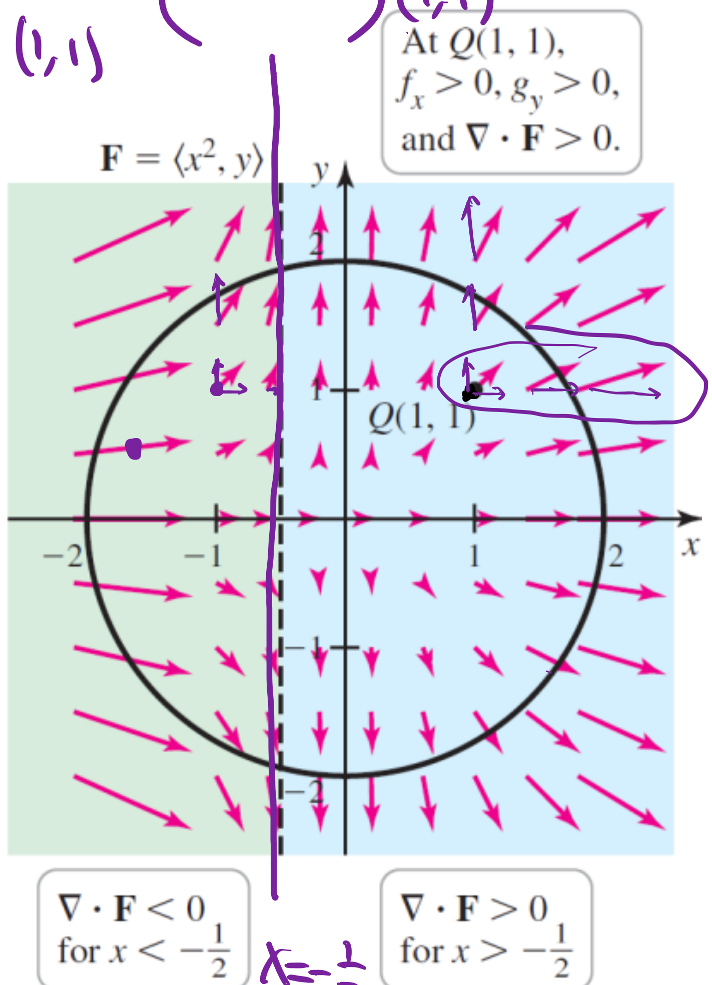
$$\nabla\cdot\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\nabla\cdot\langle f, g\rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

(c) $2x+1 > 0$

$x > -\frac{1}{2} \Rightarrow \nabla\cdot\vec{F} > 0$

$x < -\frac{1}{2}$, $\nabla\cdot\vec{F} < 0$



$f(x) \nearrow \Rightarrow f'(x) > 0$

Definition Curl of a Vector Field

The **curl** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\nabla \times \mathbf{F} = \text{curl } \mathbf{F}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)\mathbf{k}.$$

If $\nabla \times \mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

$\vec{F} = \langle f, g \rangle$

$\nabla \times \vec{F} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \vec{k}$

- General Rotation Vector Field $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{r} = \langle x, y, z \rangle$

$? = \nabla \times \vec{F}$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \langle a_2 z - a_3 y,\ a_3 x - a_1 z,\ a_1 y - a_2 x \rangle$$

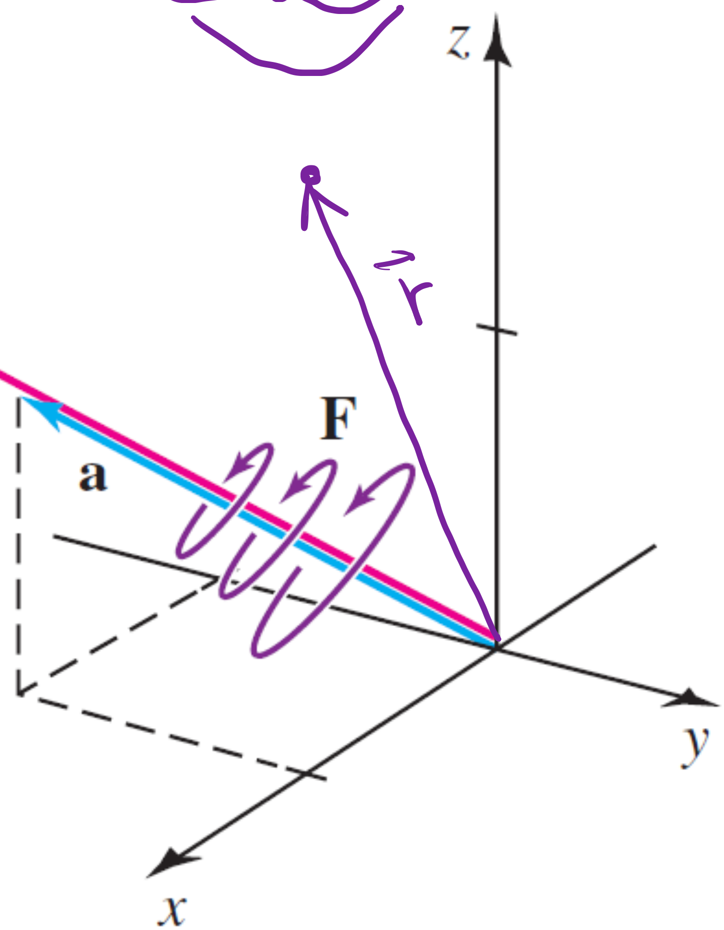
Figure 17.41

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \langle a_1 + a_1,\ a_2 + a_2,\ a_3 + a_3 \rangle$$

$$= 2\langle a_1, a_2, a_3 \rangle$$

$$= 2\vec{a}$$



General rotation field

$\mathbf{a} = \langle 1, -1, 1 \rangle$
$\mathbf{F} = \mathbf{a} \times \mathbf{r}$
$\nabla \times \mathbf{F} = 2\mathbf{a}$

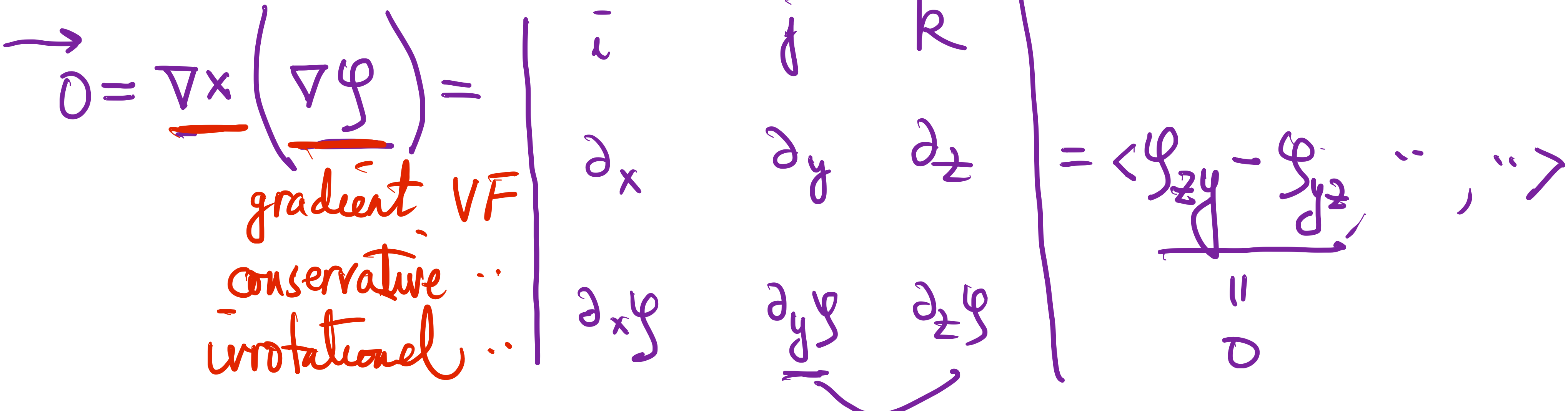
General Rotation Vector Field

The **general rotation vector field** is $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where the nonzero constant vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is the axis of of rotation and $\mathbf{r} = \langle x, y, z \rangle$. For all nonzero choices of **a**, $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$ and $\nabla \cdot \mathbf{F} = 0$. If **F** is a velocity field, then the constant angular speed of the field is

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}|.$$

Theorem 17.11 Curl of a Conservative Vector Field

Suppose **F** is a conservative vector field on an open region *D* of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on *D*. Then $\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \mathbf{0}$: The curl of the gradient is the zero vector and **F** is irrotational.



Theorem 17.12 Divergence of the Curl

Suppose $\mathbf{F} = \langle f, g, h \rangle$ where *f*, *g*, and *h* have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Theorem 17.13 Product Rule for the Divergence

Let u be a scalar-valued function that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F}).$$

Properties of a Conservative Vector Field

Let **F** be a conservative vector field whose components have continuous second partial derivatives on an open connected region *D* in

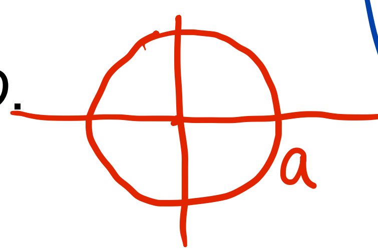
∘ ^{3}. Then **F** has the following equivalent properties.

1. There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ (definition).

2. $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points *A* and *B* in *D* and all piecewise-smooth oriented curves *C* in *D* from *A* to *B*.

3. $\oint_C \mathbf{F} \cdot dr = 0$ on all simple piecewise-smooth closed oriented curves *C* in *D*.

4. $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of *D*.



Section 17.6 Surface Integrals

① graph $y=f(x)$, $a \le x \le b$

$\vec{r}(x) = \langle x, f(x) \rangle$, $x \in [a,b]$

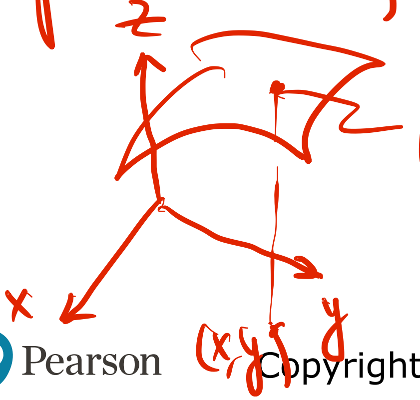
② level curve $f(x,y)=c$

③ parametrized curve.

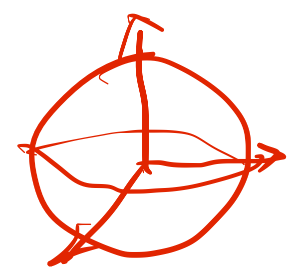
$\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$

Representation of Surfaces

- graph $z = f(x,y)$, $(x,y) \in R$



- level surfaces $f(x,y,z) = c$

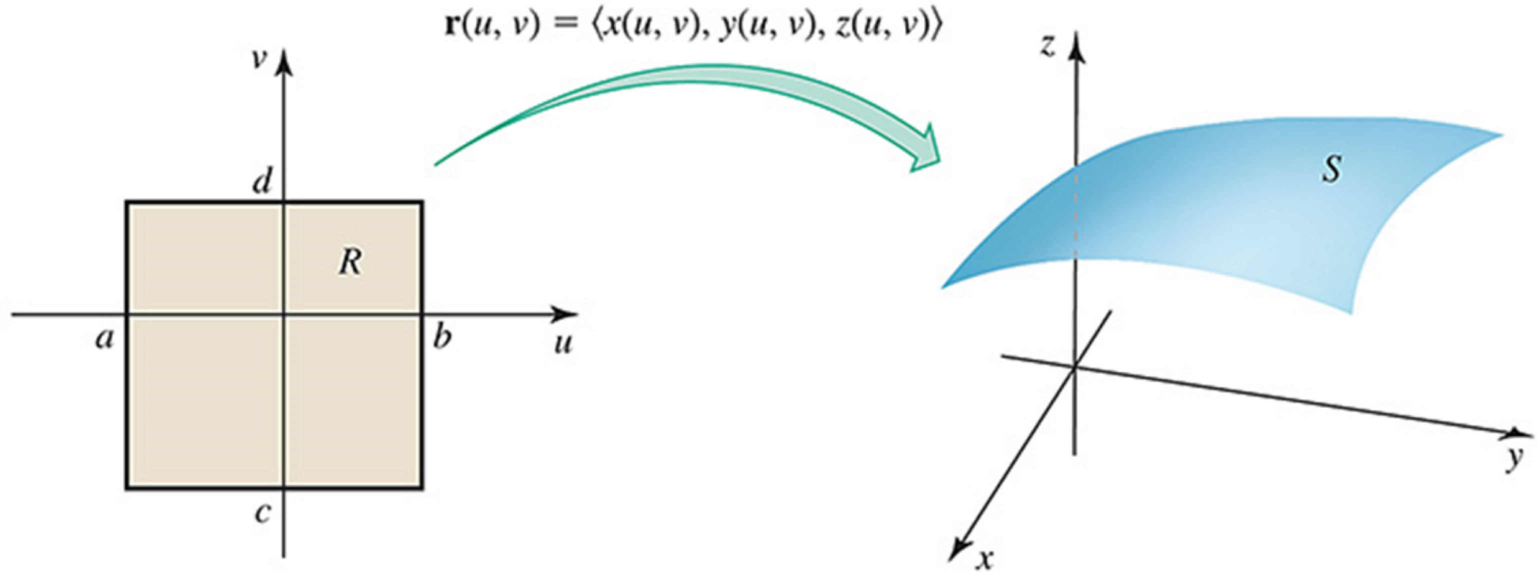


Parametrized Surfaces

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$(u,v) \in R$

Figure 17.43



A rectangle in the *uv*-plane is mapped to a surface in *xyz*-space.

Cylinders

$x^2 + y^2 = a^2 \iff r = a$

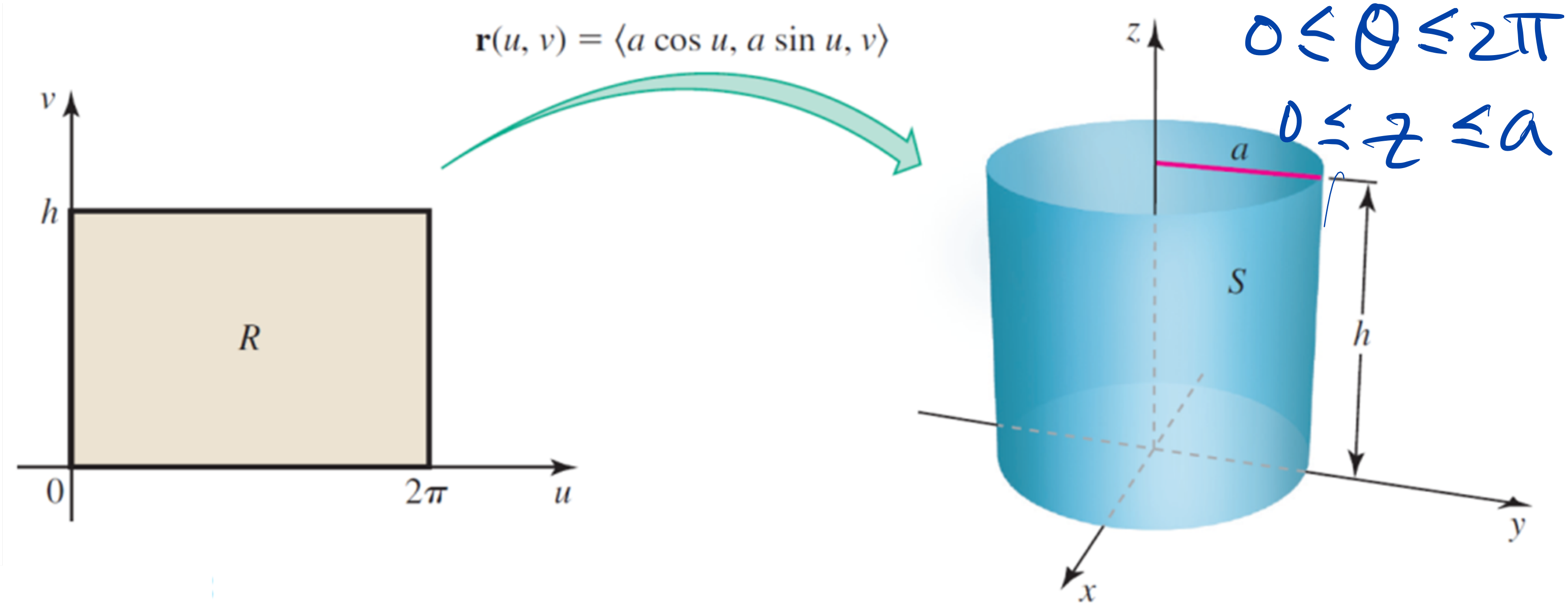
Figure 17.44

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \xRightarrow{r=a} \begin{cases} x = a\cos\theta \\ y = a\sin\theta \\ z = z \end{cases}$$

$$\vec{r}(\theta, z) = \langle a\cos\theta, a\sin\theta, z \rangle$$

$0 \le \theta \le 2\pi$

$0 \le z \le a$



Cones

$$z=\frac{h}{a}\sqrt{x^2+y^2}$$

$$\cdot\ \vec{r}(x,y)=\left\langle x,\ y,\ \frac{h}{a}\sqrt{x^2+y^2}\right\rangle,\quad (x,y)\in R$$

