Chapter 6 Nonlinear Systems and Phenomena (§6.1, 6.2, 6.3, 6.4)

§6.1 Stability and the Phase Plane

autonomous system
$$\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) = g(x(t), y(t)) \end{cases}$$

- solution $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, geometrically, it is a parametric curve in R^2.
- equilibrium solution $\begin{cases} x = x_* \\ y = y_* \end{cases}$ where $\begin{cases} f(x_*, y_*) = 0 \\ g(x_*, y_*) = 0 \end{cases}$

where (x_*, y_*) is called critical pt. Behavior of (x_*, y_*)?

Example 1 Find critical pts $(0,0), (0,8), (7,0), (4,6)$
$$\begin{cases} x'(t) = 14x - 2x^2 - xy = f(x,y) \\ y'(t) = 16y - 2y^2 - xy = g(x,y) \end{cases}$$

$$\begin{cases} 0 = f(x,y) = 14x - 2x^2 - xy = x(14 - 2x - y) \Rightarrow \begin{cases} x = 0 \\ \text{or } 14 - 2x - y = 0 \end{cases} \\ 0 = g(x,y) = 16y - 2y^2 - xy = y(16 - 2y - x) \end{cases}$$

(1) $x = 0 \Rightarrow 0 = g(0,y) = y(16 - 2y) \Rightarrow \begin{cases} y = 0 \\ \text{or } y = 8 \end{cases} \Rightarrow (0,0), (0,8)$

(2) $x = 7 - \frac{1}{2}y \Rightarrow 0 = g(7 - \frac{1}{2}y, y) = y(16 - 2y - 7 + \frac{1}{2}y)$

$= y\left[9 - \frac{3}{2}y\right] \Rightarrow \begin{cases} y = 0 \\ \text{or } y = 6 \end{cases} \xRightarrow{x = 7 - \frac{1}{2}y} \begin{cases} x = 7 & (7,0) \\ x = 7 - 3 = 4 & (4,6) \end{cases}$

- Phase Portraits (or phase plane picture)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} f \\ g \end{pmatrix}$$

solution curves (trajectories) $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$

Direction field

$$(f(x,y),\ g(x,y))$$

Example 2 $\begin{cases} x' = x - y \\ y' = 1 - x^2 \end{cases}$

- critical points
- phase plane (critical pt behavior)

$\begin{cases} 0 = x - y \\ 0 = 1 - x^2 \end{cases}$ $\Longrightarrow$ $y = x$, $\Longrightarrow$ $x = \pm 1$

$(-1,-1)$, $(1,1)$

- stability

a critical pt (x_*, y_*) is stable

$$\Longleftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } |\vec{x}(0) - \vec{x}_*| < \delta \Rightarrow |\vec{x}(t) - \vec{x}_*| < \varepsilon \ \forall t > 0$$

- asymptotic stable

a critical pt (x_*, y_*) is asymptotic stable

$$\Longleftrightarrow \text{(1) it is stable and (2) } |\vec{x}(0) - \vec{x}_*| < \delta \Rightarrow \lim_{t \to \infty} \vec{x}(t) = \vec{x}_*$$

Example 3

$\begin{cases} x'' + \omega^2 x = 0 \\ y = x' \end{cases} \iff \begin{cases} x' = y \\ y' = -\omega^2 x \end{cases}$

$\omega = \dfrac{k}{m}$ — k: Hooke's constant, m: mass

general solution $\begin{cases} x(t) = A\cos\omega t + B\sin\omega t = C\cos(\omega t - \alpha) \\ y(t) = -A\omega\sin\omega t + B\omega\cos\omega t = -\omega C\sin(\omega t - \alpha) \end{cases}$

$$\Downarrow$$

$$\frac{x^2}{C^2} + \frac{y^2}{\omega^2 C^2} = 1$$

where $A = C\cos\alpha$, $B = C\sin\alpha$.

- critical pt $(0,0)$ is stable.

