equation of a line L

given
- a point (x_0, y_0, z_0) on L
- a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ parallel to L

equation $\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\vec{v} = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle$

#1 point $(2, 1, -3)$ $\qquad \vec{r}(t) = \langle 2+2t, 1-3t, -3+7t \rangle$

vector $\vec{v} = \langle 2, -3, 7 \rangle$

#2 point $(1, -1, 0)$ $\qquad \vec{r}(t) = \langle 1-3t, -1+4t, 5t \rangle$

vector $\vec{v} = \langle -2, 3, 5 \rangle - \langle 1, -1, 0 \rangle = \langle -3, 4, 5 \rangle$

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equation of a plane $\mathcal{P}$

Given
- a point (x_0, y_0, z_0) on $\mathcal{P}$
- a vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ orthogonal to $\mathcal{P}$

equation $0 = \langle n_1, n_2, n_3 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$

#3 point $(1, -1, -1)$

vector $\vec{n} = \langle \frac{1}{2}, 2, 3 \rangle$

$$0 = \langle \tfrac{1}{2}, 2, 3 \rangle \cdot \langle x-1, y+1, z+1 \rangle$$
$$= \tfrac{1}{2}(x-1) + 2(y-1) + 3(z+1)$$
$$= \tfrac{1}{2}x + 2y + 3z + (3 - \tfrac{1}{2} - 2)$$

#4 $\vec{a} = (-5, 3, 2) - (1, 0, -1)$

$= \langle -6, 3, 3 \rangle = 3\langle -2, 1, 1 \rangle$

$\vec{b} = (2, -1, 4) - (1, 0, -1)$

$= \langle 1, -1, 5 \rangle$

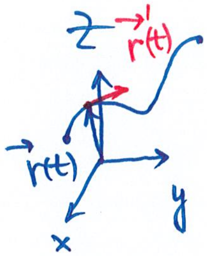
$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ -6 & 3 & 3 \\ 1 & -1 & 5 \end{vmatrix} = \langle 18, 33, 3 \rangle = 3\langle 6, 11, 1 \rangle$$

point $(1, 0, -1)$, vector $\vec{n} = \langle 6, 11, 1 \rangle$

$$0 = \langle 6, 11, 1 \rangle \cdot \langle x-1, y, z+1 \rangle$$
$$= 6(x-1) + 11y + (z+1)$$
$$= 6x + 11y + z - 5$$

parametrized curve

$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
$a \le t \le b$



arc length

$C = \int_C 1\, ds = \int_a^b |\vec{r}'(t)|\, dt$

position $\vec{r}(t)$

velocity $\vec{v}(t) = \vec{r}'(t)$

speed $v = |\vec{r}'(t)|$

acceleration $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$

#5 tangent line $\boxed{\vec{l}(t) = \vec{r}(t_0) + \cancel{t\vec{r}'(t)}\,(t-t_0)\vec{r}'(t_0)}$

$\overset{?}{t_0 = ?}$ $\quad (2, 4, 8) = \vec{r}(t_0) = \langle t_0, t_0^2, t_0^3 \rangle \Rightarrow t_0 = 2$

$\vec{r}'(t_0) = \langle 1, 2t, 3t^2 \rangle \Big|_{t=2} = \langle 1, 4, 12 \rangle$

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\vec{l}(t) = \langle 2, 4, 8 \rangle + (t-2)\langle 1, 4, 12 \rangle$

#6 $\vec{r}(t) = \langle \cos t, 3\sin t, -t^2 \rangle$

$\vec{v}(\pi) = \vec{r}'(\pi) = \langle -\sin t, 3\cos t, -2t \rangle \Big|_{t=\pi} = \langle 0, -3, -2\pi \rangle$ $\quad v(\pi) = \sqrt{9 + (2\pi)^2}$

$\vec{a}(\pi) = \vec{v}'(\pi) = \langle -\cos t, -3\sin t, -2 \rangle \Big|_{t=\pi} = \langle 1, 0, -2 \rangle$

#7 $(1, 0, 5), (0, 1, 5), (-1, 0, 5)$

$\vec{r}(t) = \langle \cos t, \sin t, 5 \rangle$

$0 \le t \le \pi$

#8 $\vec{r}(t) = \langle \frac{2}{3}(1+t)^{\frac{3}{2}}, \frac{2}{3}(1-t)^{\frac{3}{2}}, t \rangle$, $-1 \le t \le 1$