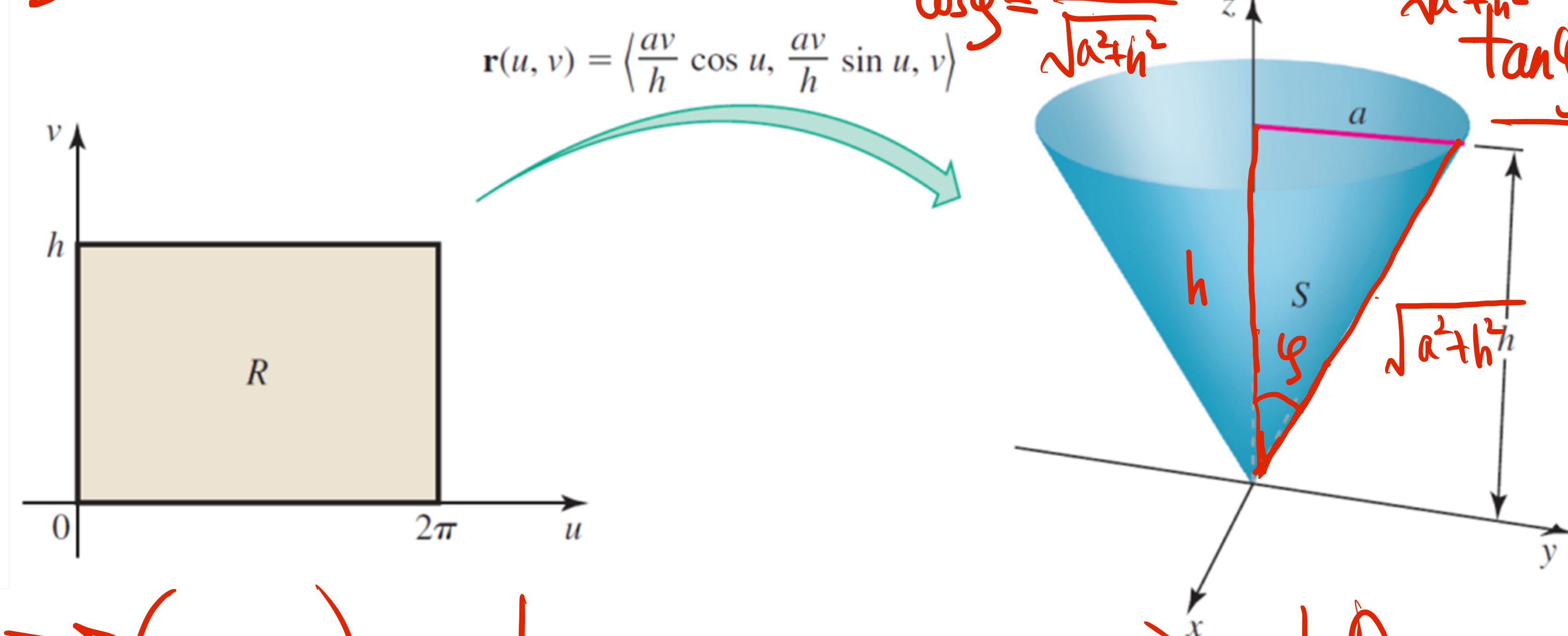
$$\varphi=\arctan\frac{a}{h}$$

Figure 17.45

$$\rho\cos\varphi=\frac{h}{a}\rho\sin\varphi$$

$$\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases} \xrightarrow{\varphi=\arctan\frac{a}{h}} \begin{cases} x=\frac{a}{\sqrt{a^2+h^2}}\rho\cos\theta \\ y=\frac{a}{\sqrt{a^2+h^2}}\rho\sin\theta \\ z=\frac{h}{\sqrt{a^2+h^2}}\rho \end{cases}$$

$$\sin\varphi=\frac{a}{\sqrt{a^2+h^2}}\qquad \cos\varphi=\frac{h}{\sqrt{a^2+h^2}}\qquad \tan\varphi=\frac{a}{h}$$



$$\vec{r}(\rho,\theta)=\frac{1}{\sqrt{a^2+h^2}}\left\langle a\rho\cos\theta,\ a\rho\sin\theta,\ h\rho\right\rangle$$

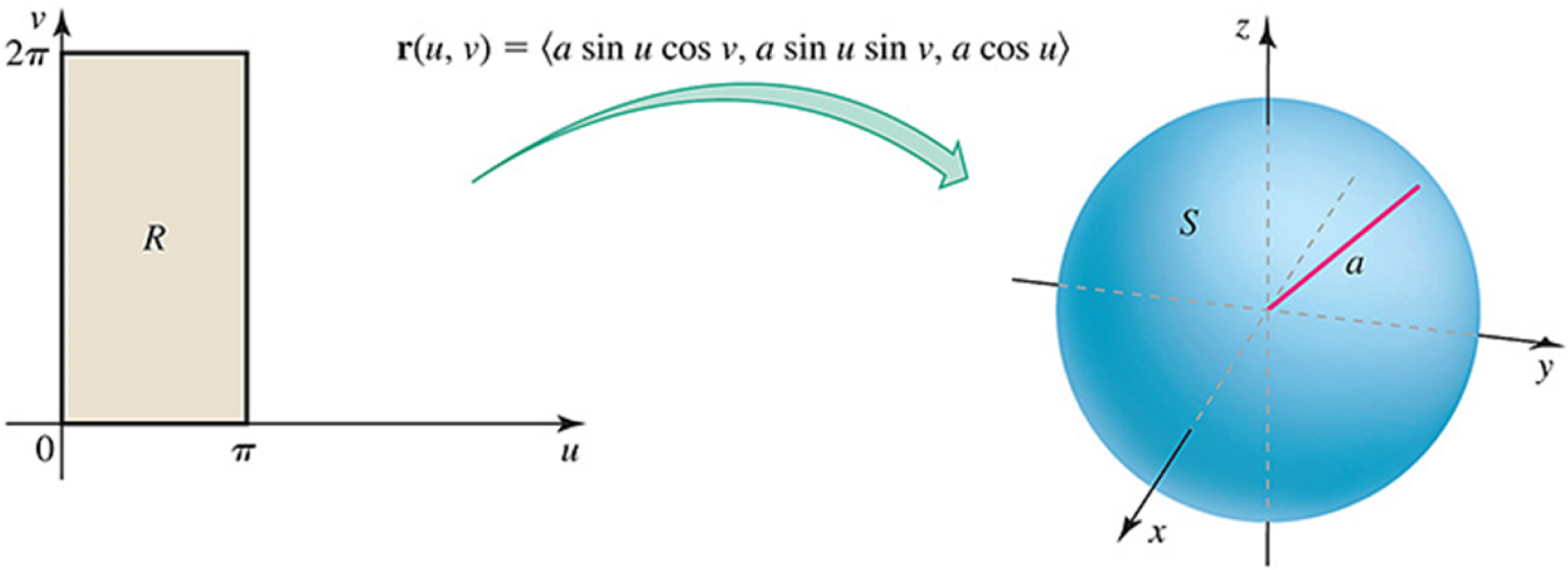
$$0\le\theta\le 2\pi,\quad 0\le\rho\le\sqrt{a^2+h^2}$$

Spheres $x^2+y^2+z^2=a^2 \iff \rho = a$

$\vec{r}(\varphi,\theta) = \langle a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi \rangle$

$0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$

Figure 17.46



Example 1 Find parametric descriptions for the following surfaces.

(a) the plane $3x - 2y + z = 2$

(b) the paraboloid $z = x^2 + y^2$, for $0 \le z \le 9$

(a) $z = 2 - 3x + 2y$

$\vec{r}(x, y) = \langle x, y, 2 - 3x + 2y \rangle$, $(x, y) \in \mathbb{R}$

(b) $\vec{r}(x, y) = \langle x, y, x^2 + y^2 \rangle$, $R: 0 \le x^2 + y^2 \le 9$

$\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$

Surface Integrals of Scalar-Valued Functions

$$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \quad (u,v) \in R$$

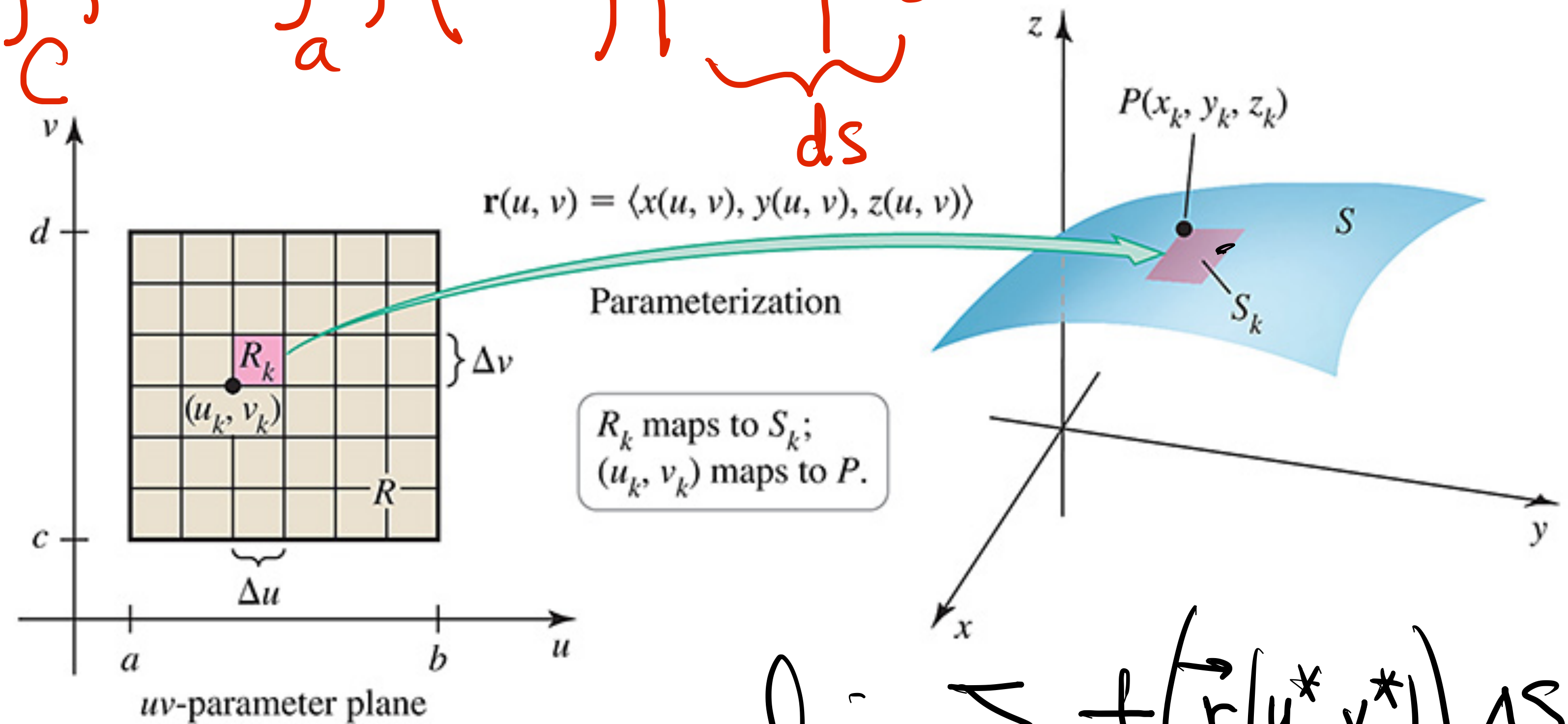
$$\iint_S f\, dS = \iint_R f(\vec{r}(u,v))\, \underbrace{|\vec{r}_u \times \vec{r}_v|\, du\, dv}_{dS}$$

$$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad t \in [a,b]$$

$$\iint_R f\, dA$$

$$\int_C f\, ds = \int_a^b f(\vec{r}(t))\, \underbrace{|\vec{r}\,'(t)|\, dt}_{ds}$$

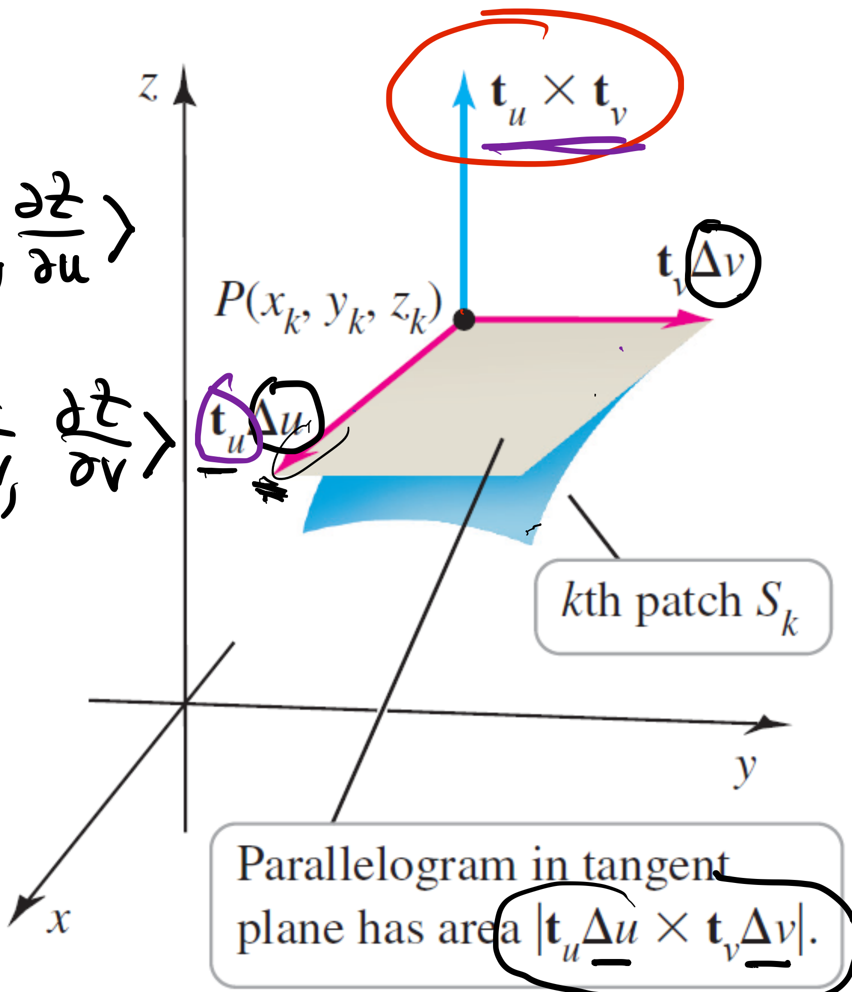
Figure 17.47



$$\lim_{\Delta S_k \to 0} \sum_k f(\vec{r}(u_k^*, v_k^*))\, \Delta S_k$$

$$\Delta S_k \approx |\vec{r}_u \times \vec{r}_v|\, \Delta u\, \Delta v$$

Figure 17.48



$$\vec{t}_u = \frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$$\vec{t}_v = \frac{\partial \vec{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

$$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

$\vec{r}'(t)$

$\vec{r}(t) = \langle x(t), y, z \rangle$

$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

Definition Surface Integral of Scalar-Valued Functions on Parameterized Surfaces

Let *f* be a continuous scalar-valued function on a smooth surface *S* given parametrically by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where *u* and *v* vary over $R = \{(u,v) : a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$ are continuous on *R* and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on *R*. Then the **surface integral** of *f* over *S* is

$$\iint_S f(x,y,z)\,dS = \iint_R f(x(u,v), y(u,v), z(u,v))\left|\mathbf{t}_u \times \mathbf{t}_v\right| dA.$$

If $f(x,y,z) = 1$, this integral equals the surface area of *S*.

Example 2 Find the surface area.