Example 4 $\begin{cases} x' = y \\ y' = -2x - 2y \end{cases} \iff \begin{cases} x'' + 2x' + 2x = 0 \\ y = x' \end{cases}$

mass-spring system

$c = 2$ — damping constant

$$0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = \lambda(\lambda+2) + 2 = \lambda^2 + 2\lambda + 2$$

$$= (\lambda+1)^2 + 1$$

$$\Rightarrow \lambda = -1 \pm i$$

phase-plane

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1+i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow b = (1-i)a$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{x}_1(t) = e^{-t}\begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix}, \quad \vec{x}_2(t) = e^{-t}\begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}$$

$$\vec{x}(t) = e^{-t}\begin{pmatrix} A\cos t + B\sin t \\ (A-B)\cos t + (A+B)\sin t \end{pmatrix}$$

#24 $\begin{cases} \dfrac{dx}{dt} = y\underline{(1+x^2+y^2)} \\ \dfrac{dy}{dt} = x\underline{(1+x^2+y^2)} \end{cases}$

(1) solve $\underline{dy/dx = g(x,y)/}\underline{f(x,y)}$

(2) phase plane; critical pts.

$$\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x(1+x^2+y^2)}{y(1+x^2+y^2)} = \boxed{\frac{x}{y} = \frac{dy}{dx}} = f(x,y) = g(x)\,h(y)$$

$$\underline{\frac{1}{h(y)}\,dy = g(x)\,dx}$$

$$y\,dy = x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

critical pt

$$\begin{cases} 0 = y(1+x^2+y^2) \Longrightarrow y=0 \\ 0 = x(1+x^2+y^2) \Longrightarrow x=0 \end{cases} \qquad (0,0)$$

<u>Summary</u> $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$

(1) $(x(t), y(t)) \longrightarrow (x_*, y_*)$ as $t \to +\infty$

(2) $(x(t), y(t))$ is unbounded with increasing t

(3) $(x(t), y(t))$ is a periodic solution with a closed trajectory

(4) $(x(t), y(t))$ spirals toward a closed trajectory as $t \to +\infty$

§6.2 Linear and Almost Linear Systems

(1) $\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$

• behavior of $(x(t), y(t))$ near an isolated critical pt (x_0, y_0)?

$\exists$ a neighborhood of (x_0, y_0) s.t. (x_0, y_0) is the only critical pt.

$$\Downarrow \quad \begin{cases} u = x - x_0 \\ v = y - y_0 \end{cases}$$

$$\begin{cases} \dfrac{du}{dt} = f(u+x_0, v+y_0) = f_1(u,v) \\ \dfrac{dv}{dt} = g(u+x_0, v+y_0) = g_1(u,v) \end{cases}$$

has a critical pt at $(0,0)$.

Linearization Near a Critical Pt

Assume that (x_0, y_0) is an isolated critical pt:

$$\begin{cases} f(x_0, y_0) = 0 \\ g(x_0, y_0) = 0 \end{cases}$$

the linearized system $\Longrightarrow$

(2) $$\begin{cases} \dfrac{du}{dt} = f_x(x_0, y_0)\, u + f_y(x_0, y_0)\, v \\ \dfrac{dv}{dt} = g_x(x_0, y_0)\, u + g_y(x_0, y_0)\, v \end{cases} = J(x_0, y_0)\begin{pmatrix} u \\ v \end{pmatrix}$$

where $J(x_0, y_0) = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}$ the Jacobi matrix.

Def. (1) is locally linear at (x_0, y_0)

$$\Longleftrightarrow \begin{cases} \bullet \begin{pmatrix} f \\ g \end{pmatrix} = J(x_0, y_0)\begin{pmatrix} u \\ v \end{pmatrix} + \vec{r}(u,v) \\ \bullet \lim\limits_{\|(u,v)\| \to 0} \|\vec{r}(u,v)\| \big/ \|(u,v)\| = 0 \end{cases}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Example 1 $\begin{cases} x' = 3x - x^2 - xy \\ y' = y + y^2 - 3xy \end{cases}$

- critical pts?
- local linear systems at $(1,2)$?

CP $0 = 3x - x^2 - xy = x(3-x-y)$

$0 = y(1+y-3x)$

$x=0$ $(0,0)$ or $(0,-1)$

$y=0$ $(3,0)$ $\begin{cases} x+y=3 \\ 3x-y=1 \end{cases} \Rightarrow (1,2)$

$f_x = 3-2x-y$, $f_y = -x$

$g_x = -3y$, $g_y = 1+2y-3x$

$$J(1,2) = \begin{pmatrix} 3-2-2 & -1 \\ -3\cdot 2 & 1+2\cdot 2-3\cdot 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -6 & 2 \end{pmatrix}$$

Isolated Critical Pts of Linear Systems

$$\vec{x}\,' = A\vec{x}, \quad \text{where } A_{2\times 2} \text{ — constant matrix}$$

- $\vec{x} = \vec{0}$ is an isolated critical pt

$\Longleftrightarrow$ $0 \neq \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\Longleftrightarrow$ $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

$$0 = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (\lambda-a)(\lambda-d) - bc = \lambda(\lambda-(a+d)) + \det A$$

Real parts of λ_1 and λ_2	type of critical pt	stability
(1) both negative	proper/improper nodal sink, or spiral sink	asymptotically stable
(2) $\boxed{\lambda = \pm \beta i}$	center	stable but no AS.
(3) at least one positive	nodal source, or spiral source, or saddle pt	unstable

Theorem (Stability of Linear Systems)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Let λ_1 and λ_2 be the eigenvalues of A. Assume that $\det(A) \neq 0$.