$\vec{r}'(t) = \langle (1+t)^{\frac{1}{2}}, -(1-t)^{\frac{1}{2}}, 1 \rangle$

$L = \int_{-1}^{1} |\vec{r}'(t)|\, dt$

$|\vec{r}'(t)| = \sqrt{(1+t) + (1-t) + 1}$

$= \sqrt{3}$

$= \int_{-1}^{1} \sqrt{3}\, dt = 2\sqrt{3}$

#9 level curve of $f(x,y)=\sqrt{1-x^2-2y^2}=k$

$f(x,y)=k$ $\quad 1-x^2-2y^2=k^2 \Rightarrow x^2+2y^2=1-k^2$ $\quad$ E

#10 $f(x,y,z)=z-x^2-y^2=k \Rightarrow z=x^2+y^2+k$

$(1,2,-3)$ $\quad -3=1+2^2+k \Rightarrow k=-8 \Rightarrow z=x^2+y^2-8$

$y=0 \Rightarrow \boxed{z=x^2-8}$ $\quad$ B

#11

1(b), 2(c), 3(d), 4(a), 5(e)

#12 $w=\dfrac{u^2}{v}$ $\quad \begin{cases} u=g_1(t) \\ v=g_2(t) \end{cases}$

~~$g_1(t)=3$~~ $g_1(1)=3$, $g_1'(1)=5$ $\quad u|_{t=1}=g_1(1)$

~~$g_2(t)=$~~ $g_2(1)=2$, $g_2'(1)=-4$ $\quad v|_{t=1}=g_2(1)$

$$\left.\frac{dw}{dt}\right|_{t=1}=\left.\frac{\partial w}{\partial u}\cdot\frac{du}{dt}+\frac{\partial w}{\partial v}\cdot\frac{dv}{dt}\right|_{t=1}=\left[\frac{2u}{v}g_1'(t)-\frac{u^2}{v^2}g_2'(t)\right]_{t=1}$$

$$=\frac{2g_1(1)}{g_2(1)}g_1'(1)-\frac{g_1^2(1)}{g_2^2(1)}\cdot g_2'(1)=\frac{2\cdot 3}{2}\cdot 5-\frac{3^2}{2^2}\cdot(-4)$$

$$=15+9=24$$ E

#13 $w=e^{uv}$ $\quad \begin{cases} u=r+s \\ v=rs \end{cases}$

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial r}+\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial r}$$

$$=ve^{uv}\cdot 1+ue^{uv}\cdot s=(rs+s(r+s))e^{(r+s)rs}$$

$$=(2rs+s^2)e^{(r+s)rs}$$

#14 $f(x,y)=\cos(\cancel{x,y})$ $\quad \dfrac{\partial f}{\partial y}=-x\sin(xy)$

$$\frac{\partial^2 f}{\partial x\partial y}=-\left[\sin(xy)+x\,y\cos(xy)\right]$$ B

#15 $\dfrac{\partial}{\partial x}$ $\boxed{xy^2+3z=\cos(z^2)}$ defines z implicitly as a function of x and y

~~$z(x,y)$~~ ~~$\dfrac{\partial z}{\partial x}=$~~ $z(x,y)$ $\quad \left(3+2z\sin(z^2)\right)\dfrac{\partial z}{\partial x}=-y^2$

$$y^2+3\frac{\partial z}{\partial x}=-2z\sin(z^2)\cdot\frac{\partial z}{\partial x}$$ D

#16 $f(x,y)=xy^2$, $\nabla f(2,3) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle\Big|_{(2,3)}$

$$= \langle y^2, 2xy\rangle\Big|_{(2,3)} = \langle 9, 2\cdot 2\cdot 3\rangle = \langle 9, 12\rangle$$ C

#17 $f(x,y)=5-4x^2-3y$

(x,y) $\vec{v} = -\langle x,y\rangle \Big/ \sqrt{x^2+y^2}$ unit vector $D_{\vec v} f = \nabla f \cdot \vec{v}$

$$= \langle -8x, -3\rangle \cdot \frac{\langle -x, -y\rangle}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}}\left(8x^2+3y\right)$$ E

#18 $f(x,y)=x^2y$, find $\vec{u}$ s.t. $\underline{D_{\vec u} f(2,3)=0}$.

$$0 = D_{\vec u} f(2,3) = \nabla f(2,3)\cdot \vec{u}$$

$$= \langle 2xy, x^2\rangle\Big|_{(2,3)} \cdot \langle u_1, u_2\rangle = \langle 12, 4\rangle \cdot \langle u_1, u_2\rangle$$