(a) A cylinder with radius $a>0$ and height h (excluding the circular ends)

Figure 17.49

$$A(S)=\iint_S 1\,dS = \iint_R a\,du\,dv = \int_0^{2\pi} du \int_0^h a\,dv = a\cdot 2\pi\cdot h$$

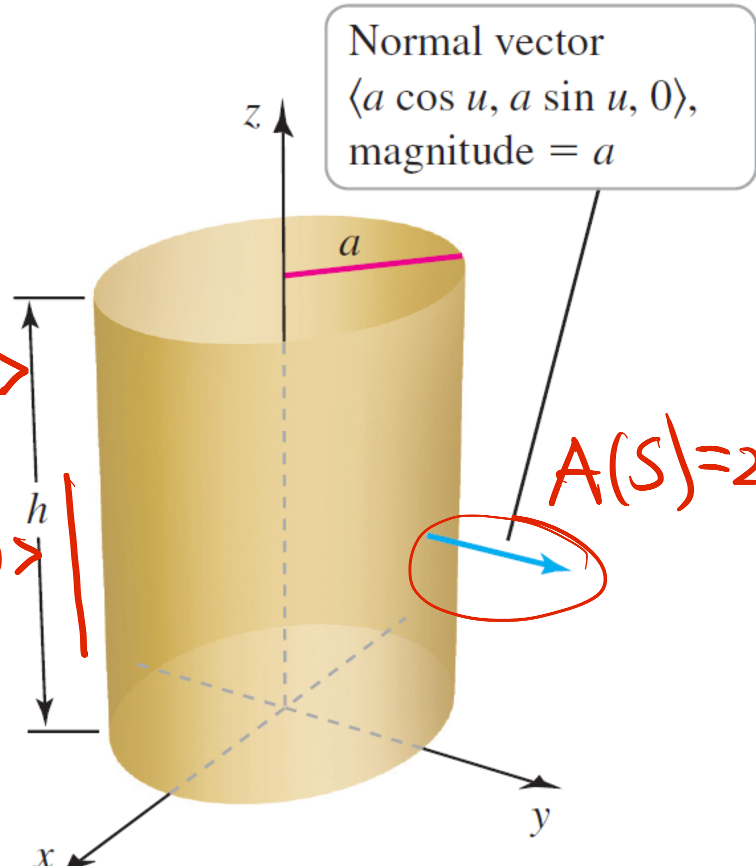
$$\vec{t}_u = \langle -a\sin u,\ a\cos u,\ 0\rangle$$

$$\vec{t}_v = \langle 0,\ 0,\ 1\rangle$$

$$\vec{t}_u \times \vec{t}_v = \langle a\cos u,\ a\sin u,\ 0\rangle$$

$$|\vec{t}_u \times \vec{t}_v| = a\,|\langle \cos u, \sin u, 0\rangle| = a$$

$A(S) = 2\pi r h +$



Cylinder: $\mathbf{r}(u, v) = \langle a\cos u, a\sin u, v\rangle$,
$0 \le u \le 2\pi$ and $0 \le v \le h$

$u=\theta,\ v=z$

(b) a sphere of radius a. $A(S) = 4\pi a^2$

$$A(S) = \iint_S 1\,dS = \int_0^{\pi}\int_0^{2\pi} 1\cdot a^2 \sin u\,dv\,du$$

$$= a^2\cdot 2\pi \int_0^{\pi} \sin u\,du$$

$$= 2\pi a^2 \cdot [-\cos u]_0^{\pi} = 4\pi a^2$$

Figure 17.50

Sphere: $u = \varphi,\ v = \theta$

$\mathbf{r}(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle,$

$0 \le u \le \pi$ and $0 \le v \le 2\pi$: R

$\vec{t}_u = \langle a\cos u\cos v, a\cos u\sin v, -a\sin u\rangle$

$\vec{t}_v = \langle -a\sin u\sin v, a\sin u\cos v, 0\rangle$

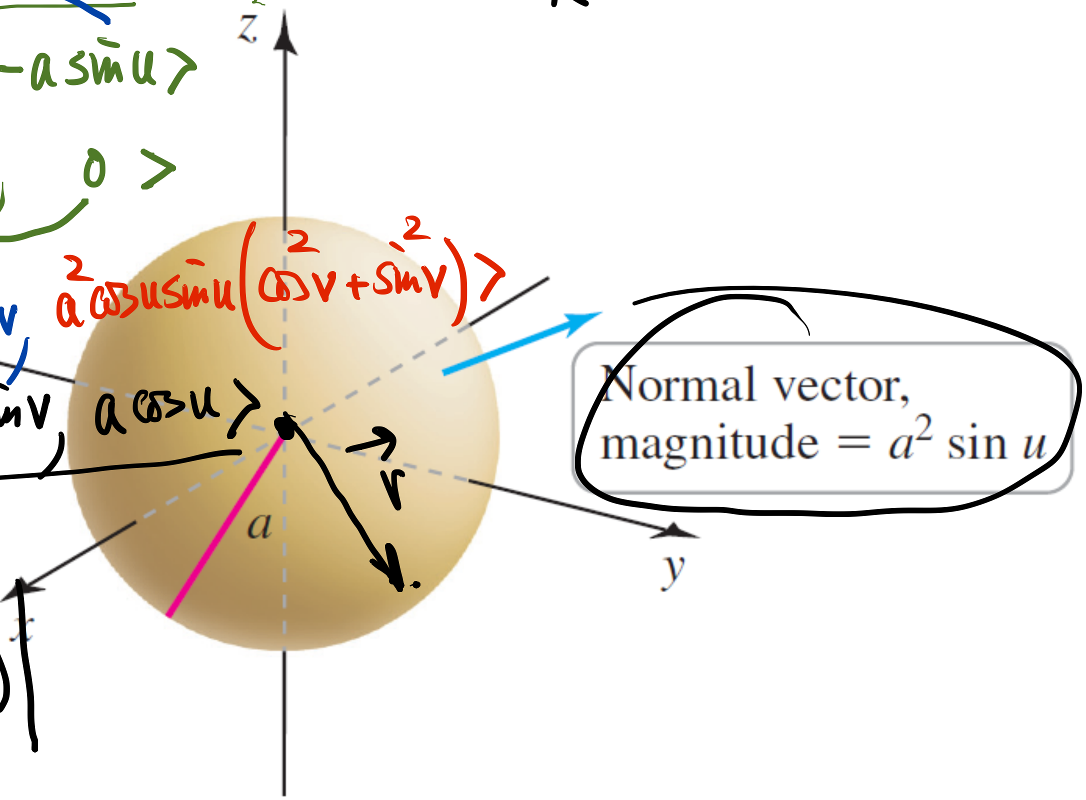
$\vec{t}_u \times \vec{t}_v = \langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u(\cos^2 v + \sin^2 v)\rangle$

$= a\sin u\langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$

$= a\sin u\,\vec{r}(u,v)$

$|\vec{t}_u \times \vec{t}_v| = a\sin u\,|\vec{r}(u,v)|$

$= a^2\sin u$



Example 3 Find the surface area of the cylinder $\{(r,\theta): r=4,\ 0\le\theta\le 2\pi\}$ between the planes $z=0$ and $z=16-2x$ (excluding the top and bottom surfaces).

$= 16-2\cdot 4\cos u$

Figure 17.51

$\vec{r}_u \times \vec{r}_v = \langle 4\cos u,\ 4\sin u,\ 0\rangle$

$|\vec{r}_u \times \vec{r}_v| = 4$

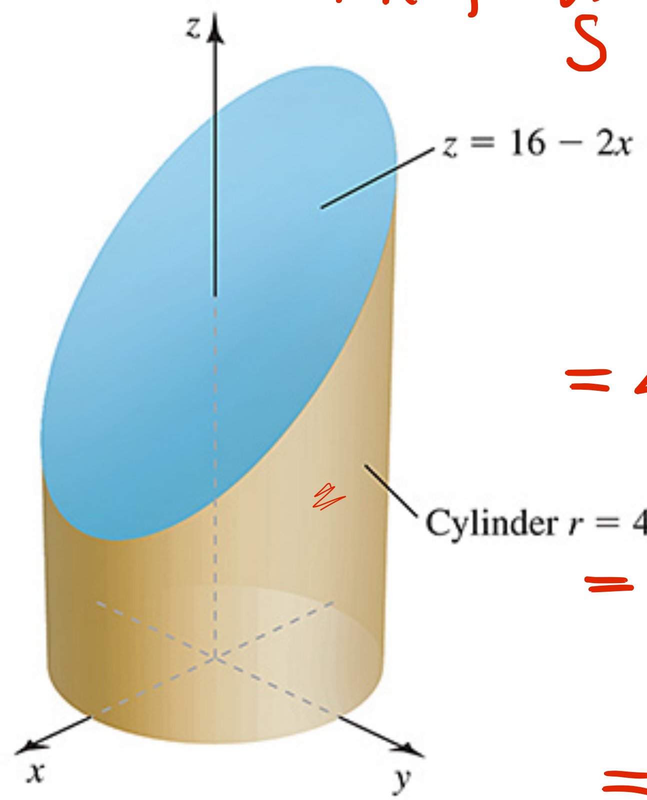
$A(S) = \iint_S 1\,dS = \iint_R 4\,du\,dv$

$= \int_0^{2\pi}\int_0^{16-8\cos u} 4\,dv\,du$

$= 4\int_0^{2\pi}(16-8\cos u)\,du$

$= 4\left[16u - 8\sin u\right]_0^{2\pi}$

$= 4\cdot 16\cdot 2\pi$



Sliced cylinder is generated by $\mathbf{r}(u,v) = \langle 4\cos u, 4\sin u, v\rangle$, where $0 \le u \le 2\pi$, $0 \le v \le 16 - 8\cos u$.

$u=\theta$, $v=z$

Surface Integrals on Graph

$S: z = g(x,y),\ (x,y) \in R$

Figure 17.52

$$\iint_S f\,dS = \iint_R f(x,y,g(x,y))\,|\vec{t}_x \times \vec{t}_y|\,dx\,dy$$

$$= \iint_R f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\,dx\,dy$$

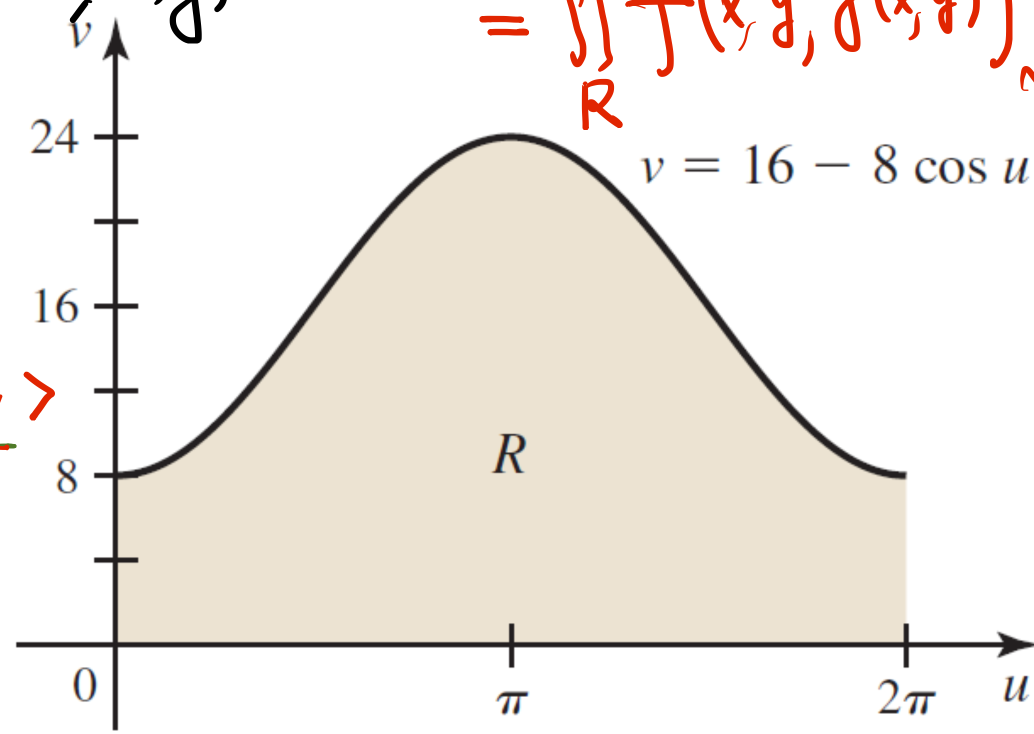
$$\sqrt{1+z_x^2+z_y^2}$$

$\vec{r}(x,y) = \langle x, y, g(x,y)\rangle,\ (x,y) \in R$

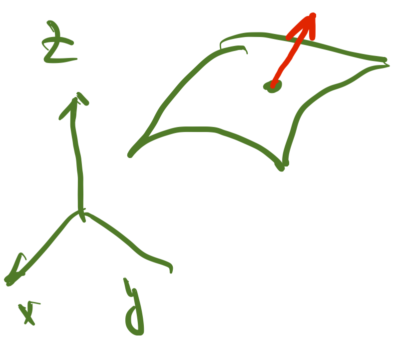
$\vec{t}_x = \frac{\partial \vec{r}}{\partial x} = \langle 1, 0, g_x\rangle$

$\vec{t}_y = \frac{\partial \vec{r}}{\partial y} = \langle 0, 1, g_y\rangle$

$\vec{t}_x \times \vec{t}_y = \langle -g_x, -g_y, 1\rangle$



Region of integration in the uv-plane is $R = \{(u, v): 0 \le u \le 2\pi,\ 0 \le v \le 16 - 8\cos u\}$.



Theorem 17.14 Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let *f* be a continuous function on a smooth surface *S* given by $z = g(x, y)$, for (x, y) in a region *R*. The surface integral of *f* over *S* is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\underbrace{\sqrt{z_x^2 + z_y^2 + 1}\, dA}_{dS}.$$

If $f(x, y, z) = 1$, the surface integral equal the area of the surface.

Example 5 Find the area of the surface S that lies in the plane $z = 12 - 4x - 3y$ directly above the region R bounded by the ellipse $\frac{x^2}{4} + y^2 = 1$.

Figure 17.53

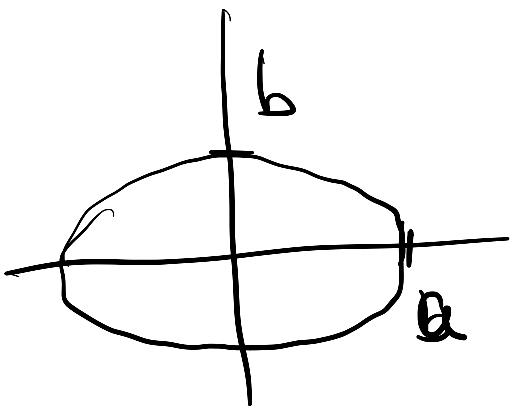
$$A(S) = \iint_S 1\,dS = \iint_R \sqrt{1 + z_x^2 + z_y^2}\,dx\,dy$$

$$= \iint_R \sqrt{1 + (-4)^2 + (-3)^2}\,dx\,dy$$

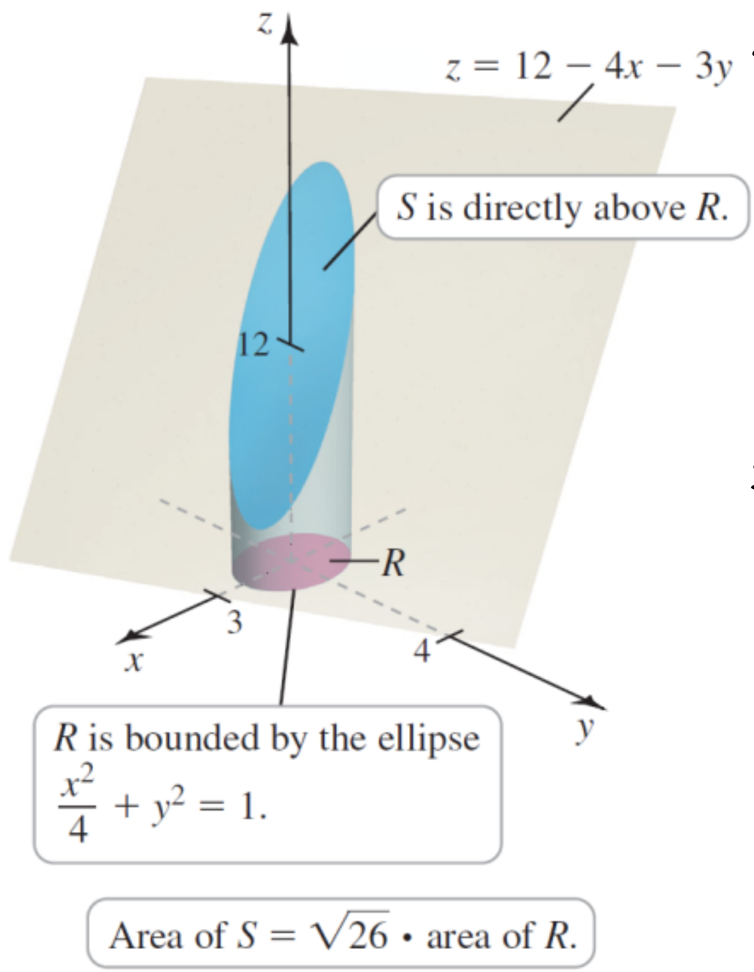
$$= \sqrt{26} \iint_R dx\,dy$$

$$= \sqrt{26}\,A(R)$$

$$= \sqrt{26}\,\pi \cdot 2 \cdot 1$$



$\pi a b$



Example 6 A thin conical sheet is described by the surface $z=(x^2+y^2)^{1/2}$, for $0\le z\le 4$. The density is $\rho(x,y,z)=8-z$ g/cm². What is the mass of the cone?