$\Longrightarrow$ the critical pt $(0,0)$ is
$$\begin{cases} \text{(1) AS if } Re(\lambda_1) < 0 \text{ and } Re(\lambda_2) < 0; \\ \text{(2) stable but not AS if } Re(\lambda_1) = Re(\lambda_2) = 0; \\ \text{(3) unstable if either } Re(\lambda_1) > 0 \text{ or } Re(\lambda_2) > 0. \end{cases}$$

- **Locally Almost Linear Systems**

$$\begin{cases} x' = f(x,y) = ax + by \boxed{+ r(x,y)} \\ y' = g(x,y) = cx + dy \boxed{+ s(x,y)} \end{cases}$$

where $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$,
$c = g_x(x_0, y_0)$, $d = g_y(x_0, y_0)$

LALS
$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$
assume that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$

Theorem 2 (stability of LALS)

(1) λ_1 and λ_2 are real and $\lambda_1 = \lambda_2$

$\Longrightarrow$ $(0,0)$ is
$$\begin{cases} \bullet \text{ asym. stable (node or spiral pt) if } \lambda_1 = \lambda_2 < 0 \\ \bullet \text{ unstable if } \lambda_1 = \lambda_2 > 0 \end{cases}$$

(2) $\lambda_1 = \xi i$, $\lambda_2 = -\xi i$

$\Longrightarrow$ $(0,0)$ is either a center or a spiral pt

and may be either AS, S, or Unstable.

(3) Otherwise, the critical pt $(0,0)$ of LALS is of the same type and stability as the critical pt $(0,0)$ of the associated linear system.

eigenvalues of the linearized system	type of CP of the Locally Almost Linear System
• $\lambda_1 < \lambda_2 < 0$	stable improper node
• $\lambda_1 = \lambda_2 < 0$	stable node or spiral pt
• $\lambda_1 < 0 < \lambda_2$	unstable saddle pt
• $\lambda_1 = \lambda_2 > 0$	unstable node or spiral pt
• $\lambda_1 > \lambda_2 > 0$	unstable improper node
• $\lambda_1, \lambda_2 = a \pm bi \quad (a<0)$ ✓	stable spiral pt
• $\lambda_1, \lambda_2 = a \pm bi \quad (a>0)$	unstable spiral pt
• $\boxed{\lambda_1, \lambda_2 = \pm bi}$	stable or unstable, center or spiral pt.

Example 2
$$\begin{cases} x' = 4x + 2y + 2x^2 - 3y^2 = f \\ y' = 4x - 3y + 7xy = g \end{cases}$$
the type and stability of the CP $(0,0)$?

$f_x = 4 + 4x \qquad f_x(0,0) = 4 \qquad g_x = 4 + 7y \qquad g_x(0,0) = 4$

$f_y = 2 - 6y \qquad f_y(0,0) = 2 \qquad g_y = -3 + 7x \qquad g_y(0,0) = -3$

$$\Rightarrow J(0,0) = \begin{pmatrix} 4 & 2 \\ 4 & -3 \end{pmatrix}$$

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 4 & -3-\lambda \end{vmatrix} = (\lambda-4)(\lambda+3) - 8 = \lambda^2 - \lambda - 20 = (\lambda-5)(\lambda+4)$$

$\lambda_1 = -4, \quad \lambda_2 = 5 \qquad$ unstable saddle

Example 3 $\begin{cases} x' = 33 - 10x - 3y + x^2 \\ y' = -18 + 6x + 2y - xy \end{cases}$ the type and stability of the CP $(4,3)$?