~~#19~~

$$0 = 12u_1 + 4u_2 = 4(3u_1+u_2) \Rightarrow \underline{3u_1+u_2=0}$$

$$\begin{cases} u_1 = 1 \\ u_2 = -3u_1 = -3 \end{cases}$$

$$\vec{u} = \langle 1, -3\rangle \Big/ \sqrt{1+9}$$ D

#19 $f(x,y,z) = 3xy - 9xz^2 + y$,

$$\nabla f(1,1,0) = \langle f_x, f_y, f_z\rangle\Big|_{(1,1,0)}$$

$$= \langle 3y-9z^2, 3x+1, -18xz\rangle\Big|_{(1,1,0)} = \langle 3, 4, 0\rangle$$ A

#20 $\underbrace{2\cos(\pi xy)}_{f(x,y)} = 1$

$$\nabla f\left(\tfrac16, 2\right) = \langle f_x, f_y\rangle\Big|_{(\frac16,2)} = 2\langle -\pi y\sin(\pi xy), -\pi x\sin(\pi xy)\rangle\Big|_{(\frac16,2)}$$

$$= -2\pi\left\langle 2\sin\frac{\pi}{3}, \frac16\sin\frac{\pi}{3}\right\rangle = -2\pi\sin\frac{\pi}{3}\underline{\left\langle 2, \frac16\right\rangle}$$

$$= -\frac13\pi\sin\frac{\pi}{3}\langle 12, 1\rangle$$ C

$$0 = \underline{\langle n_1, n_2, n_3 \rangle} \overset{?}{\cdot} \langle x-1, y-1, z+1 \rangle$$

#21 $f(x,y,z) = x^2 + 2y^2 + 3z^2$ $\quad (1, 1, -1)$

$$\vec{n} = \nabla f(1,1,-1) = \langle 2x, 4y, 6z \rangle \Big|_{(1,1,-1)} = \langle 2, 4, -6 \rangle$$

$$= 2\langle 1, 2, -3 \rangle$$

E

$$0 = \langle 1, 2, -3 \rangle \cdot \langle x-1, y-1, z+1 \rangle$$

$$= x + 2y - 3z + (-1-2-3) = x + 2y - 3z - 6$$

#22 eq. of tangent plane $\boxed{z = f(2,\pi) + \nabla f(2,\pi) \cdot \langle x-2, y-\pi \rangle}$

$f(x,y) = \pi + \sin(\pi x^2 + 2y)$

$$f(2,\pi) = \pi + \sin(4\pi + 2\pi) = \pi$$

A

$$\nabla f(2,\pi) = \langle 2\pi x \cos(\pi x^2 + 2y),\ 2\cos(\pi x^2 + 2y) \rangle \Big|_{(2,\pi)}$$

$$= \langle 4\pi \cos 6\pi,\ 2\cos 6\pi \rangle = \langle 4\pi, 2 \rangle$$

$$\Rightarrow z = \pi + 4\pi(x-2) + 2(y-\pi) = 4\pi x + 2y - 9\pi$$ ✓

#23 $\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}$ $\quad f = xe^{y^2-z^2}$

$f_y = xe^{y^2-z^2} \cdot 2y$

D

$$= e^{y^2-x^2}dx + 2xy\,e^{y^2-z^2}dy - 2xz\,e^{y^2-z^2}dz$$

critical pt.

#24 $f(x,y) = 2x^3 - 6xy - 3y^2$, $\quad \nabla f = \langle 6x^2 - 6y, -6x - 6y \rangle = \langle 0, 0 \rangle$

B

$$\begin{cases} x^2 - y = 0 \\ x + y = 0 \Rightarrow \boxed{y = -x} \end{cases} \Rightarrow 0 = x^2 + x = x(x+1) \Rightarrow x = 0, -1$$

$$\Rightarrow (0,0),\ (-1,1)$$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x & -6 \\ -6 & -6 \end{vmatrix} = -36[2x+1]$$

$D(0,0) = -36 < 0$ saddle, $\quad D(-1,1) = 36 > 0$, $f_{xx}(-1,1) = -12 < 0$ l. max