Figure 17.54

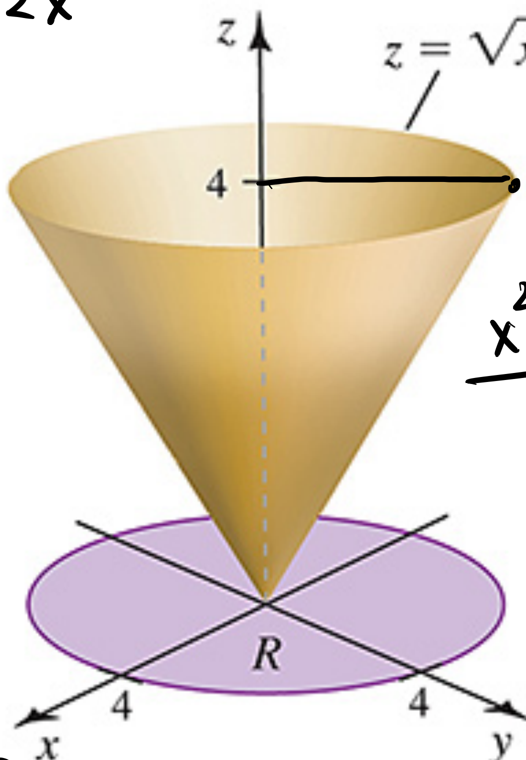
$$m=\iint_S (8-z)\,dS=\iint_R \left(8-(x^2+y^2)^{\frac12}\right)\sqrt{1+z_x^2+z_y^2}\,dx\,dy$$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2)^{\frac12}=\frac12(x^2+y^2)^{-\frac12}\cdot 2x=\frac{x}{\sqrt{x^2+y^2}}$$

$$\left(\frac{\partial z}{\partial x}\right)^2=\frac{x^2}{x^2+y^2}$$

$$\left(\frac{\partial z}{\partial y}\right)^2=\frac{y^2}{x^2+y^2}$$

$$1+z_x^2+z_y^2=1+\frac{x^2+y^2}{x^2+y^2}=2$$



Density function of sheet is $\rho = 8 - z$.

$$4=\sqrt{x^2+y^2}$$

$$x^2+y^2=4^2$$

$$=\sqrt2\iint_R \left(8-(x^2+y^2)^{\frac12}\right)dx\,dy$$

$$=\sqrt2\int_0^{2\pi}\int_0^4 (8-r)\,r\,dr\,d\theta$$

$$=\sqrt2\cdot 2\pi\int_0^4 (8r-r^2)\,dr$$

$$=2\sqrt2\,\pi\left[4r^2-\frac13 r^3\right]_0^4$$

Table 17.3

	Explicit Description $z=g(x,y)$		Parametric Description	
Surface	**Equation**	**Normal Vector; magnitude** $\pm\langle -z_x, -z_y, 1\rangle; \lvert\langle -z_x, -z_y, 1\rangle\rvert$	**Equation**	**Normal Vector; magnitude** $t_u \times t_v; \lvert t_u \times t_v\rvert$
Cylinder	$x^2+y^2=a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle; a$	$\mathbf{r} = \langle a\cos u, a\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle a\cos u, a\sin u, 0\rangle; a$
Cone	$z^2 = x^2+y^2$, $0 \le z \le h$	$\langle x/z, y/z, -1\rangle; \sqrt{2}$	$\mathbf{r} = \langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle v\cos u, v\sin u, -v\rangle; \sqrt{2}v$
Sphere	$x^2+y^2+z^2=a^2$	$\langle x/z, y/z, 1\rangle; a/z$	$\mathbf{r} = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$, $0 \le u \le \pi, 0 \le v \le 2\pi$	$\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\sin u\cos u\rangle; a^2\sin u$
Paraboloid	$z = x^2+y^2$, $0 \le z \le h$	$\langle 2x, 2y, -1\rangle; \sqrt{1+4(x^2+y^2)}$	$\mathbf{r} = \langle v\cos u, v\sin u, v^2\rangle$, $0 \le u \le 2\pi, 0 \le v \le \sqrt{h}$	$\langle 2v^2\cos u, 2v^2\sin u, -v\rangle; v\sqrt{1+4v^2}$

$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \ (u,v) \in R$

$$\iint_S f\,dS = \iint_R f(\vec{r}(u,v))\,\underbrace{|\vec{t}_u \times \vec{t}_v|\,du\,dv}_{dS}$$

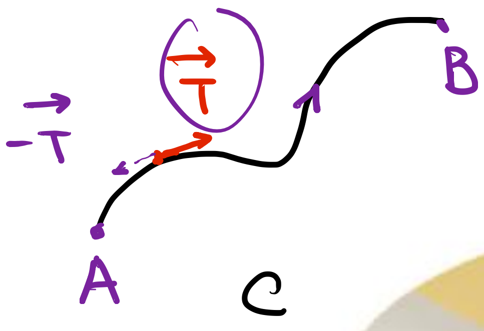
Figure 17.55

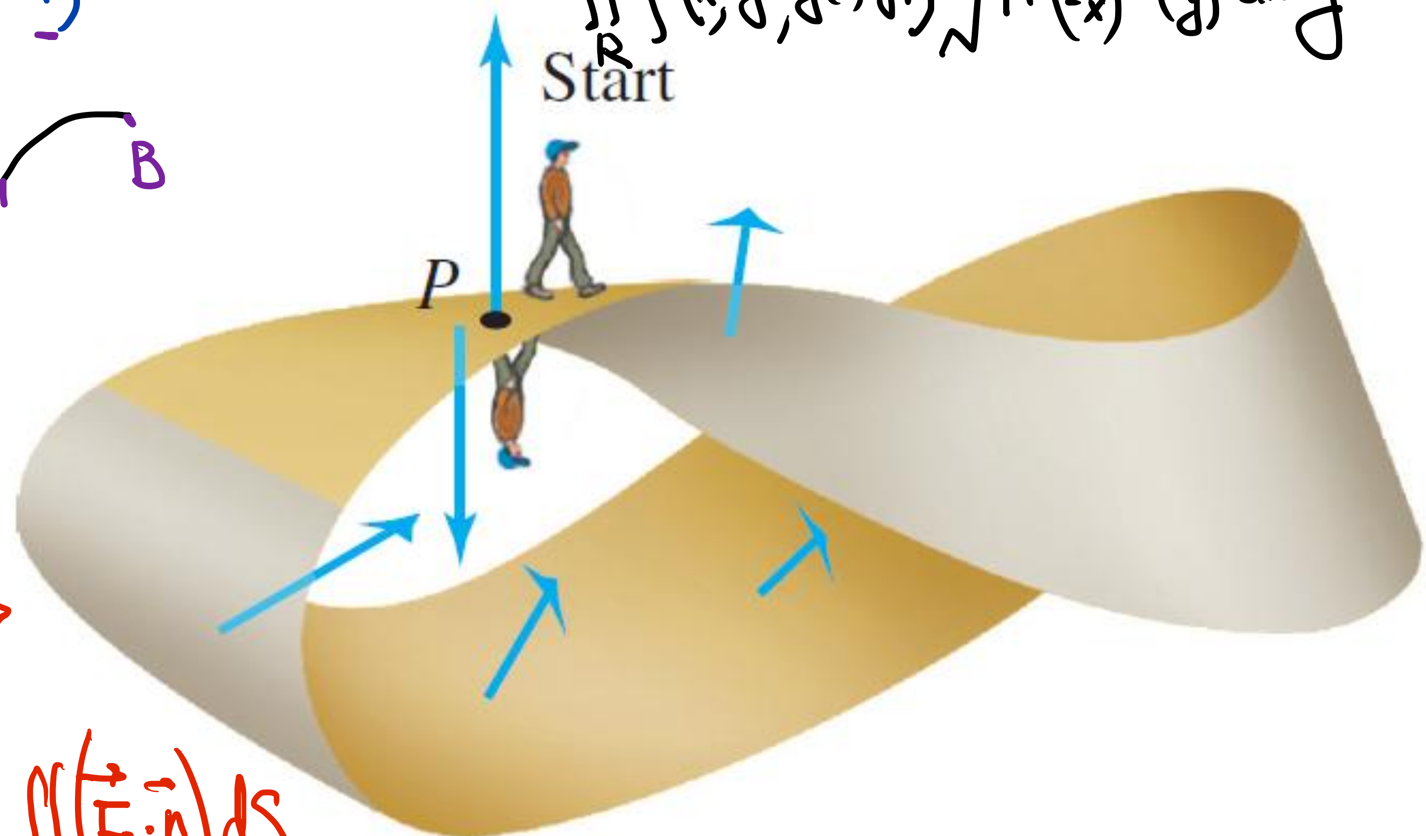
$S: z = g(x,y), \ (x,y) \in R$

$$\iint_R f(x,y,g(x,y))\sqrt{1 + (z_x)^2 + (z_y)^2}\,dx\,dy$$

$\vec{t}_x \times \vec{t}_y = \langle -z_x, -z_y, 1 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \vec{T})\,ds$$

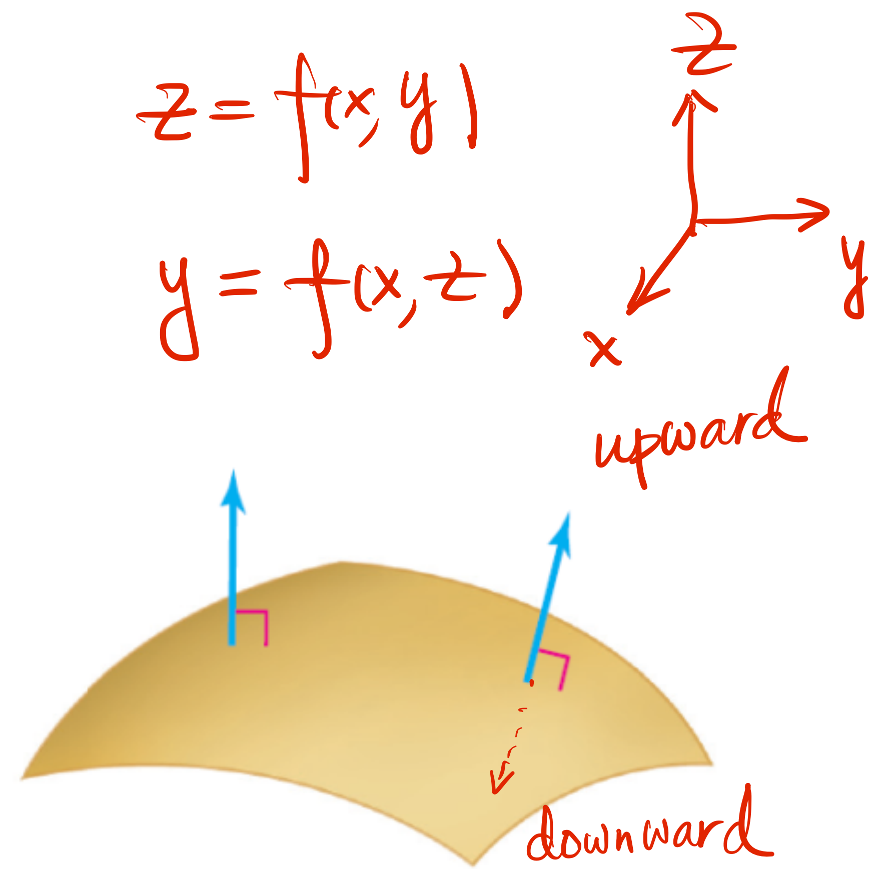


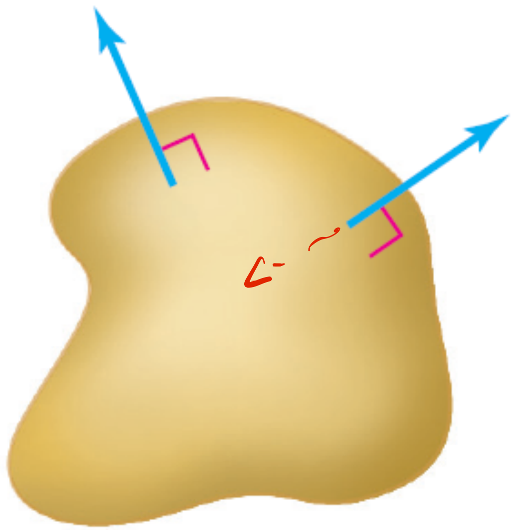


$\vec{F} = \langle f, g, h \rangle$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n})\,dS$$

Figure 17.56





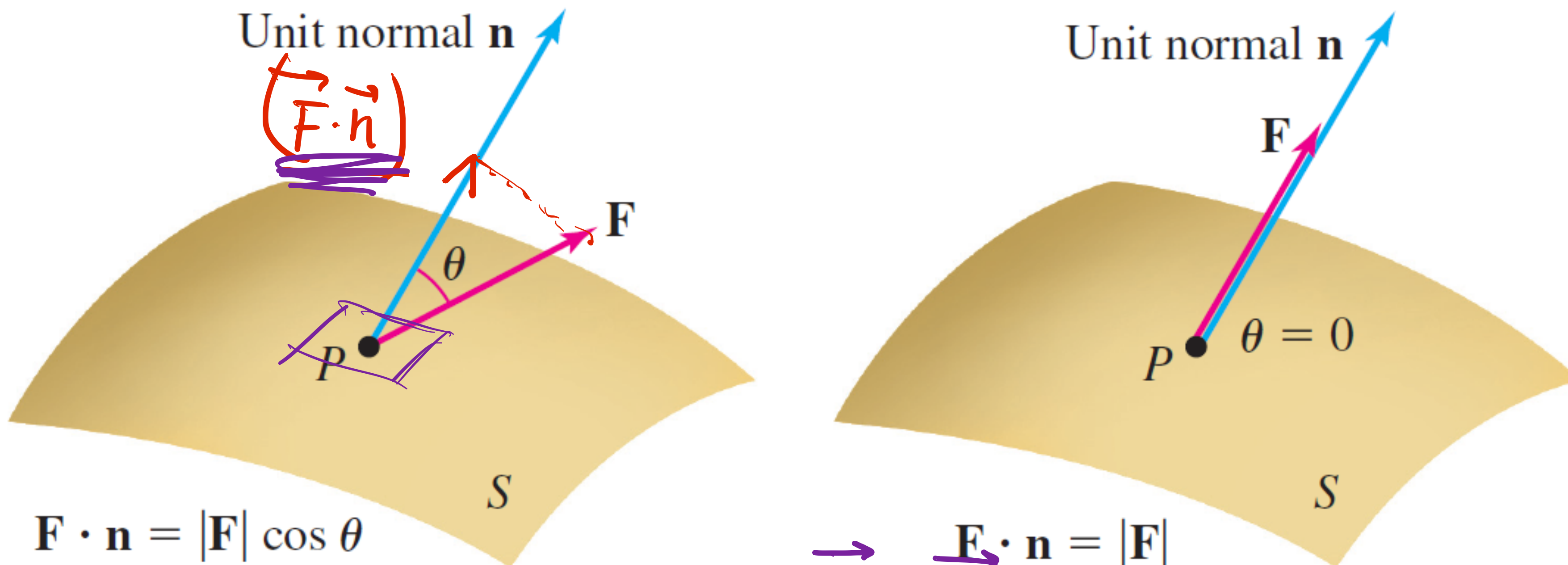
Surfaces that enclose a region are oriented so normal vectors point in the outward direction.

For other surfaces, the orientation of the surface must be specified.

Figure 17.57 (a & b)

$$\int_C (\vec{F}\cdot\vec{T})\,ds \text{ in } R^d \quad d=2,3$$

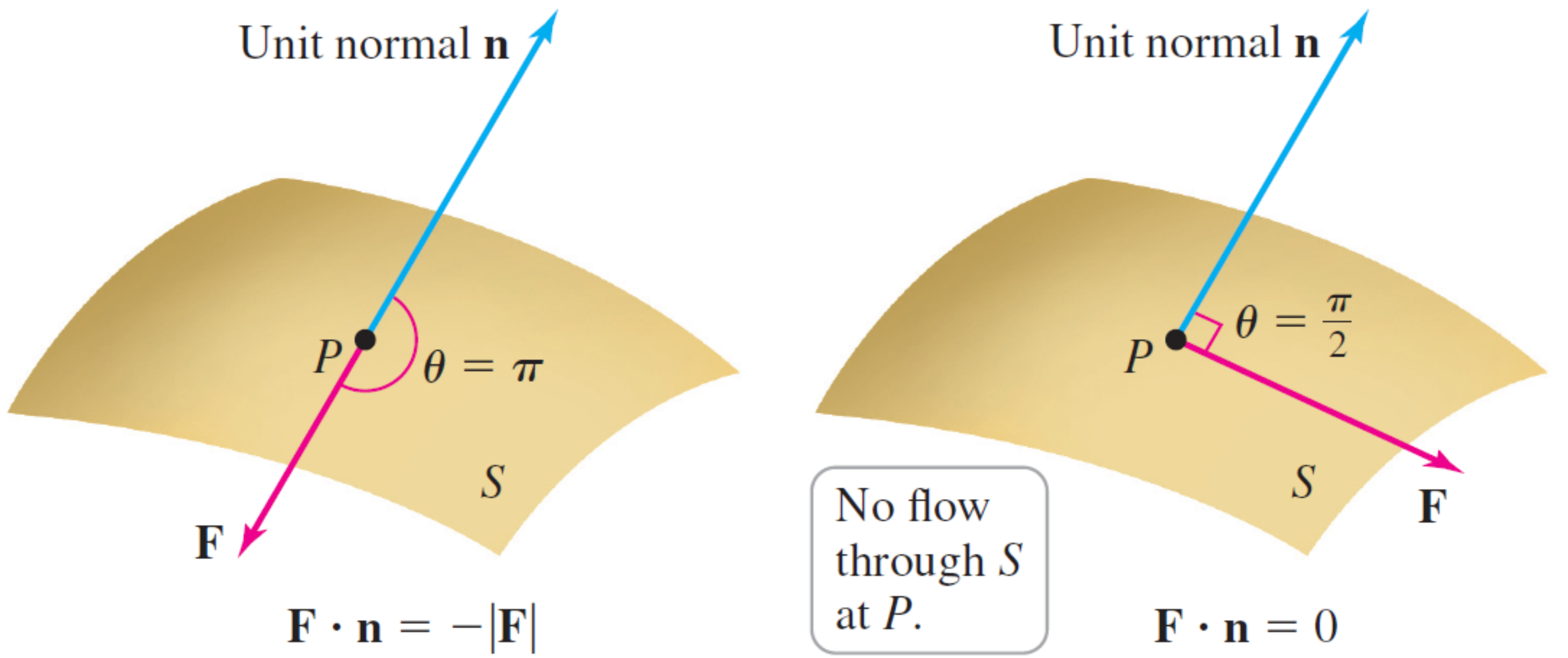
$$\int_C (\vec{F}\cdot\vec{n})\,ds$$



$$\iint_S (\vec{F}\cdot\vec{n})\,dS = \iint_R \vec{F}(\vec{r}(u,v))\cdot\frac{\vec{t}_u\times\vec{t}_v}{|\vec{t}_u\times\vec{t}_v|}\,|\vec{t}_u\times\vec{t}_v|\,du\,dv$$

$$= \iint_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{t}_u \times \vec{t}_v)\, du\, dv$$

Figure 17.57 (c & d)



Definition Surface Integral of a Vector Field

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface *S*. If *S* is defined parametrically as $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, for (u,v) in a region *R*, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA,$$

where $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$ are continuous on *R*, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on *R*, and the direction of the normal vector is consistent with the orientation of *S*. If *S* is defined in the form $z = s(x,y)$, for (x,y) in a region *R*, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R (-f z_x - g z_y + h)\, dA.$$

Example 7 $\vec{F} = \langle 0, 0, -1 \rangle$, S: $z = 4 - 2x - y$ in the 1st octant.