$f_x = -10 + 2x \quad f_x(4,3) = -2 \qquad g_x = 6 - y, \quad g_x(4,3) = 3$

$f_y = -3 \qquad g_y = 2 - x, \quad g_y(4,3) = -2$

$$J(4,3) = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$$

$$0 = \begin{vmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 + 9 \implies \lambda = -2 \pm 3i$$

stable, spiral

§6.3 Ecological Models: Predators and Competitors

- math model of a predator-prey situation

  $x(t)$ — the number of prey at time t

  $y(t)$ — the number of predator at time t

(1) in the absence of predator: $x'(t) = a\,x(t)$, $a > 0$

(2) in the absence of prey: $y'(t) = -b\,y(t)$, $b > 0$

(3) both present: $\begin{cases} \text{an interaction rate of decline } -pxy \text{ in } x \\ \text{an interaction rate of growth } qxy \text{ in } y \end{cases}$

$$\begin{cases} x' = ax - pxy = x(a - py) \\ y' = -by + qxy = y(-b + qx) \end{cases} \qquad \boxed{a, b, p, q > 0}$$

- critical pts $(0,0)$, $\left(\frac{b}{q}, \frac{a}{p}\right)$

$$\begin{cases} x(a - py) = 0 \\ y(-b + qx) = 0 \end{cases} \Longrightarrow \begin{cases} x = 0 \Longrightarrow (0,0) \\ \text{or } y = \frac{a}{p} \Longrightarrow \left(\frac{b}{q}, \frac{a}{p}\right) \end{cases}$$

- Jacobian matrix

$$J(x,y) = \begin{pmatrix} a - py & -px \\ qy & -b + qx \end{pmatrix}$$

- critical pt $(0,0)$

$$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} \qquad \lambda_1 = a > 0, \quad \lambda_2 = -b < 0 \qquad \text{unstable saddle pt.}$$

- critical pt $\left(\frac{b}{q}, \frac{a}{p}\right)$

$$J\left(\frac{b}{q}, \frac{a}{p}\right) = \begin{pmatrix} 0 & -\frac{pb}{q} \\ \frac{aq}{p} & 0 \end{pmatrix}$$

$\lambda_1 = \sqrt{ab}\, i$

$\lambda_2 = -\sqrt{ab}\, i$

linear: stable center — nonlinear: indeterminate

phase plane

Example 2

$$\begin{cases} x' = 0.2x - 0.005xy \\ y' = -0.5y + 0.01xy \end{cases}$$

Fig 6.3.1 — stable center

$x(t)$ vs t, $y(t)$ vs t — Fig 6.3.3 — $x \in (33, 72)$, $y \in (20, 70)$

with $x(0) = 70$, $y(0) = 40$

rabbit, fox

- Competing Species

$$\begin{cases} x' = a_1 x - b_1 x^2 - c_1 xy \\ y' = a_2 y - b_2 y^2 - c_2 xy \end{cases}$$

where a_i, b_i, c_i are positive

four critical pts: $(0,0)$, $\left(0, \frac{a_2}{b_2}\right)$, $\left(\frac{a_1}{b_1}, 0\right)$, (x_E, y_E)

$$\begin{cases} 0 = x(a_1 - b_1 x - c_1 y) \\ 0 = y(a_2 - b_2 y - c_2 x) \end{cases}$$

$$\begin{cases} b_1 x + c_1 y = a_1 \\ c_2 x + b_2 y = a_2 \end{cases}$$

stability at (x_E, y_E)

$$J(x_E, y_E) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}(x_E, y_E) = \begin{pmatrix} a_1 - 2b_1 x_E - c_1 y_E & -c_1 x_E \\ -c_2 y_E & a_2 - 2b_2 y_E - c_2 x_E \end{pmatrix}$$

$$= \begin{pmatrix} -b_1 x_E & -c_1 x_E \\ -c_2 y_E & -b_2 y_E \end{pmatrix}$$

$$0 = \begin{vmatrix} -b_1 x_E - \lambda & -c_1 x_E \\ -c_2 y_E & -b_2 y_E - \lambda \end{vmatrix}$$

$$= (\lambda + b_1 x_E)(\lambda + b_2 y_E) - c_1 c_2 x_E y_E$$

- $b_1 b_2$ measures inhibition

$$= \lambda^2 + (b_1 x_E + b_2 y_E)\lambda + (b_1 b_2 - c_1 c_2) x_E y_E$$

$c_1 c_2$ measures competition $\Rightarrow \lambda_{1,2} = \frac{1}{2}\left[-(b_1 x_E + b_2 y_E) \pm \sqrt{(b_1 x_E + b_2 y_E)^2 - 4(b_1 b_2 - c_1 c_2) x_E y_E}\right]$

(1) $c_1 c_2 < b_1 b_2$ the competition is small in comparison with inhibition

($\lambda_1 < 0$ and $\lambda_2 < 0$)

$\Longrightarrow (x_E, y_E)$ is an asymptotic stable $\Longrightarrow$ coexist

(2) $c_1 c_2 > b_1 b_2$ the competition is large

($\lambda_1 < 0$ and $\lambda_2 > 0$)