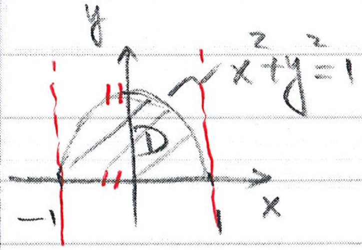
#25 $f(x,y) = 4x^2 + y^2$, $g(x,y) = xy$

$$\begin{cases} \nabla f = \langle 8x, 2y \rangle = \lambda \nabla g = \lambda \langle y, x \rangle \\ xy = 1 \end{cases} \Rightarrow \begin{cases} \boxed{\begin{matrix} 8x = \lambda y \\ 2y = \lambda x \end{matrix}} \\ xy = 1 \end{cases} \Rightarrow 16xy = \lambda^2 xy$$

$$\Rightarrow \lambda^2 = 16 \Rightarrow \lambda = \pm 4$$ E

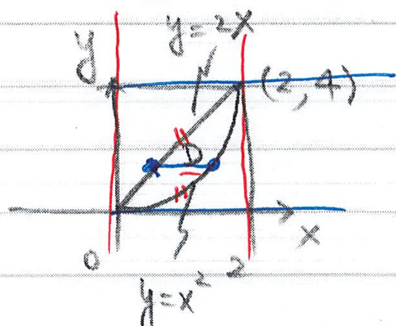
#26 $$\int_1^3 \int_0^x \frac{1}{x}\, dy\, dx = \int_1^3 \frac{1}{x} \cdot x\, dx = 2$$ B

#27



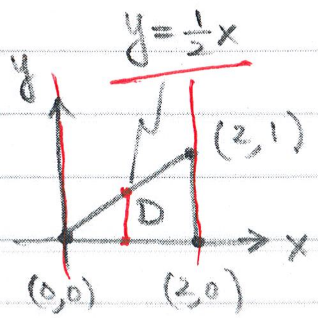
$$\begin{cases} 0 \le y \le \sqrt{1-x^2} \\ -1 \le x \le 1 \end{cases}$$ C

#28 $$\begin{cases} x^2 \le y \le 2x \\ 0 \le x \le 2 \end{cases}$$



$$\begin{cases} 0 \le y \le 4 \\ \frac{y}{2} \le x \le \sqrt{y} \end{cases}$$ B

#29



$$\iint_D y\, dA = \int_0^2 dx \int_0^{\frac{1}{2}x} y\, dy = \int_0^2 \frac{1}{2} y^2 \Big|_0^{\frac{1}{2}x} dx$$

$$= \frac{1}{2} \int_0^2 \frac{1}{4} x^2 dx = \frac{1}{8} \cdot \frac{1}{3} x^3 \Big|_0^2 = \frac{1}{3}$$ E

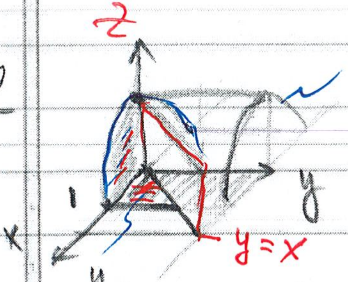
$$\int_0^2 \int_{x^2}^{2x} f\, dy\, dx$$

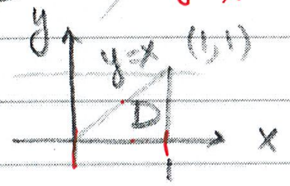
$0 \le x \le 2$

$0 \le y \le \frac{1}{2}x$

~~$x^2 \le y$~~

#30





$$\begin{cases} 0 \le z \le 1-x^2 \\ 0 \le y \le x \\ 0 \le x \le 1 \end{cases}$$

$$V=\int_0^1\int_0^x\int_0^{1-x^2} dz\,dy\,dx$$

$$=\int_0^1\int_0^x (1-x^2)\,dy\,dx$$

(A)

#31

$$\begin{cases} 0 \le r \le 5\sin(3\theta) \\ 0 \le \theta \le \frac{\pi}{3} \end{cases}$$

$$A=\int_0^{\frac{\pi}{3}}\int_0^{5\sin(3\theta)} 1\,\underbrace{r\,dr\,d\theta}_{dA}$$

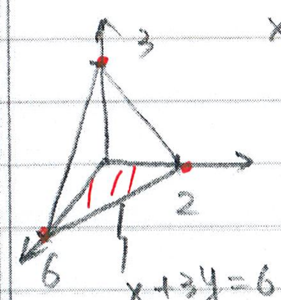
$$=\int_0^{\frac{\pi}{3}} \frac{1}{2}r^2\Big|_0^{5\sin(3\theta)} d\theta$$