Find the flux in the downward direction across the surface S.

Figure 17.58

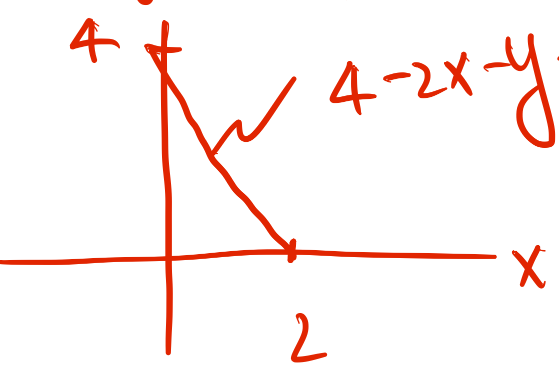
$$\iint_S \vec{F} \cdot \vec{n}\, dS = -\iint_R \big(0 + 0 + (-1)\big)\, dx\, dy$$

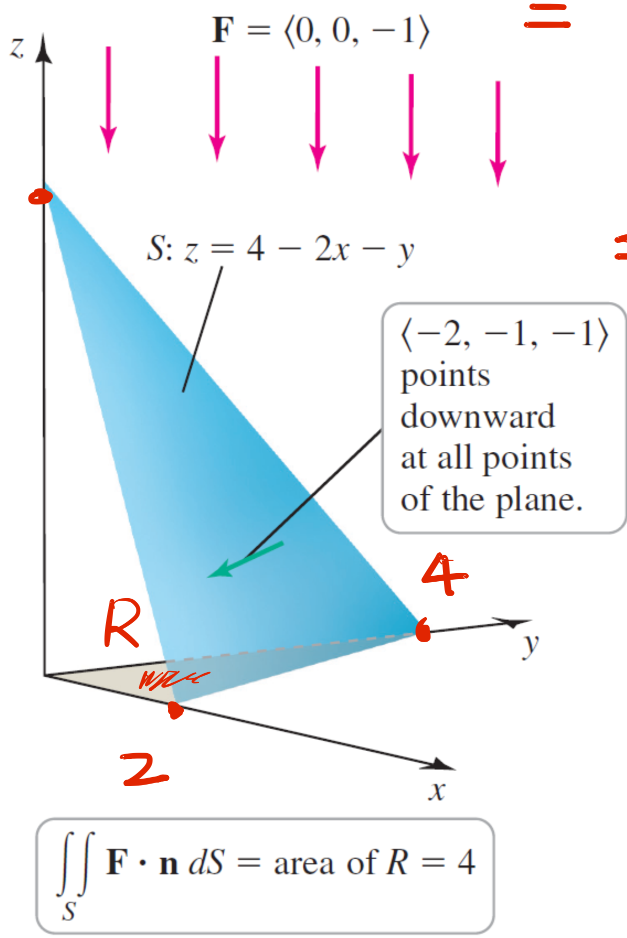
$$= \iint_R dx\, dy = \text{area}(R)$$

$$= \frac{1}{2} \cdot 2 \cdot 4 = 4$$

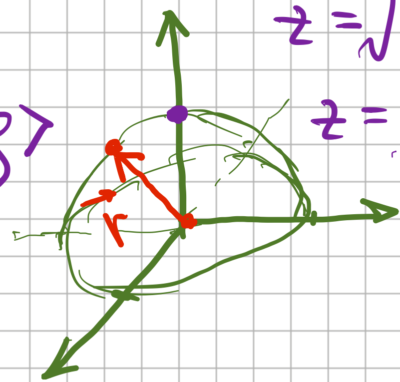
$$\begin{cases} z = 4 - 2x - y \\ z = 0 \end{cases}$$

$4 - 2x - y = 0$





Example 7 $\vec{F} = \langle x, y, z \rangle$. Is the upward flux of $\vec{F}$ greater across the hemisphere $x^2+y^2+z^2=1$, for $z \geq 0$, or across the paraboloid $z = 1-x^2-y^2$, for $z \geq 0$?



$z = \sqrt{1-x^2-y^2}$

$z = 1-x^2-y^2$

(a) $\iint_{S_1} (\vec{F}\cdot\vec{n})\, dS$ $\quad \vec{r}(\varphi, \theta) = \langle x, y, z \rangle$

$= \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$

$= \iint_R \langle x, y, z\rangle \cdot \langle x, y, z \rangle \sin\varphi \, dA$ $\quad R: \; 0 \leq \varphi \leq \frac{\pi}{2}$, $\; 0 \leq \theta \leq 2\pi$

$= \iint_R (x^2+y^2+z^2) \sin\varphi \, dA$ $\quad \vec{r}_\varphi \times \vec{r}_\theta = \sin\varphi \, \vec{r} = \sin\varphi \langle x, y, z \rangle$

$= \iint_R \sin\varphi \, dA = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\pi} d\theta \sin\varphi = 2\pi \int_0^{\frac{\pi}{2}} \sin\varphi \, d\varphi = 2\pi \left[-\cos\varphi\right]_0^{\frac{\pi}{2}} = 2\pi$

(b) $\iint_{S_2} \vec{F}\cdot\vec{n} \, dS = \iint_R \langle x, y, z \rangle \cdot \langle -z_x, -z_y, 1 \rangle \, dx\,dy$

$= \iint_R \langle x, y, 1-x^2-y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dx\,dy$

$= \iint_R (2x^2+2y^2+1-x^2-y^2) \, dx\,dy = \iint_R (1+x^2+y^2)\, dx\,dy$ $\quad R: \; x^2+y^2 \leq 1$

$= \int_0^{2\pi} \int_0^1 (1+r^2)\, r\,dr\,d\theta$

$= 2\pi \left[\frac{1}{2}r^2 + \frac{1}{4}r^4\right]_0^1 = 2\pi\left[\frac{1}{2}+\frac{1}{4}\right] = 2\pi \cdot \frac{3}{4} < 2\pi$

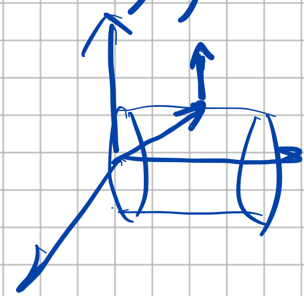
Example 8 (#53) S is the cylinder $x^2+z^2=a^2$, $|y| \le 2$, outward

$$\iint_S \frac{\langle x,0,z\rangle}{\sqrt{x^2+z^2}} \cdot \vec{n}\, dS$$

$\vec{r}(\theta, y) = \langle a\cos\theta, y, a\sin\theta\rangle$
$= \langle x, y, z\rangle$

$R: \begin{cases} 0 \le \theta \le 2\pi \\ -2 \le y \le 2 \end{cases}$ $2\pi \cdot 4$

$\vec{r}_\theta \times \vec{r}_y = \langle a\cos\theta, 0, a\sin\theta\rangle$
$= \langle x, 0, z\rangle$



$$= \iint_R \frac{\langle x,0,z\rangle}{\sqrt{x^2+z^2}} \cdot \langle x,0,z\rangle\, dA \quad (dA = d\theta\, dy)$$

$$= \iint_R \frac{x^2+z^2}{\sqrt{x^2+z^2}}\, dA$$

$$= \iint_R \sqrt{x^2+z^2}\, dA = a \iint_R dA$$

$$= a \cdot 2\pi \cdot 4$$

#65 $\vec{F} = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{p/2}} = \dfrac{\vec{r}}{|\vec{r}|^p}$

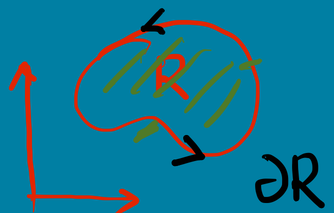
S consist of the spheres A and B centered at the origin with radii $0<a<b$, resp.
The total outward flux across S consists of the flux out of S across the outer sphere B minus the flux into S across the inner sphere A.

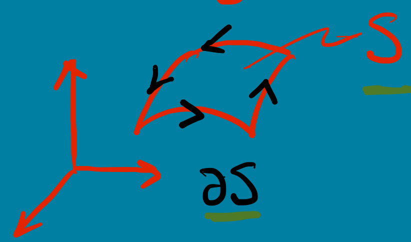
(a) Find the total flux across S with $p=0$.

(b) Show that for $p=3$, the flux across S is independent of a and b.

$$\vec{F}(x,y) = \langle f(x,y),\ g(x,y) \rangle \qquad \vec{F}(x,y,z) = \langle f(x,y,z),\ g(x,y,z),\ h(x,y,z) \rangle$$

$$\nabla \times \vec{F} = \langle 0,\ 0,\ g_x - f_y \rangle \qquad \nabla \times \vec{F} = \langle h_y - g_z,\ f_z - h_x,\ g_x - f_y \rangle$$



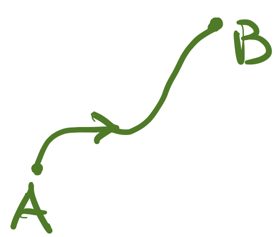


Section 17.7 Stokes' Theorem

Green's Theorem

$$\iint_R \nabla \times \vec{F} \cdot \vec{k}\, dA = \int_{\partial R} \vec{F} \cdot d\vec{r}$$

$$\iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$



$$\int_C \nabla \varphi \cdot d\vec{r} = \varphi(B) - \varphi(A)$$

Figure 17.59

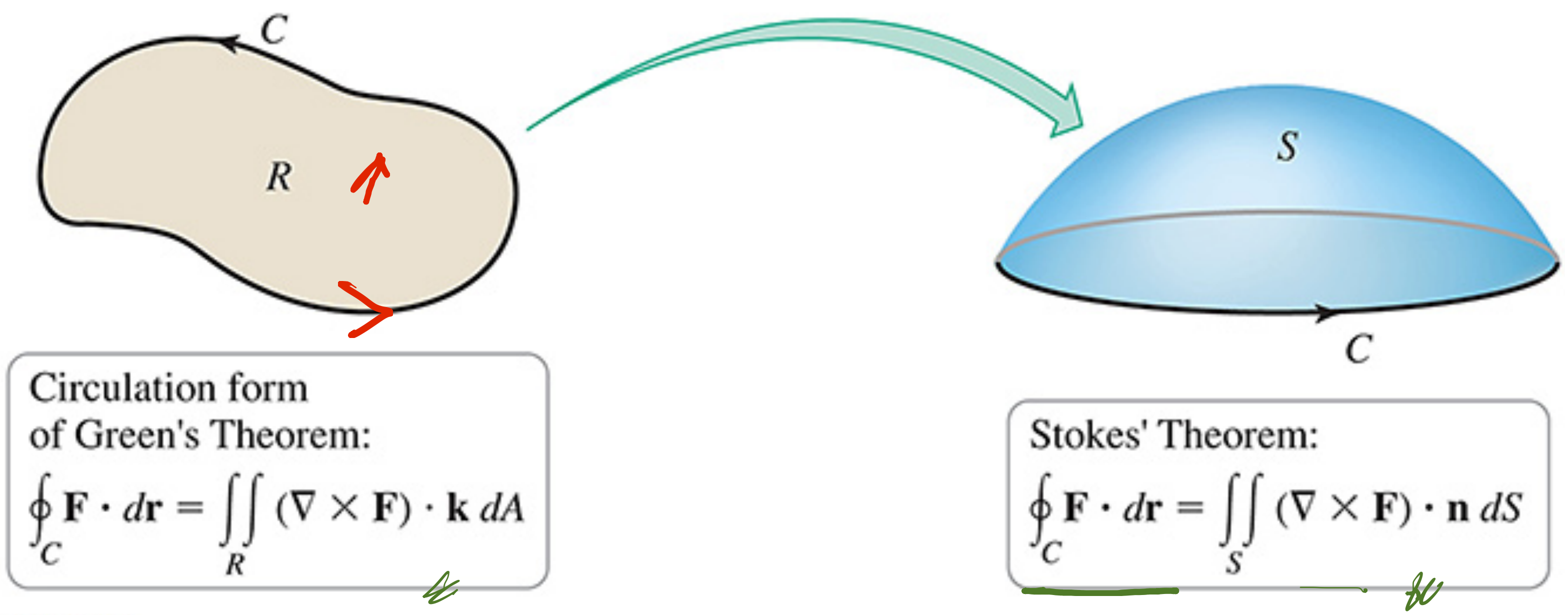
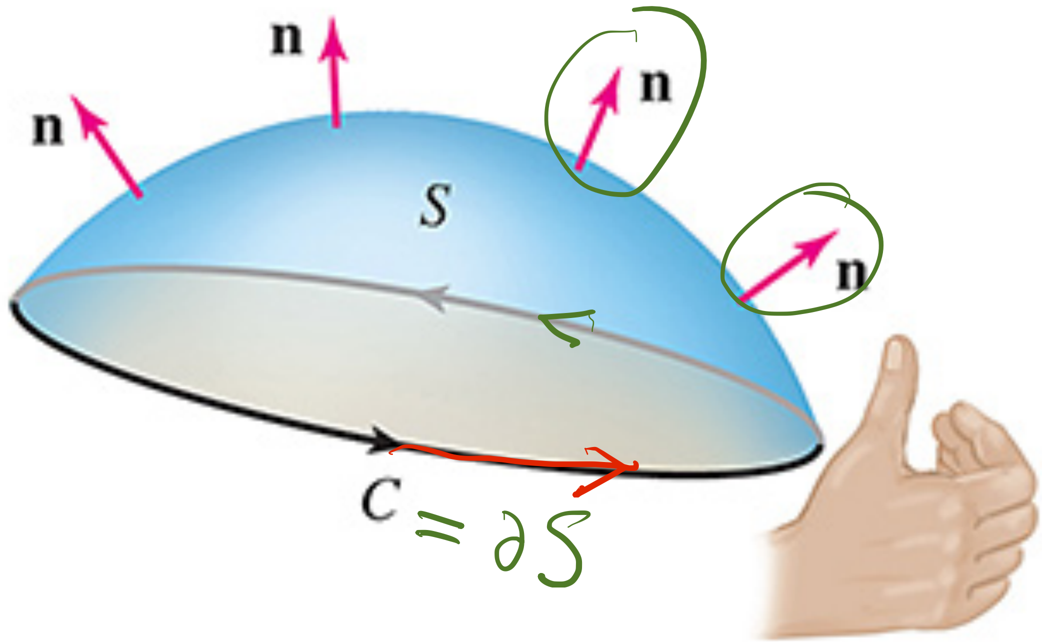

C
R
S
C
Circulation form
of Green's Theorem:
$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$
Stokes' Theorem:
$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$

Figure 17.60



Theorem 17.15 Stokes' Theorem

Let *S* be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary *C* whose orientation is consistent with that of *S*. Assume $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on *S*. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS,$$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \left(\vec{F} \cdot \vec{t_u} \times \vec{t_v} \right) du\, dv$$

where **n** is the unit vector normal to *S* determined by the orientation of *S*.

$S : \vec{r}(u,v)$

Example 1 $\vec{F} = \langle z-y, x, -x\rangle$, S: $x^2+y^2+z^2=4$ for $z \ge 0$

Confirm the Stokes Theorem.

