Name: ____________________
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Midterm 1– Math 262 (2/13/07)
SHOW ALL RELEVANT WORK!!!

1. (10pts) The general solution to $xy' + y = e^{2x}$ for $x > 0$ is

(A) $y = \frac{1}{5}e^{5x} + c$; (B) $y = \frac{1}{5x}e^{5x} + c\frac{1}{x}$; (C) $y = \frac{1}{6}e^{5x} + ce^{-x}$;

(D) $y = ce^{5x}$; (E) $y = \frac{5}{x}e^{5x} + c$.

Solution. Answer (B)

$$\begin{aligned} \mu &= e^{\int \frac{1}{x}\,dx} = e^{\ln|x|} = x. \\ y &= \frac{1}{x}\left(\int x\,\frac{1}{x}\,e^{2x}\,dx + c\right) \\ &= \frac{1}{x}\left(\frac{1}{2}\,e^{2x} + c\right). \end{aligned}$$

2. (10pts) An explicit solution of $y' = y^2 - 4$ is

(A) $y = \frac{2Ce^{4t}}{1-Ce^{4t}}$ (B) $y = \frac{2+2Ce^{4t}}{1-Ce^{4t}}$; (C) $y = \frac{2}{1-Ce^{4t}}$;

(D) $y = \frac{2+Ce^{4t}}{1-Ce^{4t}}$; (E) $\frac{1}{3}y^3 - 4y = C$.

Solution. Answer (B)

$$\begin{aligned} \frac{1}{y^2-4} &= \frac{1}{(y-2)(y+2)} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right) \\ &\left(\frac{1}{y-2} - \frac{1}{y+2}\right)\,dy = 4\,dx \\ &\ln\left|\frac{y-2}{y+2}\right| = 4x + c, \qquad \frac{y-2}{y+2} = C\,e^{4x} \\ &(1 - C\,e^{4x})y = 2 + 2\,Ce^{4x} \end{aligned}$$

3. (10pts) Solve the equation

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}.$$

Solution. Let $v = y/x$ and $F(v) = v + (1/v)$, then

$$\frac{1}{F(v) - v}\, dv = \frac{1}{x}\, dx, \qquad v\, dv = \frac{1}{x}\, dx$$

$$\frac{1}{2}\, v^2 = \ln|x| + c, \qquad v^2 = \ln|x|^2 + C$$

$$y^2 = x^2(\ln x^2 + C)$$

4. (10pts) A tank initially contains 120 liters of pure water. A mixture containing a concentration of γ g/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. The initial value problem describing the amount of salt at time t, $Q(t)$, becomes:

A. $Q' + \frac{1}{60}Q = 2\gamma, \quad Q(0) = 120;$

B. $Q' + \frac{1}{120}Q = 2\gamma, \quad Q(0) = 120;$

C. $Q' = 2\gamma, \quad Q(0) = 0;$

D. $Q' + \frac{1}{60}Q = 2\gamma, \quad Q(0) = 0;$

E. none of above.

Solution. Answer (D)

$$Q(0) = 0, \quad Q' = 2\gamma - \frac{2Q}{120}, \quad Q' + \frac{Q}{60} = 2\gamma.$$

5. (10pts) Find an implicit solution of $(y\cos x+2xe^y)+(\sin x+x^2e^y-1)y'=0$ and $y(\pi)=1$.

Solution.

$$M=y\cos x+2xe^y,\quad N=\sin x+x^2e^y-1$$

$$M_y=N_x=\cos x+2xe^y$$

$$\psi_x=M,\qquad \psi=y\sin x+x^2e^y+h(y)$$

$$\psi_y=\sin x+x^2e^y+h'(y)=N=\sin x+x^2e^y-1$$

$$h'(y)=-1,\qquad h(y)=-y+c$$

$$y\sin x+x^2e^y-y=C$$

$$C=1\cdot\sin\pi+\pi^2e^1-1=\pi^2e-1.$$

6. (10pts) Solve the equation

$$(1+x^2)y''=-2xy'.$$

Note that $\int\frac{1}{1+x^2}\,dx=\tan^{-1}x+c$.

Solution. Let $u=y'$, then $(1+x^2)u'=-2xu$

$$\frac{1}{u}\,du=-\frac{2x}{1+x^2}\,dx\qquad \ln|u|=-\ln|1+x^2|+c$$

$$|u|=e^c\frac{1}{1+x^2}\qquad u=C\,\frac{1}{1+x^2}$$

$$y=\int\frac{C}{1+x^2}\,dx+C_1=C\tan^{-1}x+C_1$$

7. (10pts) Find A^T, B^T, AB, and B^TA^T for

$$A = \begin{pmatrix} 1 & -1 & 1 & 4 \\ 2 & 0 & 2 & -3 \\ 1 & 2 & -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Solution.

$$A^T = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 1 & 2 & -1 \\ 4 & -3 & 0 \end{pmatrix} \quad \text{and} \quad B^T = \begin{pmatrix} 0 & -1 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 10 & 4 \\ -4 & 1 \\ -3 & 4 \end{pmatrix} \quad \text{and} \quad B^TA^T = (AB)^T = \begin{pmatrix} 10 & -4 & -3 \\ 4 & 1 & 4 \end{pmatrix}.$$

8. (10pts) Determine all values of the constant k for which the following system has (1) no solution, (2) an infinite number of solutions, (3) a unique solution.

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 4, \\ 2x_1 + 5x_2 + x_3 &= 7, \\ x_1 + x_2 - k^2x_3 &= -k. \end{aligned}$$

Solution.

$$\begin{pmatrix} 1 & 2 & -2 & 4 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -k^2 & -k \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & 1 & 5 & -1 \\ 0 & -1 & -k^2+2 & -k-4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -k^2+7 & -k-5 \end{pmatrix}$$

Let $-k^2 + 7 = 0$, then $k = \pm\sqrt{7}$.
(1) when $k = \pm\sqrt{7}$, it has no solution.
(1) for no values of k, it has an infinite number of solutions.
(3) when $k \neq \pm\sqrt{7}$, it has a unique solution.

9. (10pts) Let A be a 3×3 matrix. If both $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions of the linear system $A\,\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Then which of the following is also a solution?

(A) $\mathbf{x_1} - \mathbf{x_2}$; (**B**) $\mathbf{x_1} - \mathbf{2x_2}$; (**C**) $\mathbf{2x_1} - \mathbf{x_2}$; (**D**) All of the above; (**E**) None of the above.

Solution. Answer (C)

$$A(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{Ax_1} - \mathbf{Ax_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Similarly,

$$A(\mathbf{x_1} - \mathbf{2x_2}) = \mathbf{Ax_1} - \mathbf{2Ax_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \mathbf{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A(2\mathbf{x_1} - \mathbf{x_2}) = \mathbf{2Ax_1} - \mathbf{Ax_2} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

10. (10pts) If $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{pmatrix}$ and $B = A^{-1}$, what is the element b_{12} of B in row 1 and column 2?

$(A)\ -1$; $(B)\ -2$; $(C)\ -3$; $(D)\ -4$; $(E)\ -5$.

Solution. Answer (A)

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{pmatrix}$$

90 – 100	80 – 89	70 – 79	60 – 69	50 – 59	40 – 49	30 – 39	0 – 29
3	7	9	8	6	4	1	

Table 1: Grade Distribution

Grades

99,90,90,89,88,87,85,82,81,80,78,78,75,73,73,72,72,72,71,69,68,67,67,
62,61,61,60,57,56,52,50,50,50,