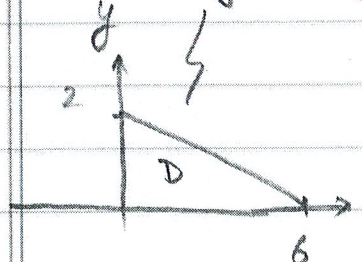
$$=\frac{1}{2}5^2\int_0^{\frac{\pi}{3}}\sin^2(3\theta)\,d\theta=\frac{25}{2}\cdot\int_0^{\frac{\pi}{3}}\frac{1-\cos(6\theta)}{2}\,d\theta$$

$$=\frac{25}{4}\left[\theta-\frac{1}{6}\sin(6\theta)\right]_0^{\frac{\pi}{3}}=\frac{25}{12}\pi$$

(B)

#32





$$x+3y+2z=6 \Rightarrow z=3-\frac{1}{2}x-\frac{3}{2}y$$

$$\vec{r}(x,y)=\langle x, y, 3-\frac{1}{2}x-\frac{3}{2}y\rangle,\ (x,y)\in D$$

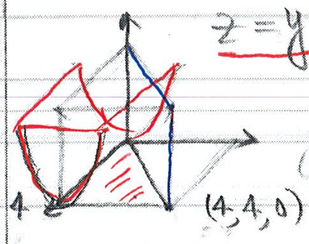
$$A=\iint_S 1\,dS=\iint_D\sqrt{1+\left(-\frac{1}{2}\right)^2+\left(-\frac{3}{2}\right)^2}\,dA$$

$$=\frac{\sqrt{14}}{2}\iint_D dA=\frac{\sqrt{14}}{2}\cdot\frac{1}{2}\cdot 2\cdot 6$$

$$=3\sqrt{14}$$

(D)

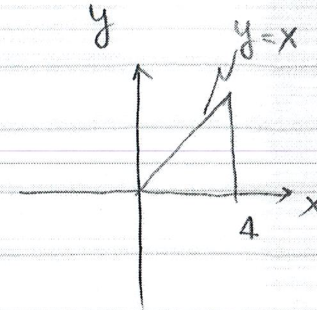
#33



$z = y^2$

$0 \le z \le y^2$

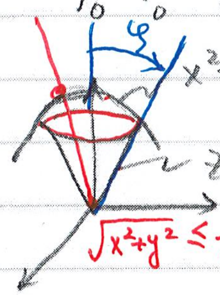
$\begin{cases} 0 \le y \le x \\ 0 \le x \le 4 \end{cases}$



B

$V = \int_0^4 \int_0^x \int_0^{y^2} 1 \, dz\, dy\, dx = \int_0^4 \int_0^x y^2\, dy\, dx = \int_0^4 \frac{1}{3} y^3 \Big|_0^x dx = \frac{1}{12} x^4 \Big|_0^4 = \frac{64}{3}$

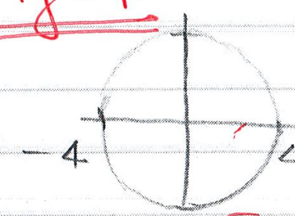
#34



$\left.\begin{array}{l} x^2+y^2+z^2 = 32 \\ z^2 = x^2+y^2 \end{array}\right\}$ $2(x^2+y^2) = 32 \Longrightarrow x^2+y^2 = 4^2$

$2z^2 = 32 \Rightarrow z = 4$

$\sqrt{x^2+y^2} \le z \le \sqrt{32-x^2-y^2}$ $D = \{ x^2+y^2 \le 4^2 \}$



B

$\rho(x,y,z) = z$

$m = \iiint_E z\, dV = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z\, dz\, dy\, dx$

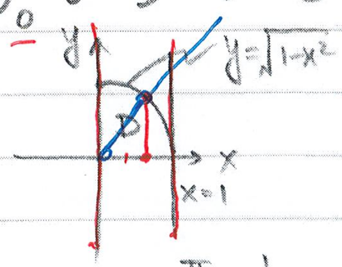
$z = \rho\cos\varphi = r = \rho\sin\varphi$

#35 $z^2 = x^2+y^2 \Longrightarrow \cos\varphi = \sin\varphi \Longrightarrow \varphi = \frac{\pi}{4}$ $0 \le \varphi \le \frac{\pi}{4}$