Figure 17.61

$$\iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS = \oint_{\partial S}\vec{F}\cdot d\vec{r} = \int_0^{2\pi}\langle -2\sin t, 2\cos t, -2\cos t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle$$

$$= \int_0^{2\pi}\left(4\sin^2 t + 4\cos^2 t\right)dt$$

$$= 4\int_0^{2\pi} dt = 8\pi$$

$$\text{LHS} = \iint_R \nabla\times\vec{F}\cdot\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle dx\,dy$$

$\vec{F} = \langle 0,1,1\rangle \times \langle x,y,z\rangle$

$\nabla\times\vec{F} = 2\vec{a}$

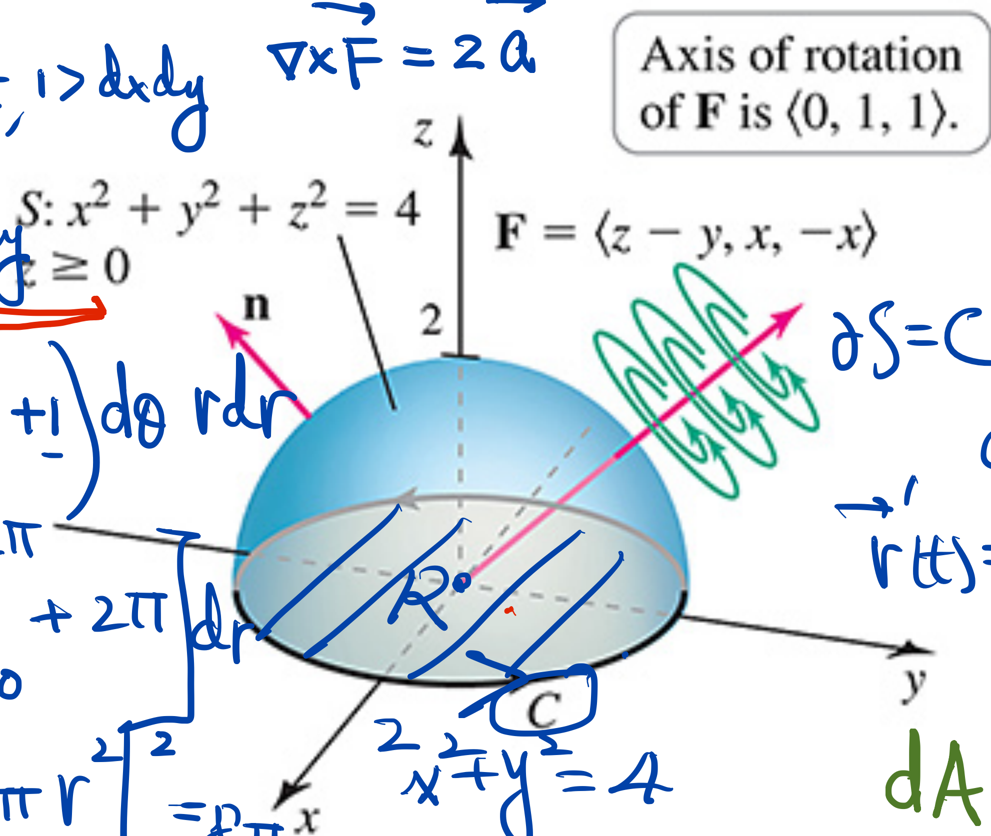
$$= \iint_R 2\langle 0,1,1\rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle dx\,dy$$

$$= 2\iint_R \left(\frac{y}{\sqrt{a^2-x^2-y^2}} + 1\right) dx\,dy$$

$$= 2\int_0^2\int_0^{2\pi}\left(\frac{r\sin\theta}{\sqrt{4-r^2}} + 1\right) d\theta\, r\,dr$$

$$= 2\int_0^2 r\left[\frac{r}{\sqrt{4-r^2}}(-\cos\theta)\Big|_0^{2\pi} + 2\pi\right] dr$$

$$= 2\int_0^2 2\pi r\,dr = 2\pi r^2\Big|_0^2 = 8\pi$$



$\partial S = C$: $\vec{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle$

$0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

$x^2+y^2=4$

$dA = dx\,dy = r\,dr\,d\theta$

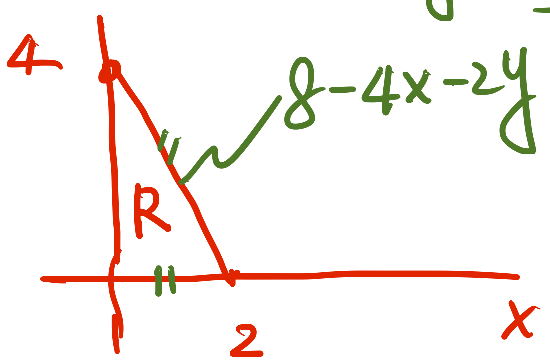
Example 2 $\vec{F} = \langle z, -z, x^2-y^2 \rangle$, C: three segments

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\vec{n}\, dS = \iint_R \nabla\times\vec{F}\cdot(\vec{t}_x\times\vec{t}_y)\, dx\,dy$$

Figure 17.62

$S: \vec{r}(x,y) = \langle x, y, 8-4x-2y \rangle$, $(x,y)\in R$

$\vec{t}_x\times\vec{t}_y = \langle -z_x, -z_y, 1 \rangle = \langle 4, 2, 1 \rangle$

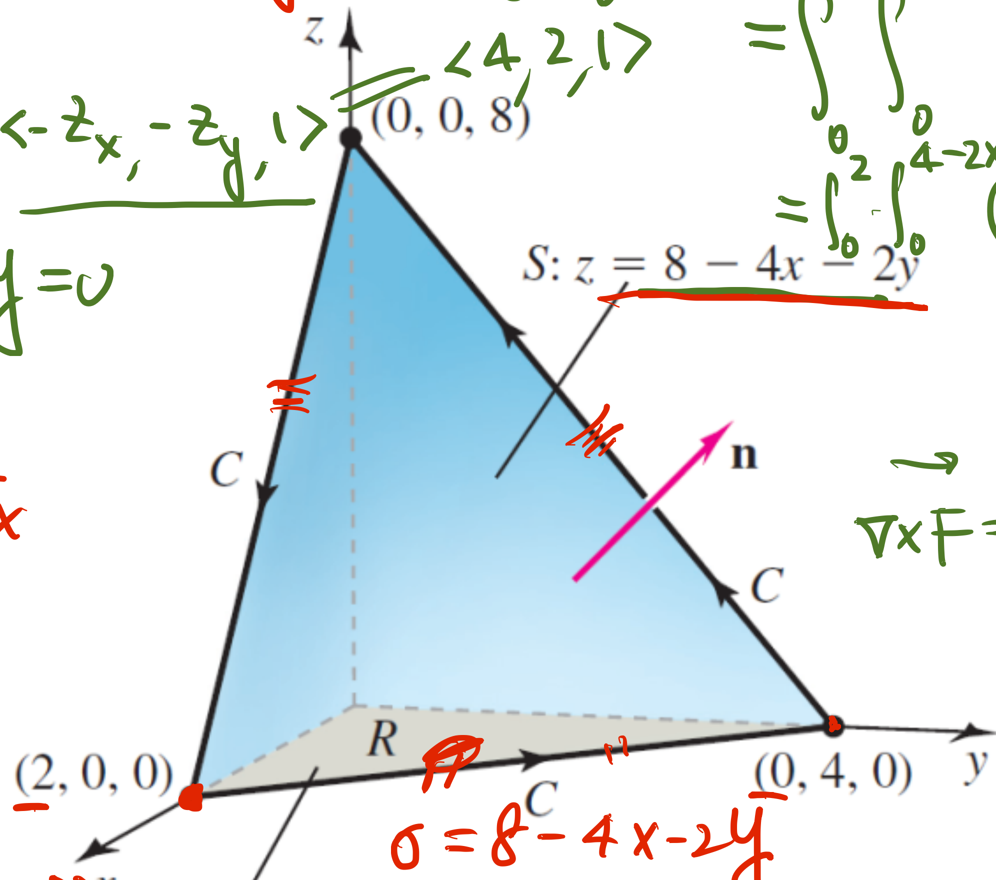


$R: 0 \le x \le 2$

$0 \le y \le 4-2x$

$C_1: \vec{r}(x) = \langle x, 4-2x, 0 \rangle$

$x: 2 \to 0$



$R = \{(x, y): 0 \le x \le 2, 0 \le y \le 4 - 2x\}$

$0 = 8-4x-2y$

$y = 4-2x$

$$= \int_0^2\int_0^{4-2x} \langle 1-2y, 1-2x, 0\rangle\cdot\langle 4,2,1\rangle\, dx\,dy$$

$$= \int_0^2\int_0^{4-2x} \left[4(1-2y) + 2(1-2x)\right] dy\,dx$$

$$= \int_0^2\int_0^{4-2x} (6-8y-4x)\, dy\,dx$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z & -z & x^2-y^2 \end{vmatrix} = \langle -2y+1, 1-2x, 0 \rangle$$

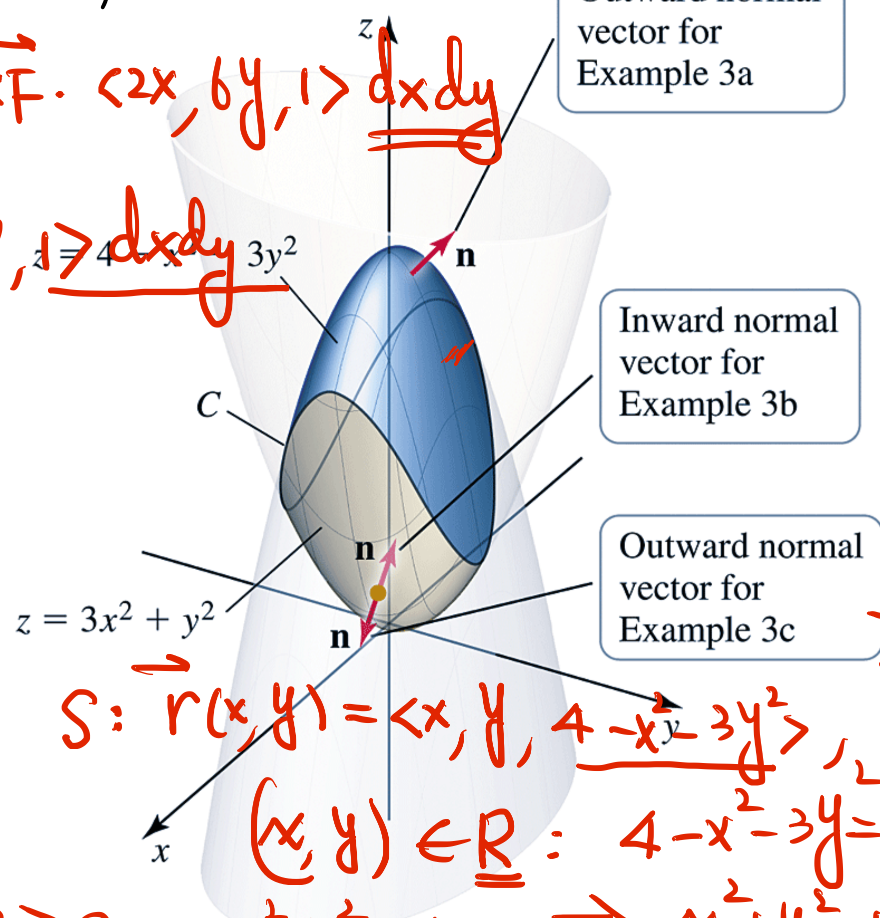
Example 3 $\vec{F} = \langle -y, x, z \rangle$, evaluate $\iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS$, $\vec{n}$ — upward

(a) S is the part of the paraboloid $z = 4 - x^2 - 3y^2$ that lies within the paraboloid $z = 3x^2 + y^2$

Figure 17.63

(b) S is the part of $z = 3x^2 + y^2$ that lies $z = 4 - x^2 - 3y^2$, $\vec{n}$ — upward

(c) S is the surface in part (b), $\vec{n}$ — downward.



(a) $\iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS = \iint_R \nabla\times\vec{F}\cdot\langle 2x, 6y, 1\rangle\,dx\,dy$

$= \iint_R \langle 0,0,2\rangle\cdot\langle 2x, 2y, 1\rangle\,dx\,dy$

$= 2\iint_R dx\,dy = 2\pi$

$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & z \end{vmatrix}$

$= \langle 0, 0, 1+1\rangle = \langle 0,0,2\rangle$

$S: \vec{r}(x,y) = \langle x, y, 4 - x^2 - 3y^2\rangle$,

$(x,y)\in R: \; 4 - x^2 - 3y^2 = z = 3x^2 + y^2$

$R: x^2 + y^2 \le 1 \quad \Rightarrow \quad x^2 + y^2 = 1$

$\vec{t}_x \times \vec{t}_y = \langle -z_x, -z_y, 1\rangle = \langle 2x, 6y, 1\rangle$

$\iint_{S_1} \nabla\times\vec{F}\cdot\vec{n}\,dS = \oint_{\partial S_1} \vec{F}\cdot d\vec{r}$

$\oint_{\partial S_2} \vec{F}\cdot d\vec{r}$

$\iint_{S_2} \nabla\times\vec{F}\cdot\vec{n}\,dS$

Interpreting the Curl

average circulation $\frac{1}{A(S)} \oint_{\partial S} \vec{F} \cdot d\vec{r}$

Figure 17.64

Example $\vec{F} = \vec{a} \times \vec{r} \Rightarrow \nabla \times \vec{F} = 2\vec{a}$

$\vec{r} = \langle x, y, z \rangle$

$\vec{a} = \langle 0, 0, 1 \rangle$

$\vec{F} = \langle -y, x, 0 \rangle$

$$\frac{1}{A(S)} \oint_{\partial S} \vec{F} \cdot d\vec{r} = \frac{1}{A(S)} \iint_S (\nabla \times \vec{F} \cdot \vec{n})\, dS$$

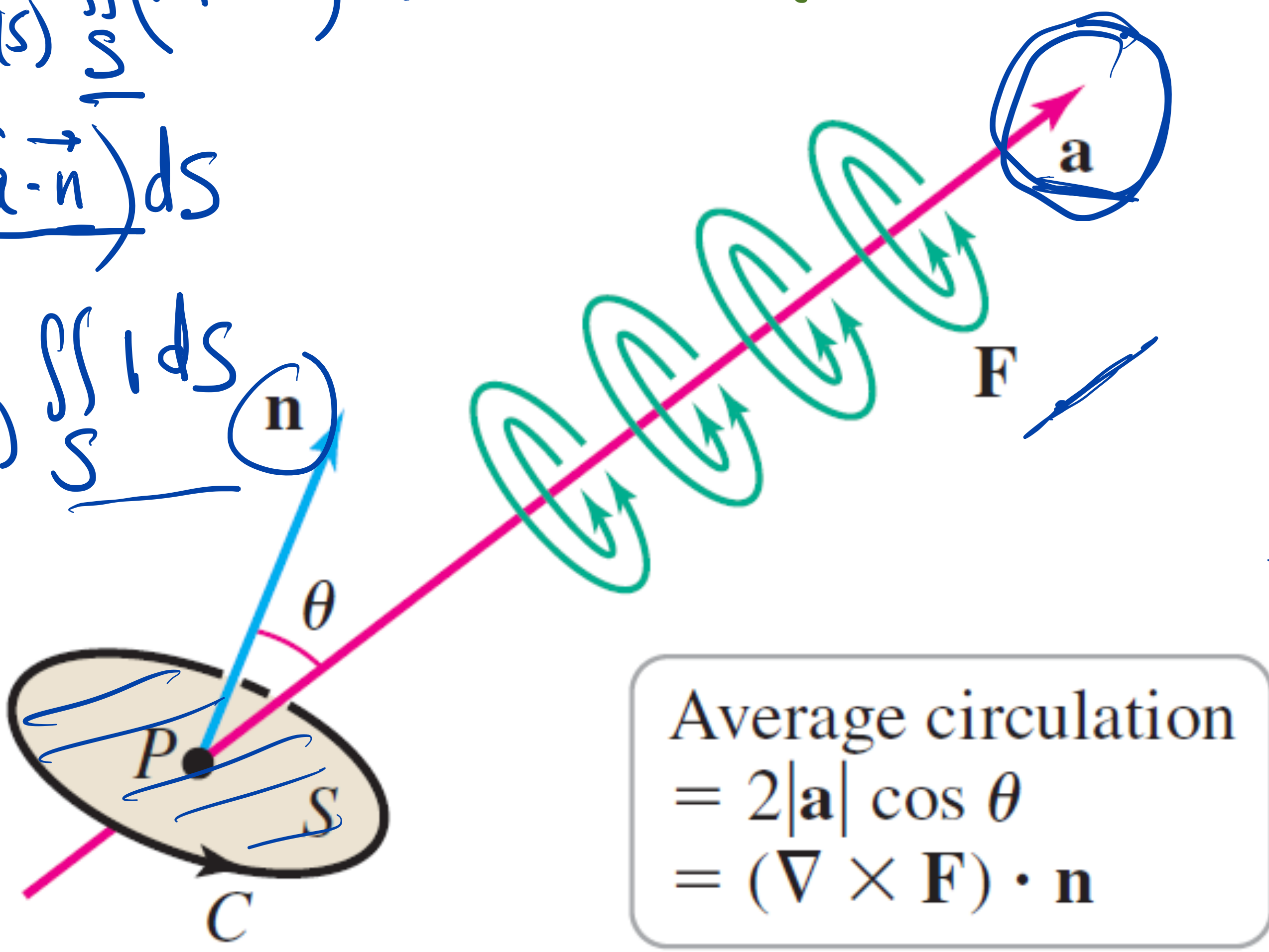
$$= \frac{1}{A(S)} \iint_S (2\vec{a} \cdot \vec{n})\, dS$$

$$= 2\vec{a} \cdot \vec{n} \frac{1}{A(S)} \iint_S 1\, dS$$

$$= 2\vec{a} \cdot \vec{n}$$

$$= \nabla \times \vec{F} \cdot \vec{n}$$

$$= |\nabla \times \vec{F}| \cos\theta$$



Example 4 velocity field $\vec{v} = \langle 0, 1-x^2, 0 \rangle$, for $|x| \le 1$ and $|z| \le 1$

(a) A paddle wheel is placed at $P(\frac{1}{2}, 0, 0)$, in which of the coordinate directions should the axis of the wheel point in order for the wheel to spin? In which direction does it spin? At $Q(-\frac{1}{2}, 0, 0)$?