$\Longrightarrow (x_E, y_E)$ is an unstable CP $\Longrightarrow$ one survives and one becomes extinct

Example 3 (Survival of a Single Species)

$$\begin{cases} x' = 14x - \frac{1}{2}x^2 - xy \\ y' = 16y - \frac{1}{2}y^2 - xy \end{cases}$$

$(0,0)$, $(0,32)$, $(28,0)$, $(12,8)$

$c_1 c_2 = 1 \cdot 1 = 1 > b_1 b_2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$$J(x,y) = \begin{pmatrix} 14 - x - y & -x \\ -y & 16 - y - x \end{pmatrix}$$

- critical pt $(0,0)$

$$J(0,0) = \begin{pmatrix} 14 & 0 \\ 0 & 16 \end{pmatrix}$$

$\lambda_1 = 14$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 16$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

unstable nodal source

- **critical pt (0, 32)**

$$J(0,32) = \begin{pmatrix} -18 & 0 \\ -32 & -16 \end{pmatrix}, \quad \lambda_1 = -18, \ \vec{v}_1 = \begin{pmatrix} 1 \\ 16 \end{pmatrix}, \quad \lambda_2 = -16, \ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

stable nodal sink

- **critical pt (28, 0)**

$$J(28,0) = \begin{pmatrix} -14 & -28 \\ 0 & -12 \end{pmatrix}, \quad \lambda_1 = -14, \ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = -12, \ \vec{v}_2 = \begin{pmatrix} -14 \\ 1 \end{pmatrix}$$

stable nodal sink

- **critical pt (12, 8)**

$$J(12,8) = \begin{pmatrix} -6 & -12 \\ -8 & -4 \end{pmatrix}, \quad \lambda_1 = -5-\sqrt{97} < 0, \ \vec{v}_1 = \begin{pmatrix} \frac{1}{8}(1+\sqrt{97}) \\ 1 \end{pmatrix}, \quad \lambda_2 = -5+\sqrt{97} > 0, \ \vec{v}_2 = \begin{pmatrix} \frac{1}{8}(1-\sqrt{97}) \\ 1 \end{pmatrix}$$

unstable saddle pt.

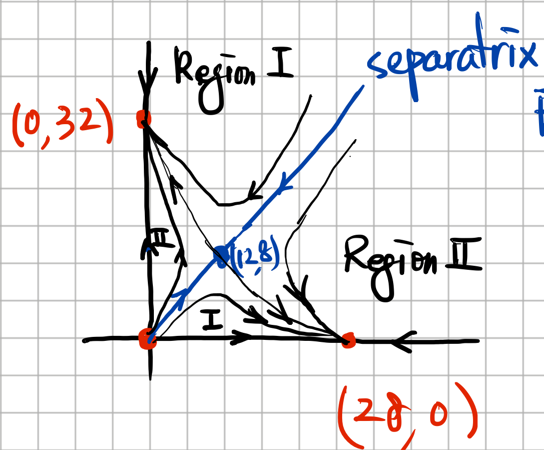


Fig. 6.3.8, 6.3.9

Example 4 (Peaceful Coexistence of Two Species)

$$\begin{cases} x' = 14x - 2x^2 - xy \\ y' = 16y - 2y^2 - xy \end{cases}$$

critical pts: $(0,0), (0,8), (7,0), (4,6)$

$$J(x,y) = \begin{pmatrix} 14 - 4x - y & -x \\ -y & 16 - 4y - x \end{pmatrix}$$

$c_1 c_2 = 1 \cdot 1 = 1 < b_1 b_2 = 2 \cdot 2 = 4$

- CP $(0,0)$

$$J(0,0) = \begin{pmatrix} 14 & 0 \\ 0 & 16 \end{pmatrix} \quad \lambda_1 = 14,\ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \lambda_2 = 16,\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

unstable nodal source

- CP $(0,8)$

$$J(0,8) = \begin{pmatrix} 6 & 0 \\ -8 & -16 \end{pmatrix} \quad \lambda_1 = 6,\ \vec{v}_1 = \begin{pmatrix} 11 \\ -4 \end{pmatrix} \quad \lambda_2 = -16,\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

unstable saddle pt

- CP $(7,0)$

$$J(7,0) = \begin{pmatrix} -14 & -7 \\ 0 & 9 \end{pmatrix} \quad \lambda_1 = -14\ \ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \lambda_2 = 9\ \ \vec{v}_2 = \begin{pmatrix} -7 \\ 23 \end{pmatrix}$$