A

$m = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\sqrt{32}} \rho\cos\varphi \cdot \rho\sin\varphi\, d\rho$

#36 $\begin{cases} 0 \le y \le \sqrt{1-x^2} \\ 0 \le x \le 1 \end{cases}$



$0 \le \theta \le \frac{\pi}{2}$

$0 \le r \le 1$

E

$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 (x^2+y^2)^3\, dy\, dx = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 dr\, r^2\sin^2\theta \cdot (r^2)^3 \cdot r = \int_0^{\frac{\pi}{2}} \int_0^1 r^9 \sin^2\theta\, dr\, d\theta$

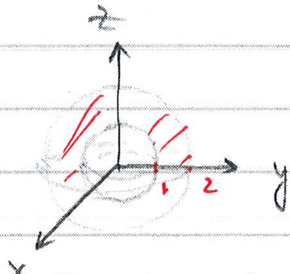
#37 $\sqrt{x^2+y^2} \le z \le 2$

$$\left.\begin{cases} \boxed{\begin{array}{c} -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \\ -2 \le x \le 2 \end{array}} \end{cases}\right\} D = \{ x^2+y^2 \le 4 \} \qquad \begin{cases} r \le z \le 2 \\ \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{cases} \end{cases}$$

$$I = \int_0^{2\pi} d\theta \int_0^2 r\,dr \int_r^2 dz$$

B

#38



above xy-plane

C

$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \frac{\pi}{2}$$
$$1 \le \rho \le 2$$

$$\iiint_D \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_1^2 \rho \cdot \rho^2 \sin\varphi\, d\rho$$

$$= 2\pi\, \frac{1}{4}\rho^4 \Big|_1^2 \left[-\cos\varphi\right]_0^{\frac{\pi}{2}} = \frac{15\pi}{2}$$

#39 1. $\text{curl}\, \langle xy^2+x,\ x^2y-y^2 \rangle = \frac{\partial}{\partial x}(x^2y-y^2) - \frac{\partial}{\partial y}(xy^2+x) = 2xy - 2xy = 0$ ✓

2. $\text{curl}\, \langle \frac{x}{y},\ \frac{y}{x} \rangle = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) - \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{y}{x^2} + \frac{x}{y^2} \ne 0$ ✗

3. $\text{curl}\, \langle ye^z,\ xe^z+e^y,\ (xy+1)e^z \rangle$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ ye^z & xe^z+e^y & (xy+1)e^z \end{vmatrix} = \langle xe^z - xe^z,\ ye^z - ye^z,\ e^z - e^z \rangle = \vec{0}$$ ✓

B

#40 c. $\text{grad}(\text{div}\, \vec{F}) \ne 0$ C

$\vec{F} = \langle x^2, y, t \rangle$ $\quad \nabla(2x+2) = \langle 2,0,0 \rangle$

$\nabla \cdot \vec{F} = 2x+2$ $\quad \ne \langle 0,0,0 \rangle$

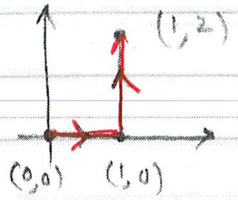
#41 $C: \vec{r}(t) = \langle t, t^2 \rangle, \quad t \in [0,2]$ $\quad \vec{r}'(t) = \langle 1, 2t \rangle$

$$m = \int_C x\, ds = \int_0^2 t\, |\langle 1, 2t \rangle|\, dt = \int_0^2 t\sqrt{1+4t^2}\, dt \xlongequal[du = 8t\,dt]{u = 1+4t^2} \int_1^{17} \frac{1}{8} u^{\frac{1}{2}}\, du$$

$$= \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{17} = \frac{1}{12}\left[17\sqrt{17} - 1\right]$$

A

#42



$$\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx + x^2\,dy$$

$$= \int_0^1 0\,dx + \int_0^2 1^2\,dy = 2$$

(D)

or $\vec{r}_1(t) = \langle t, 0\rangle,\ 0\le t\le 1$

$\vec{r}_2(t) = \langle 1, t\rangle,\ 0\le t\le 2$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}_1(t))\cdot \vec{r}_1{}'(t)\,dt + \int_0^2 \vec{F}(\vec{r}_2(t))\cdot \vec{r}_2{}'(t)\,dt$$

$$= \int_0^1 \langle 0, t^2\rangle\cdot\langle 1, 0\rangle\,dt + \int_0^2 \langle t, 1\rangle\cdot\langle 0, 1\rangle\,dt$$

$$= 0 + \int_0^2 1\,dt = 2$$

$\vec{F} = \langle x, y, xy\rangle$

#43

$$\int_C x\,dx + y\,dy + xy\,dz = \int_{-\frac{\pi}{2}}^0 \langle \cos t, \sin t, \cos t\sin t\rangle\cdot\langle -\sin t, \cos t, -\sin t\rangle\,dt$$