Figure 17.65 (a & b)

(b) Compute and graph $\nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & 1-x^2 & 0 \end{vmatrix} = \langle 0, 0, -2x \rangle$

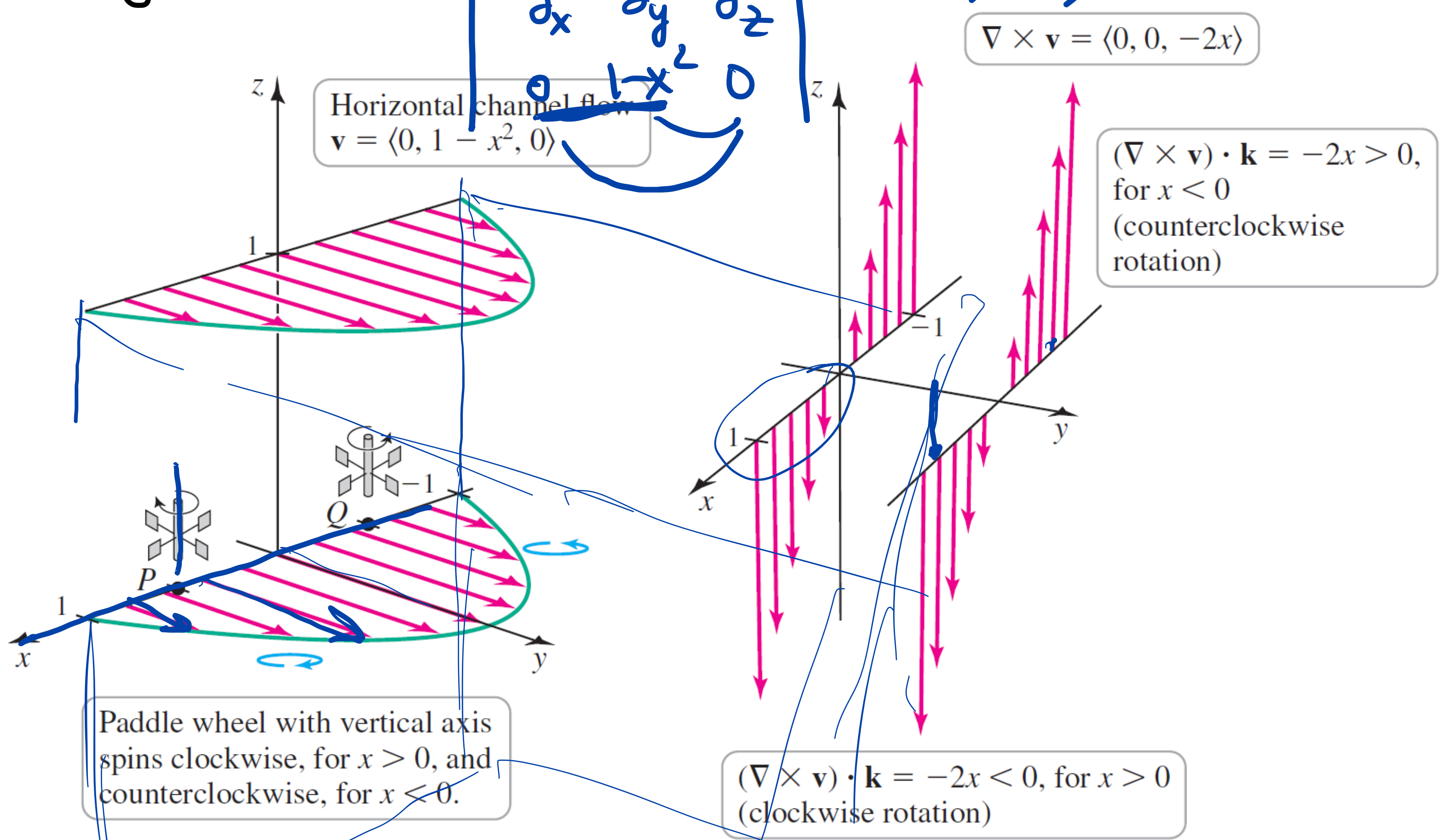
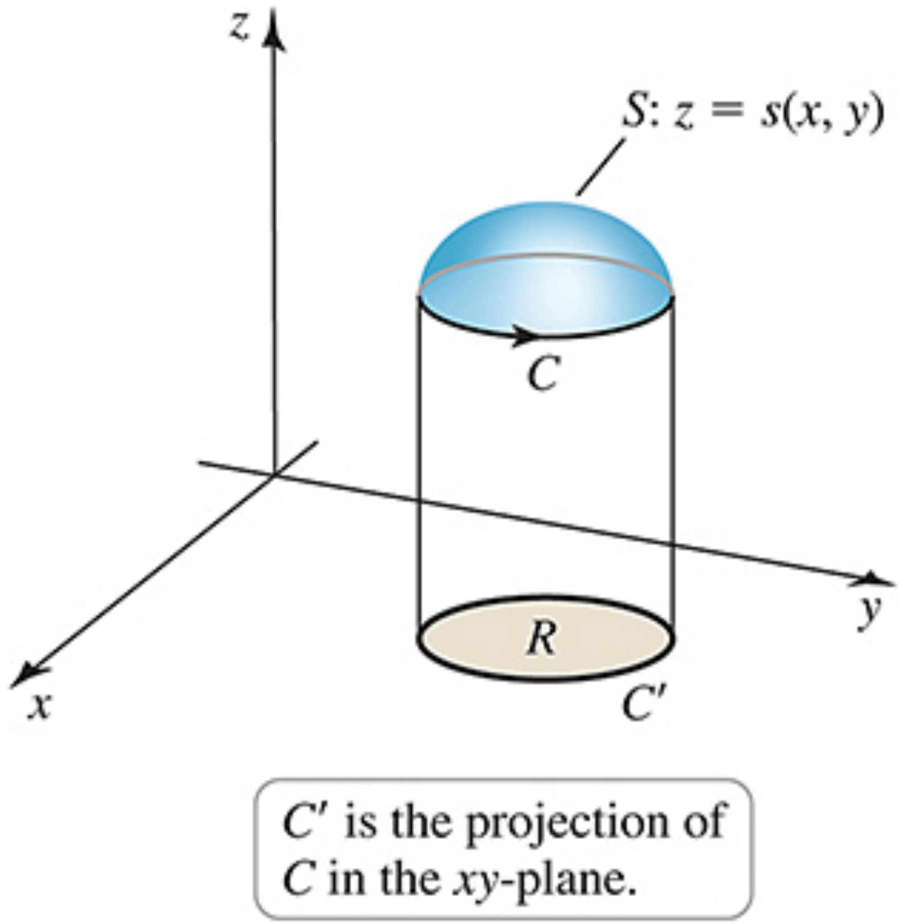


Figure 17.66



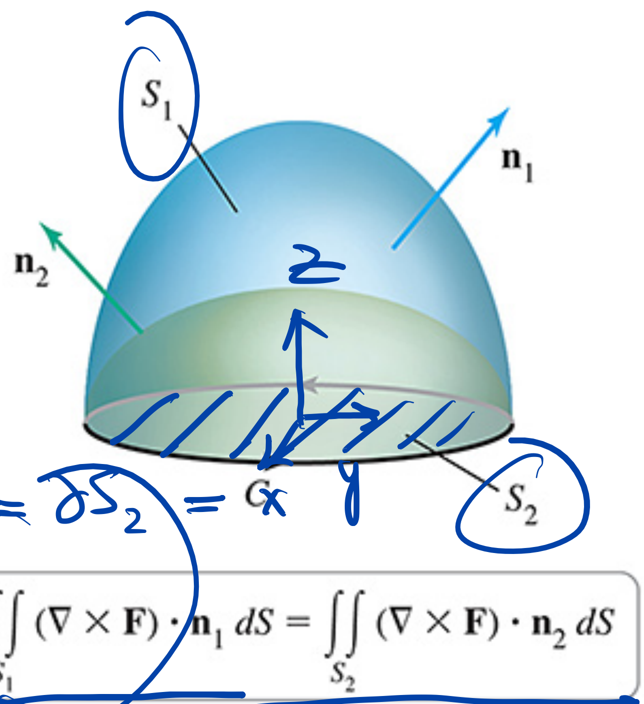
$$\iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS = \int_{\partial S}\vec{F}\cdot d\vec{r}$$

$$\int_C \nabla\varphi\cdot d\vec{r}$$

path indep.

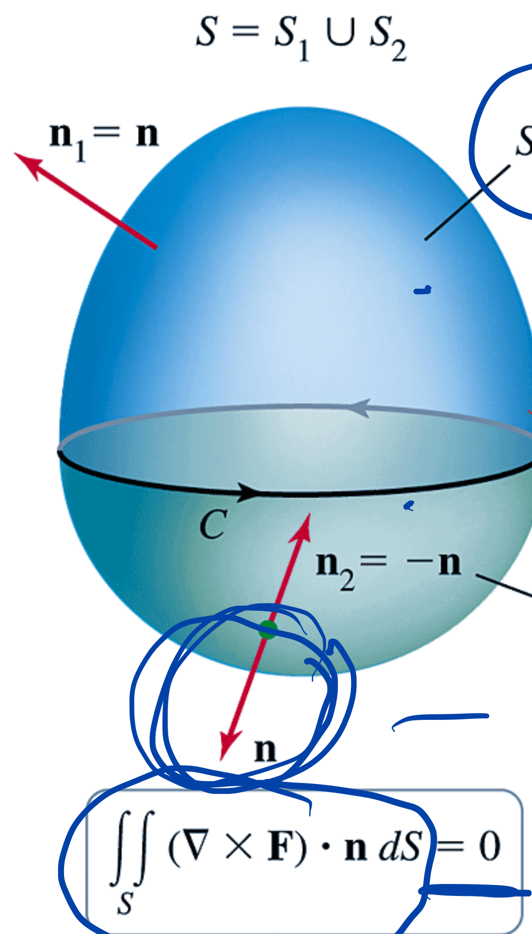
Figure 17.67 (a & b)

surface indep.



$$\partial S_1 = \partial S_2 = C$$

$$= \iint_R \nabla\times\vec{F}\cdot k\,dx\,dy$$



$$\iint_{S_1} \nabla\times\vec{F}\cdot\vec{n}\,dS = -\iint_{S_2} \nabla\times\vec{F}\cdot\vec{n}\,dS$$

Theorem 17.16 Curl F = 0 Implies F Is Conservative

Suppose $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region *D* of $\circ^{3}$. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves *C* in *D*, and **F** is conservative vector field on *D*.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS = 0$$

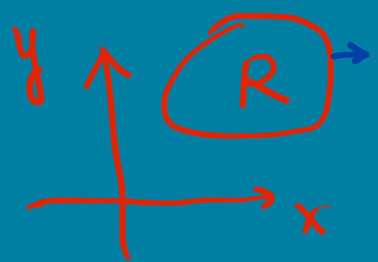
$$\partial S = C$$

$$\vec{F} = \langle f(x,y),\ g(x,y) \rangle$$

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \quad \text{scalar}$$

$$\vec{F} = \langle f(x,y,z),\ g(x,y,z),\ h(x,y,z) \rangle$$

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

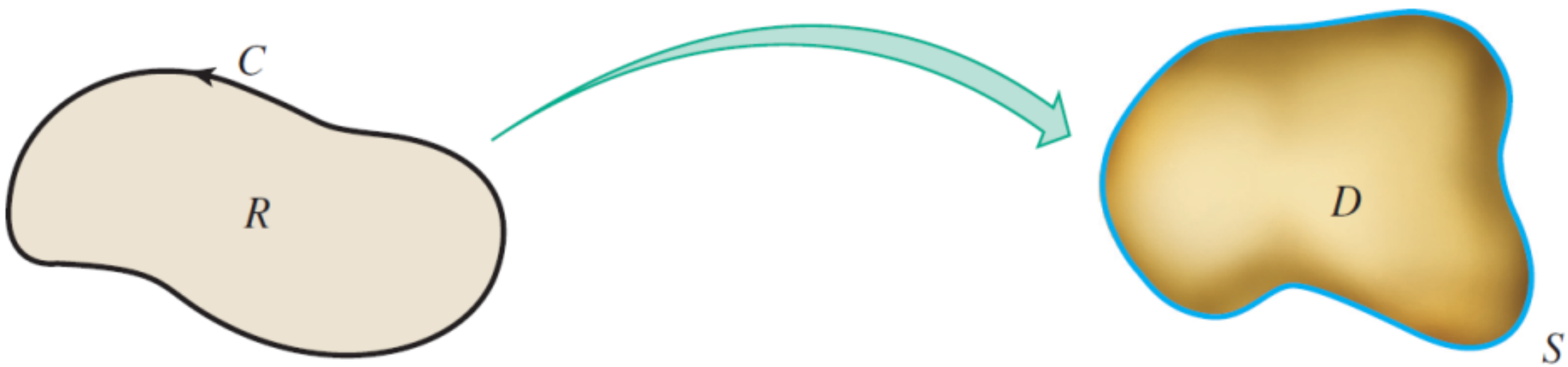


Section 17.8 Divergence Theorem

$$\iint_R \nabla \cdot \vec{F}\, dA = \oint_{\partial R} (\vec{F} \cdot \vec{n})\, ds$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_{\partial D} (\vec{F} \cdot \vec{n})\, dS$$

Figure 17.68



Flux form of Green's Theorem

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$

Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV$$

Theorem 17.17 Divergence Theorem

Let **F** be a vector field whose components have continuous first partial derivatives in a connected and simply connected region *D* in $\circ^{3}$ enclosed by an oriented surface *S*. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV,$$

where **n** is the outward unit normal vector on *S*.

Example 1 $S: x^2+y^2+z^2=a^2$, $\vec{n}$ is the outward unit normal vector to S

$\vec{F}=\langle x,y,z\rangle$, $\nabla\cdot\vec{F}=1+1+1=3$ $\quad D: x^2+y^2+z^2\le a^2$

$$\iiint_D \nabla\cdot\vec{F}\,dV = \iiint_D 3\,dV = 3\,\mathrm{Vol}(D) = 3\cdot\frac{4}{3}\pi a^3 = 4\pi a^3$$

$\vec{r}(\varphi,\theta)=\langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi\rangle = \langle x,y,z\rangle$

$R: 0\le\varphi\le\pi,\ 0\le\theta\le 2\pi$

$\vec{r}_\varphi\times\vec{r}_\theta = a\sin\varphi\,\vec{r}(\varphi,\theta) = a\sin\varphi\langle x,y,z\rangle$ outward

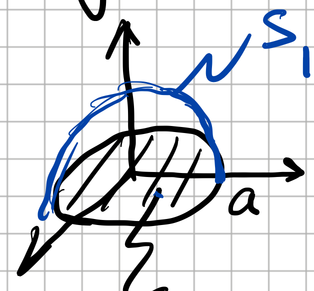
$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \langle x,y,z\rangle\cdot a\sin\varphi\langle x,y,z\rangle\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi} a\sin\varphi\,(x^2+y^2+z^2)\,d\varphi\,d\theta$$

$$= \int_0^{2\pi} a^3\,d\theta \int_0^{\pi}\sin\varphi\,d\varphi = 2\pi a^3\left[-\cos\varphi\right]_0^{\pi} = 4\pi a^3$$

Example 2

$\vec{F}=\vec{a}\times\vec{r}=\langle 1,0,1\rangle\times\langle x,y,z\rangle=\langle -y, x-z, y\rangle$

S is the hemispher $x^2+y^2+z^2=a^2$, for $z\ge 0$, together with its base in the xy-plane. Find the net outward flux across S.



$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1}\vec{F}\cdot\vec{n}\,dS + \iint_{S_2}\vec{F}\cdot\vec{n}\,dS$$

$\iint_S = \iiint_D \nabla\cdot\vec{F}\,dV = 0$

$S=S_1\cup S_2$

$$\iint_{S_1}\vec{F}\cdot\vec{n}\,dS = -\iint_{S_2}\vec{F}\cdot\vec{n}\,dS$$

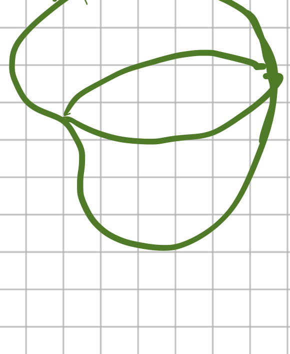
$$= -\iint_R \langle -y,x,y\rangle\cdot\langle 0,0,-1\rangle\,dx\,dy = \iint_R y\,dx\,dy$$

$\vec{r}(x,y)=\langle x,y,0\rangle$, $(x,y)\in R$

$\vec{r}_x\times\vec{r}_y=\langle 0,0,1\rangle$

$\nabla\cdot\vec{F}=\nabla\cdot\langle -y, x-z, y\rangle = 0+0+0$

$$\iiint_D \nabla\cdot\vec{F}\,dV = \iint_{\partial D} \vec{F}\cdot\vec{n}\,dS$$



Example 3 $\vec{F} = xyz\langle 1,1,1\rangle$. $S = \partial D$, $D=[0,1]\times[0,1]\times[0,1]$ point outward

Find the net outward flux.