unstable saddle pt

- CP $(4,6)$

$$J(4,6) = \begin{pmatrix} -8 & -4 \\ -6 & -12 \end{pmatrix} \quad \lambda_1 = 2(-5-\sqrt{7})\ \ \vec{v}_1 = \begin{pmatrix} \frac{1}{3}(-1+\sqrt{7}) \\ 1 \end{pmatrix} \quad \lambda_2 = 2(-5+\sqrt{7})\ \ \vec{v}_2 = \begin{pmatrix} \frac{1}{3}(-1-\sqrt{7}) \\ 1 \end{pmatrix}$$

stable nodal sink

Fig. 6.3.13

- interactions of Logistic Populations

(1) $c_1 = c_2 = 0$

two separate logistic populations

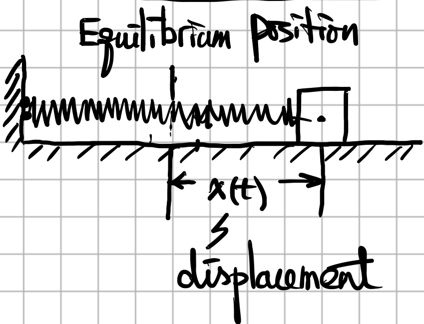
(2) $c_1 > 0$ and $c_2 > 0$ competition

$\Longrightarrow$ each population is "hurt" by their mutual interactions.

(3) $\underline{c_1 < 0 \text{ and } c_2 < 0}$ cooperation

(4) $\underline{c_1\, c_2 < 0}$ predation

§6.4 Nonlinear Mechanical Systems



- $F(x) = -kx + \beta x^3$ nonlinear spring force
- $m x''(t) = -kx + \beta x^3$

- **Position - Velocity Phase Plane**

velocity $y(t) = x'(t)$ $\Longrightarrow$ $\begin{cases} \dfrac{dx}{dt} = y \\ m\dfrac{dy}{dt} = -kx + \beta x^3 \end{cases}$

$$\Longrightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-kx+\beta x^3}{my} \Longrightarrow my\,dy = \left(-kx+\beta x^3\right)dx$$

$$\Longrightarrow \underbrace{\frac{1}{2}my^2}_{\text{the kinetic energy}} + \underbrace{\frac{1}{2}kx^2 - \frac{1}{4}\beta x^4}_{\text{the potential energy}} = E \sim \text{constant}$$

- <u>hard spring oscillations</u> $\beta < 0$

$$my' = -kx + \beta x^3 = -x\left(k + |\beta| x^2\right) = 0 \Longrightarrow x = 0$$

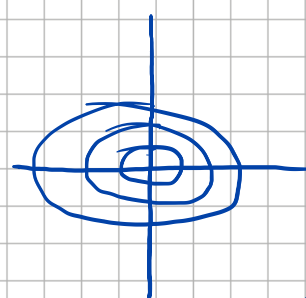
$$\Longrightarrow \begin{cases} x' = y \\ my' = -kx + \beta x^3 \end{cases}$$ has only critical pt $(0,0)$

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} - \frac{3|\beta|}{m}x^2 & 0 \end{pmatrix}, \quad J(0,0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \text{ with } \omega^2 = \frac{k}{m}$$

$\omega > 0$

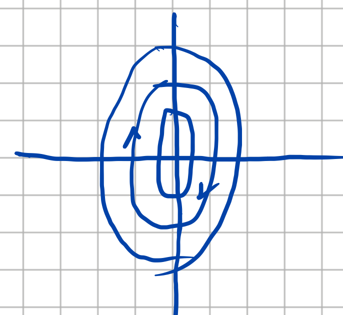
$$\Longrightarrow \lambda_{1,2} = \pm \omega i$$

$\Longrightarrow$ linear system $\begin{cases} x' = y \\ y' = -\omega^2 x \end{cases}$



has a stable center at $(0,0)$

nonlinear system (Fig. 6.4.2)



- <u>soft spring oscillations</u> $\beta > 0$

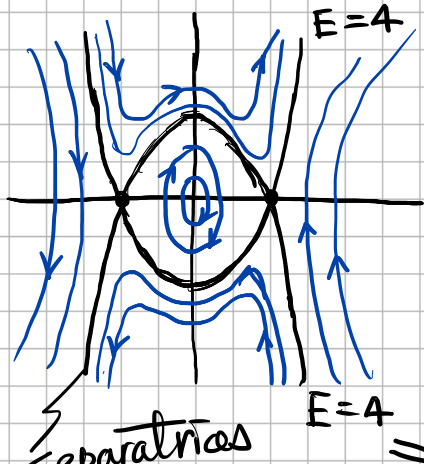
$$my' = -kx + \beta x^3 = x\left(\beta x^2 - k\right)$$

