Chapter 16 Vector Calculus

line integral

curve C : $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $t \in [a, b]$

- $$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t))\, |\vec{r}\,'(t)|\, dt$$

- $$\int_C \vec{F}(x,y,z) \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\, dt$$

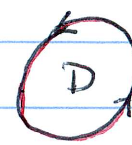
  $\vec{F} = \langle P, Q, R \rangle$

conservative field $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

- $\nabla \times \vec{F} = \vec{0} \iff \vec{F} = \nabla f$

- $$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Green's Thrm $C = \partial D$ oriented counterclockwise

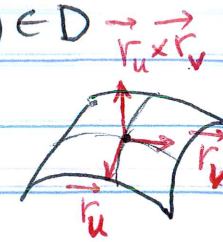


- $$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \qquad C = \partial D$$

- $$A(D) = \frac{1}{2} \int_C -y\,dx + x\,dy$$

parametrized surface

- $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \ (u,v) \in D$



- normal vector to S at $\vec{r}(u,v)$: $\vec{r}_u \times \vec{r}_v$
- eq of tangent plane to S at $\vec{r}(u_0,v_0) = \langle x_0, y_0, z_0\rangle$:

$$\left(\langle x,y,z\rangle - \langle x_0, y_0, z_0\rangle\right) \cdot \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0) = 0$$

surface integral

- $$\iint_S f(x,y,z)\, dS = \iint_D f(x(u,v), y(u,v), z(u,v))\, \left|\vec{r}_u \times \vec{r}_v\right| du\,dv$$
- $$\iint_S \vec{F}(x,y,z) \cdot d\vec{S} = \iint_D \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot \left(\vec{r}_u \times \vec{r}_v\right) du\,dv$$

parametrization of special surfaces

- graph $z = f(x,y)$, $(x,y) \in D$: $\vec{r}(x,y) = \langle x, y, f(x,y)\rangle$, $(x,y) \in D$

$$\vec{r}_x \times \vec{r}_y = \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\rangle$$

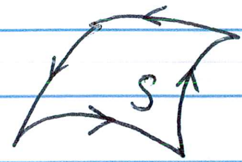
- sphere $x^2 + y^2 + z^2 = a^2$

$$\vec{r}(\theta,\varphi) = \langle a\cos\theta\sin\varphi, a\sin\theta\sin\varphi, a\cos\varphi\rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi = -a\sin\varphi\, \vec{r}(\theta,\varphi), \quad \left|\vec{r}_\theta \times \vec{r}_\varphi\right| = a^2 \sin\varphi$$

integral theorems

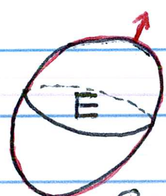
- Stoke's Thrm



$C = \partial S$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \, dS$$

- Gauss' Thrm



$S = \partial E$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F} \, dV$$