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \nabla\cdot\vec{F}\,dV$$

Figure 17.69

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(xyz)+\frac{\partial}{\partial y}(xyz)+\frac{\partial}{\partial z}(xyz)$$

$$= yz + xz + xy$$

$$= \int_0^1\int_0^1\int_0^1 (yz + xz + xy)\,dx\,dy\,dz$$

$$= 3\int_0^1\int_0^1\int_0^1 yz\,dx\,dy\,dz$$

$$= 3\cdot 1\cdot\left[\tfrac{1}{2}y^2\right]_0^1\cdot\left[\tfrac{1}{2}z^2\right]_0^1$$

$$= 3\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}.$$

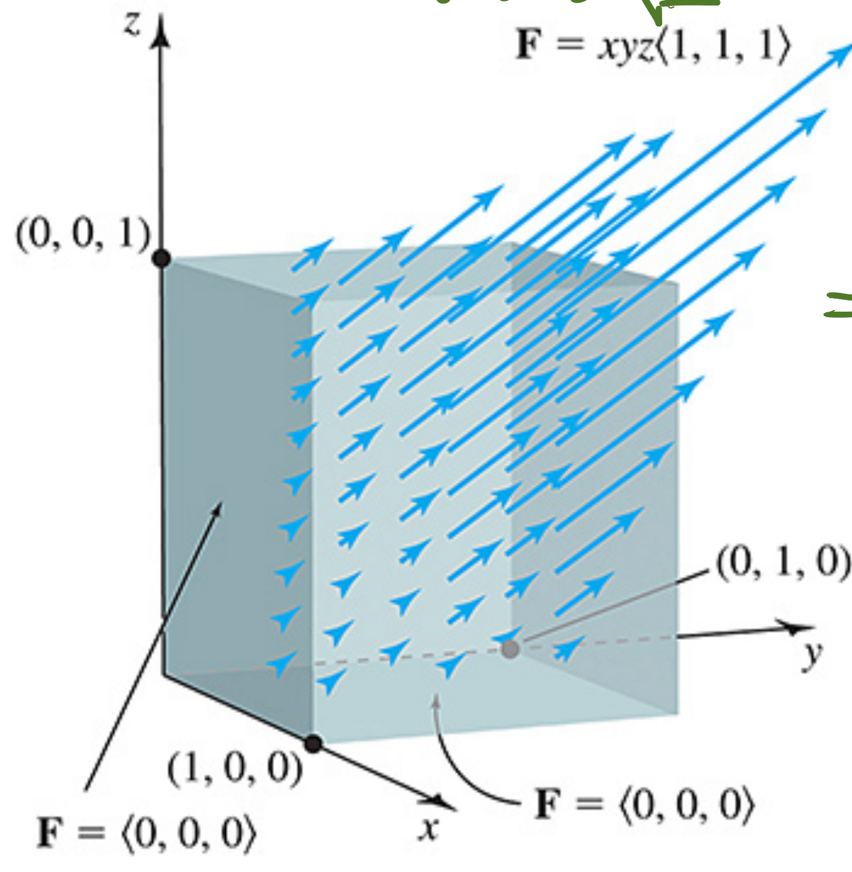
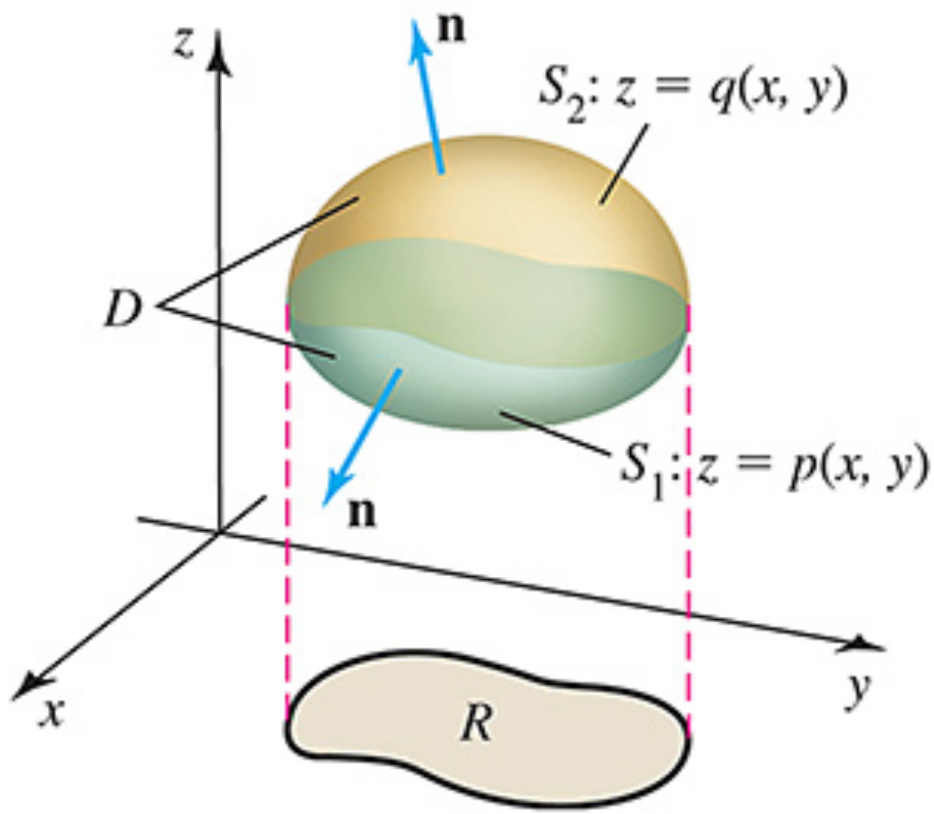


Figure 17.71



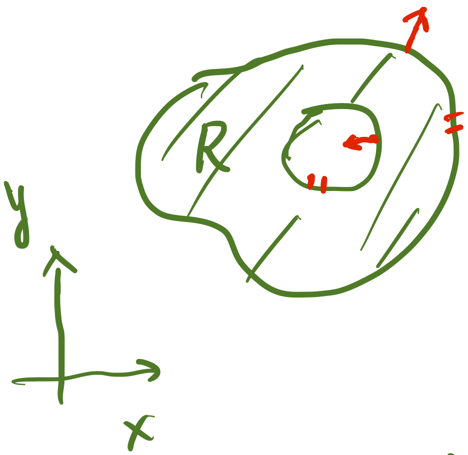
Divergence Theorem for Hollow Regions

Figure 17.72

$$\iiint_D \nabla\cdot\vec{F}\,dV = \iint_{\partial D} \vec{F}\cdot\vec{n}\,dS$$

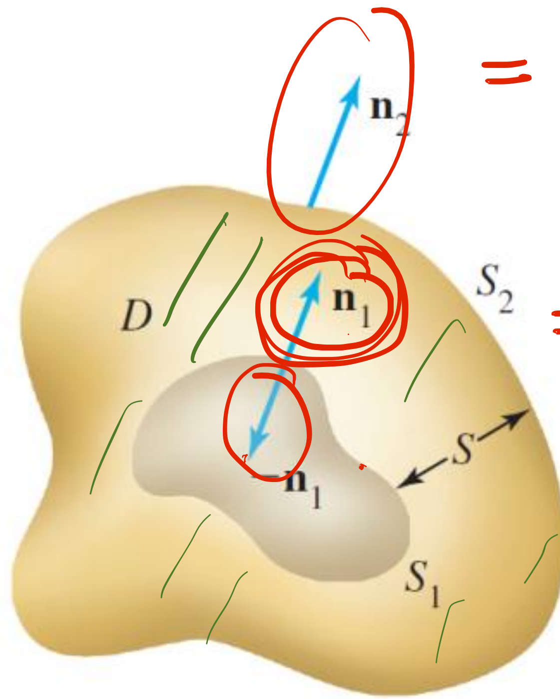
$$= \iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS + \iint_{S_1} \vec{F}\cdot(-\vec{n}_1)\,dS$$

$$= \iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS - \iint_{S_1} \vec{F}\cdot\vec{n}_1\,dS$$



$$\iint_R \nabla\cdot\vec{F}\,dA = \oint_{\partial R} \vec{F}\cdot\vec{n}\,ds$$

outward



$\mathbf{n}_1$ is the outward normal to S_1 and points into D. The outward normal to S on S_1 is $-\mathbf{n}_1$.

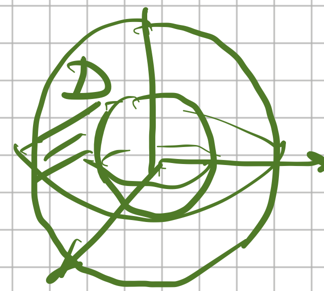
Theorem 17.18 Divergence Theorem for Hollow Regions

Suppose the vector field **F** satisfies the conditions of the Divergence Theorem on a region *D* bounded by two oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let *S* be the entire boundary of $D(S = S_1 \cup S_2)$ and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS.$$

Example 4

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$$



(a) $D = \{(x,y,z) : a^2 \le x^2+y^2+z^2 \le b^2\}$

$$\iint_{\partial D} \vec{F}\cdot\vec{n}\,dS = \iiint_D \nabla\cdot\vec{F}\,dV = 0$$

$$\frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-\frac{3}{2}}\right] = (x^2+y^2+z^2)^{-\frac{3}{2}} + x\cdot\left(-\frac{3}{2}\right)\cdot(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2x$$

$$= (x^2+y^2+z^2)^{-\frac{5}{2}}\left[(x^2+y^2+z^2) - 3x^2\right]$$

$$\nabla\cdot\vec{F} = (x^2+y^2+z^2)^{-\frac{5}{2}}\left[(y^2+z^2-2x^2) + (x^2+z^2-2y^2) + (x^2+y^2-2z^2)\right] = 0$$

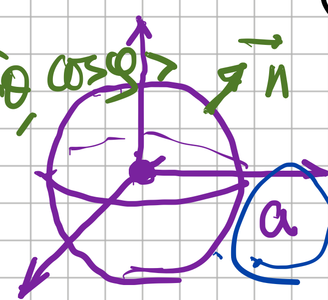
(b) Find the outward flux of $\vec{F}$ across any sphere that encloses the origin.

$$\iint_S \vec{F}\cdot\vec{n}\,dS \neq \iiint_B \nabla\cdot\vec{F}\,dV$$

$\vec{r}(\varphi,\theta) = a\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle$

$R: 0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$

$\vec{r}_\varphi \times \vec{r}_\theta = a\sin\varphi\,\vec{r}(\varphi,\theta)$



$$= \int_0^\pi \int_0^{2\pi} \frac{\langle x,y,z\rangle}{a^3}\cdot a\sin\varphi\langle x,y,z\rangle\,d\theta\,d\varphi$$

$$= \frac{1}{a^2}\int_0^\pi\int_0^{2\pi} (x^2+y^2+z^2)\sin\varphi\,d\theta\,d\varphi = \int_0^\pi\int_0^{2\pi}\sin\varphi\,d\theta\,d\varphi$$

$$= 2\pi\cdot\left[-\cos\varphi\right]_0^\pi = \boxed{4\pi}$$

Gauss' Law

$$\vec{E}(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$ the electric field due to a point charge Q located at the origin

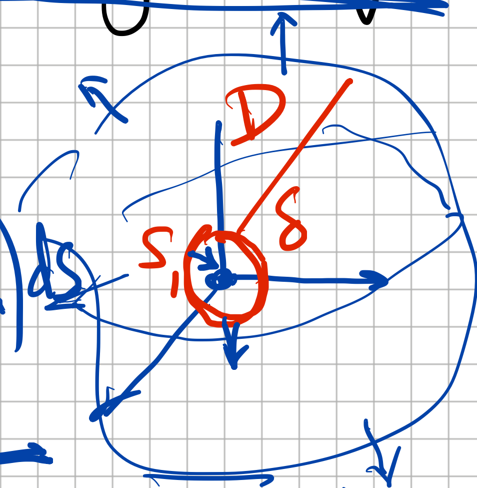
$$\iint_S \vec{E}\cdot\vec{n}\,dS = \begin{cases} \frac{Q}{\varepsilon_0}, & (0,0,0) \text{ inside } S \\ 0, & \text{otherwise} \end{cases}$$

- $\vec{r} = \langle x, y, z\rangle$
- ε_0 is the permittivity of free space
- S is any close surface

$$\iiint_D \nabla\cdot F\,dV = 0$$

$$\parallel$$

$$\iint_{\partial D} \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}\cdot\vec{n}\,dS + \iint_{S_1} \vec{F}\cdot(-\vec{n}_1)\,dS$$



$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS$$

$$= \frac{Q}{4\pi\varepsilon_0}\cdot 4\pi$$

$$= \frac{Q}{\varepsilon_0}$$

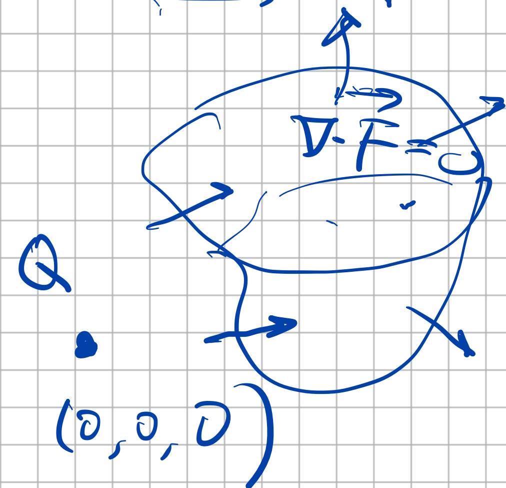
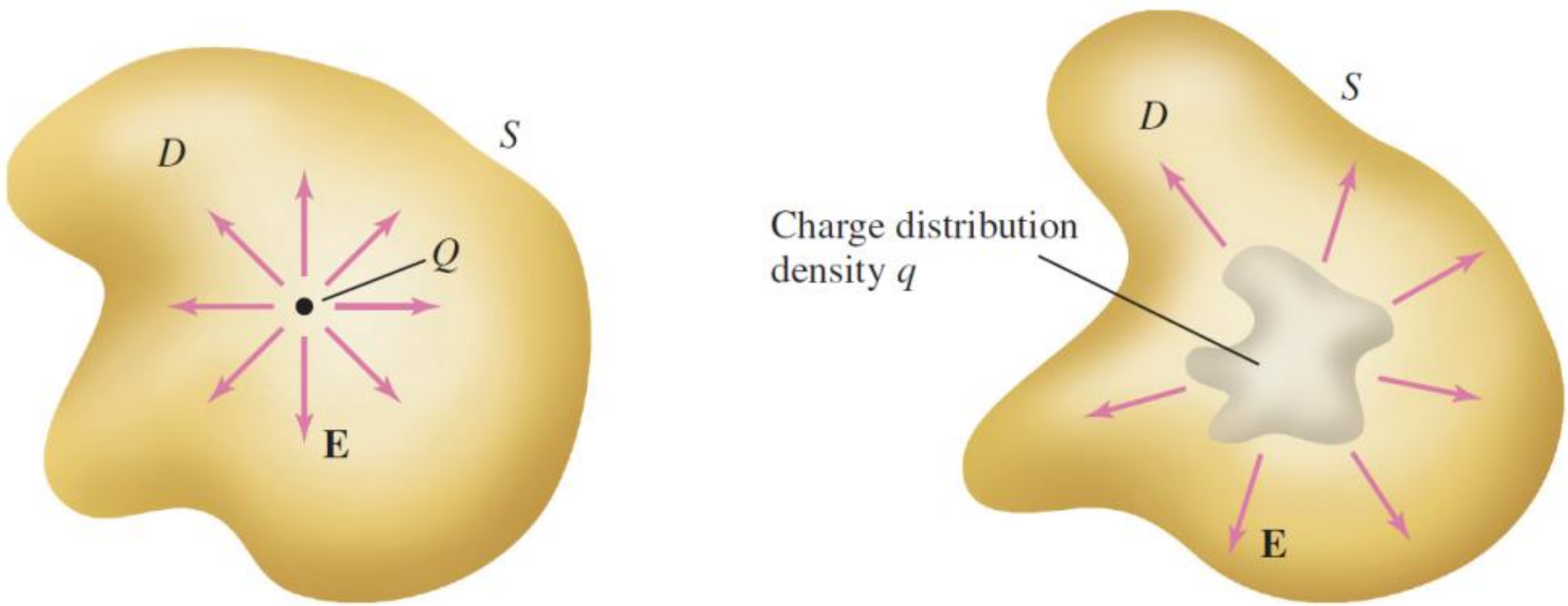


Figure 17.73 (a & b)



Gauss' Law:
Flux of electric field across S due to point charge Q

$$= \iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\varepsilon_0}$$

Gauss' Law:
Flux of electric field across S due to charge distribution q

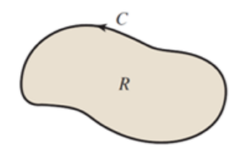
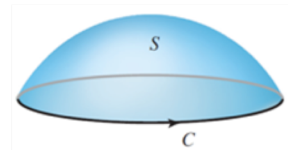
$$= \iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{1}{\varepsilon_0} \iiint_D q\, dV$$

$f'(x)$

$$\nabla f(x,y,z) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Table 17.4

Fundamental Theorem of Calculus	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
Fundamental Theorem of Line Integral	$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$	
Green's Theorem (circulation form)	$\iint_R (g_x - f_y)\,dA = \oint_C f\,dx + g\,dy$	
Stokes' Theorem	$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$