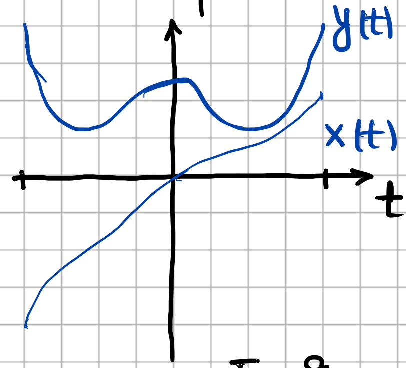
$$\begin{cases} x' = y \\ y' = x\left(\beta x^2 - k\right) \end{cases}$$ critical pts $(0,0)$, $\left(\pm\sqrt{\frac{k}{\beta}}, 0\right)$

<u>Example 1</u> $(m=1, k=4, \beta=1)$ $(0,0)$, $(-2,0)$, $(2,0)$ <u>CP</u>

$$y = \pm\sqrt{2E - 4x^2 + \frac{1}{2}x^4}, \quad E - \text{arbitrary const.}$$

$$E=\left[\tfrac{1}{2}my^2+\tfrac{1}{2}kx^2-\tfrac{1}{4}\beta x^4\right]_{\left(\pm\sqrt{\frac{k}{\beta}},0\right)}=\frac{k^2}{4\beta}\Bigg|_{k=4,\beta=1}=4$$





$E=4 \Longrightarrow$ $E=8$

$$y=\pm\sqrt{8-4x^2+\tfrac{1}{2}x^4}=\pm\frac{1}{\sqrt{2}}\left(x^2-4\right)$$

$(0,0)$ a stable center, $(\pm 2,0)$ saddle pts

$$\begin{cases} x'=y \\ y'=-4x+x^3 \end{cases} \Longrightarrow J(x,y)=\begin{pmatrix} 0 & 1 \\ -4+3x^2 & 0 \end{pmatrix}$$

$$J(0,0)=\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix},\quad \lambda_{1,2}=\pm 2i,\quad \text{stable center}$$

$$J(\pm 2,0)=\begin{pmatrix} 0 & 1 \\ 8 & 0 \end{pmatrix},\quad \lambda_{1,2}=\pm\sqrt{8},\quad \text{saddle pts.}$$

- **Damped Nonlinear Vibrations**

$$mx''=-cx'-kx+\beta x^3$$

$$\begin{cases} x'=y \\ y'=-\frac{c}{m}y-\frac{k}{m}x\left(1-\frac{\beta}{k}x^2\right) \end{cases}$$

critical pts: $(0,0)$, $\left(\pm\sqrt{\frac{k}{\beta}},0\right)$ if $\beta>0$

$$J(x,y)=\begin{pmatrix} 0 & 1 \\ -\frac{k}{m}+\frac{3\beta}{m}x^2 & -\frac{c}{m} \end{pmatrix}$$

$$J(0,0)=\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \qquad \lambda_{1,2}=\frac{-c\pm\sqrt{c^2-4km}}{2m}$$

(1) $c^2-4km>0 \Longrightarrow \lambda_1, \lambda_2<0 \Longrightarrow$ nodal sink

(2) $c^2-4km<0 \Longrightarrow Re(\lambda_i)=-\frac{c}{2m}<0 \Longrightarrow$ spiral sink

<u>Example 2</u> $(m=1, c=2, k=5, \beta=\frac{5}{4})$ $c^2-4km=2^2-4\cdot5\cdot1=-16<0$

$$\begin{cases} x'=y \\ y'=-5x-2y+\frac{5}{4}x^3 \end{cases}$$

critical pts: $(0,0)$, $(\pm2,0)$

$$J(x,y)=\begin{pmatrix} 0 & 1 \\ -5+\frac{15}{4}x^2 & -2 \end{pmatrix}$$

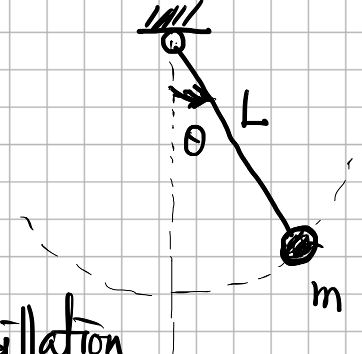
$J(0,0)=\begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$ $\lambda_{1,2}=-1\pm2i$ spiral sink

$J(\pm2,0)=\begin{pmatrix} 0 & 1 \\ 10 & -2 \end{pmatrix}$ $\begin{array}{l}\lambda_1=-1-\sqrt{11}<0 \\ \lambda_2=-1+\sqrt{11}>0\end{array}$ saddle pts.