(D)

$$= \int_{-\frac{\pi}{2}}^0 \left[-\cos t\sin t + \sin t\cos t - \cos t\sin^2 t\right] dt$$

$$= -\int_{-\frac{\pi}{2}}^0 \sin^2 t\, d\sin t = -\frac{1}{3}\sin^3 t\Big|_{-\frac{\pi}{2}}^0 = -\frac{1}{3}\left[0 - \sin^3\left(-\frac{\pi}{2}\right)\right] = -\frac{1}{3}$$

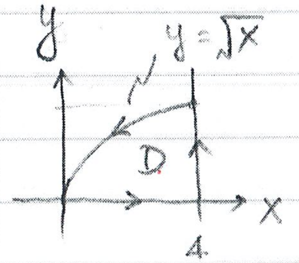
#44

$\vec{F} = \langle x^3 + 2xy, x^2 - y^2\rangle$, $\nabla\times\vec{F} = \frac{\partial}{\partial x}(x^2 - y^2) - \frac{\partial}{\partial y}(x^3 + 2xy) = 2x - 2x = 0$

(E)

$\Rightarrow \vec{F}$ is conservative.

#45



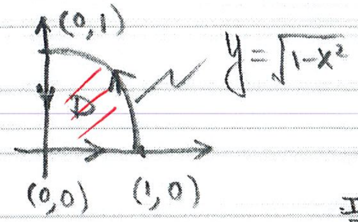
$$\int_C xy\,dy = \int_C \langle 0, xy\rangle\cdot d\vec{r}$$

$$= \iint_D \left[\frac{\partial}{\partial x}(xy) - 0\right] = \iint_D y\,dA$$

(B)

$$= \int_0^4\int_0^{\sqrt{x}} y\,dy\,dx = \int_0^4 \frac{1}{2}y^2\Big|_0^{\sqrt{x}} dx = \frac{1}{2}\int_0^4 x\,dx = \frac{1}{4}x^2\Big|_0^4 = 4$$

#46



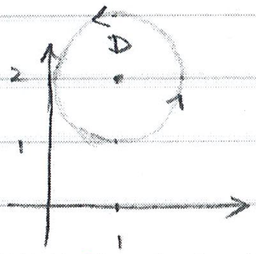
$$= \int_C \langle 0, xy\rangle \cdot d\vec{r}$$

$$\int_C \underline{xy}\, dy = \iint_D \underline{\frac{\partial Q}{\partial x}}\, dx\,dy = \iint_D y\, dx\,dy = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r\sin\theta\; r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^1 r^2\,dr \qquad \text{(B)}$$

$$= \Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} \; \frac{1}{3} r^3 \Big|_0^1 = \frac{1}{3}$$

#47



$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D \Big[(2xy-1) - (2xy)\Big]\, dx\,dy$$

$$= -\iint_D dx\,dy = -|D| = -\pi \qquad \text{(D)}$$

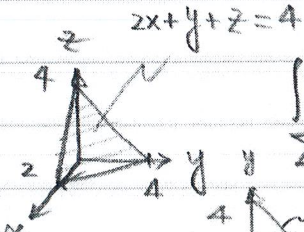
#48

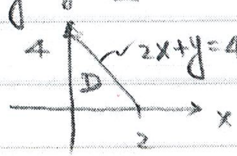
$$= \int_C \langle -y, 0\rangle \cdot d\vec{r}$$

$$-\int_C y\,dx = \iint_R -\frac{\partial P}{\partial y}\, d_A = \iint_R 1\, d_A = |R| \qquad \text{(A)}$$

#49

$2x+y+z=4$ $\boxed{z = 4-2x-y}$





(B)

$\vec{r}(x,y) = \langle x, y, 4-2x-y\rangle$

$(x,y)\in D$

$$\iint_\Sigma x\,dS = \iint_D x \left|\vec{r}_x \times \vec{r}_y\right| dx\,dy = \iint_D x \underline{\sqrt{1+(-2)^2+(-1)^2}\, dx\,dy}$$

$$= \sqrt{6}\int_0^2 x \int_0^{4-2x} dy\,dx = \sqrt{6}\int_0^2 (4x-2x^2)\,dx$$

$$= \sqrt{6}\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = \sqrt{6}\left(2^3 - \frac{2}{3}2^3\right) = \frac{8\sqrt{6}}{3}$$

#50

$z = x^2+y^2$

$\boxed{0 \le z \le 4}$

$4 = x^2+y^2$

$\boxed{x^2+y^2 \le 4}$

$\vec{r}(x,y) = \langle x, y, x^2+y^2\rangle, \quad (x,y)\in D = \{x^2+y^2 \le 4\}$

$\vec{r}_x \times \vec{r}_y = \langle -2x, -2y, 1\rangle$

$$\underline{\iint_\Sigma \vec{F}\cdot\vec{n}\,dS} = \iint_D \langle x, y, \underbrace{z}_{=x^2+y^2}\rangle \cdot \langle -2x, -2y, 1\rangle\, dx\,dy = \iint_D \left[-2x^2 - 2y^2 + z\right] dx\,dy$$

$$= \iint_D -(x^2+y^2)\,dx\,dy = -\int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta = -2\pi\cdot\frac{1}{4}r^4\Big|_0^2 = -8\pi \qquad \text{(E)}$$

$$\iiint_D \nabla\cdot\vec{F}\,dV$$

$$\#51 \quad \iint_{\Sigma} \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^1\int_0^{\frac{\pi}{2}} \operatorname{div}\vec{F}\,dz\,dy\,dx = 0 \quad \text{since } \operatorname{div}\vec{F} = 0$$

(A)

$$\vec{F} = \langle x, y, z\rangle, \quad \operatorname{div}\vec{F} = 3$$

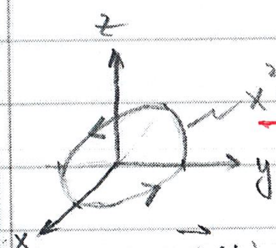
$$\#52 \quad \iint_{\Sigma} \vec{F}\cdot\vec{n}\,dS = \iiint_{R} \operatorname{div}\vec{F}\,dV = 3\,|R| = 3\int_0^1\int_0^{\pi}\int_0^{2\pi} \rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho$$

R = unit sphere

$$= 6\pi\int_0^1 \rho^2\,d\rho \int_0^{\pi}\sin\varphi\,d\varphi = 6\pi\,\frac{1}{3}\rho^3\Big|_0^1\cdot\Big[-\cos\varphi\Big]_0^{\pi} = 4\pi$$

(E)

#53



$$x^2+y^2=4$$

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{r} = \int_0^{2\pi}\langle 4\cos^2 t, 4\sin^2 t, 0\rangle\cdot\langle -2\sin t, 2\cos t, 0\rangle$$

$$\iint_S (\nabla\times\vec{F}\cdot\vec{n})\,dS$$

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle$$

$$= \int_0^{2\pi}\left[-8\cos^2 t\sin t + 8\sin^2 t\cos t\right]dt$$

$$= 8\int_0^{2\pi}\left[\cos^2 t\,d\cos t + \sin^2 t\,d\sin t\right]$$

$$= \frac{8}{3}\Big[\cos^3 t + \sin^3 t\Big]_0^{2\pi} = 0$$

(C)

#54

$$z = 1-x-y, \quad (x,y)\in D = \{x^2+y^2\le 9\}$$

$$\vec{r}(x,y) = \langle x, y, 1-x-y\rangle, \quad (x,y)\in D$$

$$\vec{r}_x\times\vec{r}_y = \langle 1, 1, 1\rangle$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 z & xy^2 & z^2 \end{vmatrix} = \langle 0, x^2, y^2\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_{\Sigma}\nabla\times\vec{F}\cdot\vec{n}\,dS = \iint_D \underbrace{\langle 0, x^2, y^2\rangle}_{\nabla\times\vec{F}}\cdot\underbrace{\langle 1,1,1\rangle}_{\vec{r}_x\times\vec{r}_y}\,dx\,dy$$

$$= \iint_D (x^2+y^2)\,dx\,dy = \int_0^3\int_0^{2\pi} r^2\cdot r\,d\theta\,dr = 2\pi\cdot\frac{1}{4}r^4\Big|_0^3$$

$$= \frac{\pi}{2}\cdot 3^4 = \frac{81\pi}{2}$$

(A)