<u>Fig. 6.4.6</u>

- <u>Nonlinear Pendulum</u>

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0$$



the undamped oscillation

$\underline{\theta \approx 0} \Longrightarrow \sin\theta \approx \theta \Longrightarrow \theta'' + \omega^2\theta = 0$ linear model $\omega^2 = \frac{g}{L}$.

$\Longrightarrow \theta(t) = A\cos\omega t + B\sin\omega t$

ω - circular frequency, $C = \sqrt{A^2+B^2}$ — amplitude

- <u>general nonlinear pendulum equation</u>

$$\boxed{\theta''(t) + c\,\theta'(t) + \omega^2 \sin\theta(t) = 0} \qquad \omega^2 = \frac{g}{L}$$

$$\begin{cases} x(t) = \theta(t) \\ y(t) = \theta'(t) \end{cases} \Longrightarrow \begin{cases} x' = y \\ y' = -cy - \omega^2 \sin x \end{cases}$$

- <u>$c = 0$ (the undamped case)</u>

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\Longrightarrow \begin{cases} x' = y \\ y' = -\omega^2 x + g(x), \end{cases} \text{ where } g(x) = -\omega^2(\sin x - x)$$

<u>critical pts</u>

$$\begin{cases} 0 = y \\ 0 = -\omega^2 \sin x \end{cases} \Longrightarrow (n\pi, 0) \text{ for } n = 0, \pm1, \pm2, \cdots$$

$\sin x = 0 \Longrightarrow x = n\pi$

<u>Jacobian matrix</u>

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -\omega^2\cos x & 0 \end{pmatrix}$$

even case $(n = 2m)$ $\cos(2m\pi) = 1$

$$J(2m\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm \omega i$$

stable center

odd case $(n = 2m+1)$ $\cos((2m+1)\pi) = -1$

$$J((2m+1)\pi, 0) = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm \omega$$

saddle pt.

trajectories

$$\frac{dy}{dx} = \frac{y'}{x'} = -\frac{\omega^2 \sin x}{y} \Rightarrow y\,dy + \omega^2 \sin x\,dx = 0$$

$\int_0^x$

$$\Rightarrow \underbrace{\frac{1}{2} y^2}_{\text{kinetic energy}} + \underbrace{\omega^2 (1 - \cos x)}_{\text{potential energy}} = E$$

arbitrary const. (total energy)

$$\Rightarrow y = \pm \sqrt{2E - 4\omega^2 \sin^2 \tfrac{1}{2} x}$$

Fig. 6.4.8

$$\begin{cases} x' = y \\ y' = -\sin x \end{cases}$$

- damped pendulum oscillations

$$\begin{cases} x' = y \\ y' = -\omega^2 \sin x - cy \end{cases} \qquad J(x,y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -c \end{pmatrix}$$

critical pts $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \cdots$

(1) $n = 2m+1$ n odd

$$J((2m+1)\pi, 0) = \begin{pmatrix} 0 & 1 \\ \omega^2 & -c \end{pmatrix}$$

$$0 = \det\begin{pmatrix} -\lambda & 1 \\ \omega^2 & -c-\lambda \end{pmatrix} = \lambda(\lambda+c) - \omega^2 = \lambda^2 + c\lambda - \omega^2$$

$$= \left(\lambda + \frac{c}{2}\right)^2 - \left(\frac{c^2}{4} + \omega^2\right)$$

$$\Rightarrow \lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 + \omega^2} \Rightarrow \lambda_1 < 0, \lambda_2 > 0 \text{ saddle pt}$$

(2) $n = 2m$ n even

$$J(2m\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -c \end{pmatrix} \Rightarrow \lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - \omega^2}$$

(i) $\left(\frac{c}{2}\right)^2 > \omega^2 \Rightarrow \lambda_1 < 0$ and $\lambda_2 < 0 \Rightarrow$ nodal sink

(ii) $\left(\frac{c}{2}\right)^2 < \omega^2 \Rightarrow \mathrm{Re}(\lambda_i) < 0 \Rightarrow$ spiral sink.

Fig. 6.4.10

$$\begin{cases} x' = y \\ y' = -\sin x - \frac{1}{4} y \end{cases} \qquad c = \frac{1}{4},\ \omega = 1 \qquad \left(\frac{c}{2}\right)^2 = \frac{1}{64} < 1 = \omega^2$$